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S. Cerreia *Vioglio, F. Maccheroni, and M. Marinacci*

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IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
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Hilbert A-Modules*

S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci

Department of Decision Sciences and IGIER, Università Bocconi

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Abstract

We consider real pre-Hilbert modules H on Archimedean f-algebras A with unit e. We provide conditions on A and H such that a Riesz representation theorem for bounded/continuous A-linear operators holds.

1 Introduction

Let A be an Archimedean f-algebra with (multiplicative) unit e. It is well known that Archimedean f-algebras are commutative. We next proceed by defining the objects we study in this paper.

Definition 1 *An abelian group* $(H,+)$ *is an* A-module *if and only if an outer product* $\cdot : A \times H \to H$ *is well defined with the following properties, for each* $a, b \in A$ *and for each* $x, y \in H$*:*

(1) $a \cdot (x+y) = a \cdot x + a \cdot y$*;*

(2) $(a+b) \cdot x = a \cdot x + b \cdot x$*;*

(3) $a \cdot (b \cdot x) = (ab) \cdot x$*;*

(4) $e \cdot x = x$*.*

An A*-module is a* pre-Hilbert A-module *if and only if an inner product* $\langle \ , \ \rangle_H : H \times H \to A$ *is well defined with the following properties, for each* $a \in A$ *and for each* $x, y, z \in H$*:*

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(5) $\langle x,x\rangle_H \geq 0$*, with equality if and only if* $x=0$*;*

(6) $\langle x,y\rangle_H = \langle y,x\rangle_H$*;*

(7) $\langle x+y,z\rangle_H = \langle x,z\rangle_H + \langle y,z\rangle_H$*;*

(8) $\langle a\cdot x,y\rangle_H = a\langle x,y\rangle_H$*.*

For $A=\mathbb{R}$ conditions (1)-(4) define vector spaces, while (5)-(8) define pre-Hilbert spaces. We will use Latin letters a,b,c to denote elements of A, Latin letters x,y,z to denote elements of H, and Greek letters α,β to denote elements of $\mathbb{R}$.

It is well known that[1]

$$\langle x,y\rangle_H^2 \leq \langle x,x\rangle_H \langle y,y\rangle_H \qquad \forall x,y\in H.$$

We can thus conclude that each $z\in H$ induces a map $f:H\to A$, via the formula

$$f(x) = \langle x,z\rangle_H \qquad \forall x\in H,$$

with the following properties:

- **A-linearity** $f(a\cdot x+b\cdot y) = af(x)+bf(y)$ for all $a,b\in A$ and for all $x,y\in H$;
- **Boundedness** There exists $c\in A_+$ such that $f(x)^2 \leq c\langle x,x\rangle_H$ for all $x\in H$.

In light of this fact, we give the following definition:

Definition 2 *Let A be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module. We say that H is* self-dual *if and only if for each $f:H\to A$ which is A-linear and bounded there exists $y\in H$ such that*

$$f(x) = \langle x,y\rangle_H \qquad \forall x\in H.$$

The goal of this paper is to provide conditions on A and H that will allow us to conclude that a pre-Hilbert A-module H is self-dual. Our initial motivation comes from Finance. There, Hilbert modules are the extension of the notion of Hilbert spaces that the analysis of conditional information requires, as first shown by Hansen and Richard [19]. In particular, self-duality is key to represent price operators through traded stochastic discount factors. Our results provide the general mathematical framework where conditional asset pricing can be performed.

[1] See Huijsmans and de Pagter [23, Theorem 3.4] and also Proposition 4 below.

Examples Consider a probability space $(\Omega, \mathcal{F}, P)$ and assume that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Denote by $\mathcal{L}^0(\mathcal{F}) = \mathcal{L}^0(\Omega, \mathcal{F}, P)$ and $\mathcal{L}^\infty(\mathcal{F}) = \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$, respectively, the space of $\mathcal{F}$-measurable functions and the space of $\mathcal{F}$-measurable and essentially bounded functions. Similarly, define $\mathcal{L}^0(\mathcal{G})$ and $\mathcal{L}^\infty(\mathcal{G})$. Define also

$$\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P) = \left\{ f \in \mathcal{L}^0(\mathcal{F}) : \mathbb{E}\left(f^2 || \mathcal{G}\right) \in \mathcal{L}^0(\mathcal{G}) \right\}$$

and

$$\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P) = \left\{ f \in \mathcal{L}^0(\mathcal{F}) : \mathbb{E}\left(f^2 || \mathcal{G}\right) \in \mathcal{L}^\infty(\mathcal{G}) \right\}.$$

The inner product, in both cases, can be defined by $(f, g) \mapsto \mathbb{E}(fg||\mathcal{G})$. In Section 6, we show that $\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$ is a pre-Hilbert $\mathcal{L}^0(\mathcal{G})$-module and $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$ is a pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module. Both spaces are of particular interest in Finance. The first space is the one originally used in the seminal paper of Hansen and Richard [19]. On the other hand, in Filipovic, Kupper, and Vogelpoth [14] (see also [30]), the second space has been shown to represent the family of all continuous and $\mathcal{L}^\infty(\mathcal{G})$-linear operators from $\mathcal{L}^2(\mathcal{F})$ to $\mathcal{L}^2(\mathcal{G})$.[2] In other words, $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$ can be interpreted as the space of all conditional stochastic discount factors (state price densities).

Related literature The literature on self-dual modules can be roughly divided in two main streams. The first one introduced the notion of Hilbert A-modules and considers *complex* C^*-algebras A. In particular, it considers algebras that admit a concrete representation as a space of continuous functions over a compact space. The second focuses on a particular algebra of functions, namely, $\mathcal{L}^0(\mathcal{G}) = \mathcal{L}^0(\Omega, \mathcal{G}, P)$. The notion of pre-Hilbert A-modules was introduced by Kaplansky [25]. Kaplansky [25] considers modules over commutative (complex) AW^*-algebras A with unit and shows that a pre-Hilbert A-module H is self-dual if and only if H satisfies some extra algebraic property (Definition 9). Paschke [33] investigates the properties of self-dual modules defined over complex B^*-algebras. Two other related papers are Frank [15] and [16] (see also [29], for a textbook exposition). In both papers, when A is assumed to be a W^* complex algebra, a pre-Hilbert A-module H is shown to be self-dual if and only if the unit ball (properly defined) of H is complete with respect to some linear topology. On the other hand, Guo, in [17] and [18], studies pre-Hilbert $\mathcal{L}^0(\mathcal{G})$-modules H and shows that they are self-dual if and only if H is complete with respect to a particular metrizable topology.[3]

[2] $\mathcal{L}^2(\mathcal{F}) = \mathcal{L}^2(\Omega, \mathcal{F}, P)$ is the space of $\mathcal{F}$-measurable and square integrable functions.

[3] In this paper, we focus on Hilbert modules. For the Banach case, we refer to Cerreia-Vioglio, Kupper, Maccheroni, Marinacci, and Vogelpoth [11] and the references therein. A pioneer work on the subject is Haydon, Levy, and Raynaud [20].

Our Contributions We provide (topological) conditions on A and H that will allow us to conclude that a pre-Hilbert A-module H is self-dual. We start by considering A to be an algebra of $\mathcal{L}^\infty$ type (Subsection 2.1). In this case, H can be suitably topologized with several norm topologies. In particular, two norms stand out: $\| \|_H$ and $\| \|_m$ (Subsection 3.1). When A is of $\mathcal{L}^\infty$ type and H is a pre-Hilbert A-module, in Theorem 3, we show that the following conditions are equivalent:

(i) H is self-dual;

(ii) B_H is "weakly" compact (where B_H is the unit ball induced by $\| \|_H$);

(iii) H is "weakly" sequentially complete;

(iv) B_H is complete with respect to $\| \|_m$.

Conditions (ii) and (iii) are novel conditions. On the other hand, a condition of completeness, similar to Condition (iv), has been found also in the complex case by Frank [15] (see the proof of [29, Theorem 3.5.1]). When $A = \mathbb{R}$, it is easy to show that $\| \|_H$ and $\| \|_m$ are equivalent (Proposition 9). Thus, in this case, properties (i)-(iv) are well known to be equivalent and we can conclude that our Theorem 3 is a generalization of the classical Riesz representation theorem for Hilbert spaces.

We then move to consider A to be an f-algebra of $\mathcal{L}^0$ type (Subsection 5.3). In this case, H can be topologized with an invariant metric d_H. When A is of $\mathcal{L}^0$ type and H is a pre-Hilbert A-module, in Theorem 5, we show that the following conditions are equivalent:

(i') H is self-dual;

(ii') H is complete with respect to d_H.

We are thus able to obtain Guo's self-duality result ([17] and [18]). The contribution to the literature of our Theorem 5 is to show the connection with the self-duality result for modules on algebras of $\mathcal{L}^\infty$ type. In fact, we show that each pre-Hilbert $\mathcal{L}^0$-module H contains a *dense* pre-Hilbert $\mathcal{L}^\infty$-module H_e. Thus, verifying the self-duality of H amounts to verify the self-duality of H_e, which can then be extended to H via a density argument.

Outline of the paper Section 2 introduces the two kinds of algebras A we will consider in studying the self-duality problem. Subsection 2.1 deals with Arens algebras, that is, *real* Banach algebras which admit a concrete representation as a space of continuous functions. Algebras of $\mathcal{L}^{\infty}$ type will belong to this class (Definition 5). Instead, Subsection 2.2 deals with f-algebras of $\mathcal{L}^{0}$ type (Definition 6).

In Section 3, we show how a pre-Hilbert A-module naturally turns out to be a vector space that can also be topologized in several different and useful ways. In Subsection 3.2, we study the corresponding topological duals.

Section 4 deals with the study of the dual module, that is, the set $H^{\sim}$ of all A-linear and bounded operators from H to A. The set $H^{\sim}$ turns out to be an A-module which can also be topologized and its study is key in dealing with the self-duality problem. From a topological point of view, the structure of $H^{\sim}$ differs depending if A is of $\mathcal{L}^{\infty}$ type or of $\mathcal{L}^{0}$ type. In Subsection 4.1, we study the first case. In Subsection 4.2, we study the second case. Finally, in Subsection 4.3, we show that $H^{\sim}$ can be identified with the norm dual of some Banach space when A is of $\mathcal{L}^{\infty}$ type.

Section 5 contains our results on self-duality. First, we discuss the case when A is of $\mathcal{L}^{\infty}$ type. An important subcase is when A is finite dimensional, which we discuss right after. We conclude the section by discussing the case in which A is an f-algebra of $\mathcal{L}^{0}$ type. In Section 6 we discuss five examples of pre-Hilbert A-modules that, given our results, turn out to be self-dual. We relegate to the Appendix the proofs of some ancillary facts.

2 Function algebras

2.1 Arens algebras

Given a commutative *real normed* algebra A with multiplicative unit e, we denote by $\| \, \|_A$ the norm of A. We denote by A^* the norm dual of A and by $\langle \, , \, \rangle$ the dual pairing of the algebra A, that is, $\langle a, \varphi \rangle = \varphi(a)$ for all $a \in A$ and $\varphi \in A^*$. Unless otherwise specified, the norm dual A^* of A is endowed with the weak* topology and all of its subsets are endowed with the relative weak* topology. In the first part of the paper, we will mostly consider commutative real Banach algebras A that admit a concrete representation. These *real* Banach algebras were first studied by Arens [8] and Kelley and Vaught [26].[4]

Definition 3 *A commutative real Banach algebra A with unit e such that*

$$\|e\|_A = 1 \text{ and } \|a\|_A^2 \leq \left\|a^2 + b^2\right\|_A \quad \forall a, b \in A$$

is called an Arens algebra.

[4]For two more recent studies, see also Albiac and Kanton [2] and [3]. Recall that a Banach algebra is such that $\|ab\|_A \leq \|a\|_A \|b\|_A$ for all $a, b \in A$.

Given an Arens algebra A, define

$$S = \{\varphi \in A^* : \|\varphi\|_{A^*} = \varphi(e) = 1\}$$
$$K = \{\varphi \in S : \varphi(ab) = \varphi(a)\varphi(b) \quad \forall a, b \in A\}.$$

The set K is compact and Hausdorff. Denote by $C(K)$ the space of real valued continuous functions on K. We endow $C(K)$ with the supnorm. It is well known that A admits a concrete representation, that is, the map $T : A \to C(K)$, defined by

$$T(a)(\varphi) = \langle a, \varphi \rangle \qquad \forall \varphi \in K, \forall a \in A,$$

is an isometry and an algebra isomorphism (see [26], [2], and [3]).

The cone generated by the squares of A induces a natural order relation on A itself: $a \geq b$ if and only if $a - b$ belongs to the norm closure of $\{c^2 : c \in A\}$. By using standard techniques, the above concrete representation of A implies that $(A, \geq)$ is a Riesz space with *strong order* unit e and T is also a lattice isomorphism. In particular, K coincides with the set of all nonzero lattice homomorphisms and A is an Archimedean f-algebra with unit e. We also have that each element $\varphi \in K$ is positive. Finally, $\| \, \|_A$ is a lattice norm such that

$$\|a\|_A = \min\{\alpha \geq 0 : |a| \leq \alpha e\} \text{ and } \left\|a^2\right\|_A = \|a\|_A^2 \qquad \forall a \in A.$$

In light of these observations, note that for each $a \geq 0$, there exists a unique $b \geq 0$ such that $b^2 = a$. From now on, we will denote such an element by $a^{\frac{1}{2}}$ or $\sqrt{a}$.

Note that if A admits a strictly positive linear functional $\bar{\varphi} : A \to \mathbb{R}$, then we could also renorm A with the norm $\| \, \|_1 : A \to [0, \infty)$, defined by $\|a\|_1 = \bar{\varphi}(|a|)$ for all $a \in A$. It is immediate to see that $\|a\|_1 \leq \|\bar{\varphi}\|_{A^*} \|a\|_A$ for all $a \in A$, and so the $\| \, \|_A$ norm topology τ_A is finer than the $\| \, \|_1$ norm topology τ_1; i.e., $\tau_1 \subseteq \tau_A$. We will denote the norm dual of A with respect to $\| \, \|_1$ by A'. Finally, we have that $A' \subseteq A^*$. If A admits a strictly positive linear functional $\bar{\varphi} : A \to \mathbb{R}$, then we could also consider A endowed with the invariant metric $d : A \times A \to [0, \infty)$, defined by $d(a, b) = \bar{\varphi}(|b - a| \wedge e)$ for all $a, b \in A$. It is immediate to see that $d(a, b) = \bar{\varphi}(|b - a| \wedge e) \leq \bar{\varphi}(|b - a|) = \|b - a\|_1$ for all $a, b \in A$, and so the $\| \, \|_1$ norm topology τ_1 is finer than the d metric topology τ_d; i.e., $\tau_d \subseteq \tau_1$.

The existence of a strictly positive linear functional $\bar{\varphi} : A \to \mathbb{R}$ will play a key role in the rest of the paper.[5] We conclude the section by exploring the extent of this assumption and its relation with the existence of a measure m on K whose support separates the points of A. Before presenting the formal result, we provide a definition:

[5] Without loss of generality, $\bar{\varphi}$ can always be assumed to be such that $\|\bar{\varphi}\|_{A^*} = 1$.

Definition 4 *Let A be an Arens algebra and m a finite measure on the Borel σ-algebra of K. The measure m separates points if and only if the support of m separates the points of A.*

Proposition 1 *Let A be an Arens algebra and $\bar{\varphi} \in A^*$. The following statements are equivalent:*

(i) The functional $\bar{\varphi}$ is strictly positive and such that $\|\bar{\varphi}\|_{A^*} = 1$;

(ii) There exists a (unique) probability measure $m_{\bar{\varphi}} = m$ on the Borel σ-algebra of K such that $\operatorname{supp} m = K$ *and*

$$\bar{\varphi}(a) = \int_K \langle a, \varphi \rangle \, dm(\varphi) \qquad \forall a \in A; \tag{1}$$

(iii) There exists a probability measure $m_{\bar{\varphi}} = m$ that separates points and satisfies (1).

Remark 1 *From now on, when we will be dealing with an Arens algebra that admits a strictly positive linear functional $\bar{\varphi}$ on A such that $\|\bar{\varphi}\|_{A^*} = 1$, the measure m will be meant to be $m_{\bar{\varphi}}$. Viceversa, if A admits a measure that separates points, then $\bar{\varphi}$ will be meant to be defined as in (1).*

We conclude by defining a particular class of Arens algebras which are isomorphic to some space $\mathcal{L}^\infty(\Omega, \mathcal{G}, P)$ (see [1, Corollary 2.2]).

Definition 5 *Let A be an Arens algebra. We say that A is of $\mathcal{L}^\infty$ type if and only if A is* Dedekind complete *and admits a* strictly positive order continuous *linear functional $\bar{\varphi}$ on A.*

2.2 f-algebras

Assume that A is an Archimedean f-algebra with unit e (see Aliprantis and Burkinshaw [6, Definition 2.53]). It is well known that e is a *weak order* unit. If A is Dedekind complete and $a \geq \frac{1}{n}e$ for some $n \in \mathbb{N}$, then there exists a unique $b \in A_+$ such that $ab = e$. We denote this element a^{-1}. If $a \geq 0$ is such that there exists a^{-1} and $b \in A$, then we alternatively denote ba^{-1} by b/a. By [22, Theorem 3.9], if A is also Dedekind complete, for each $a \geq 0$, there exists a unique $b \geq 0$ such that $b^2 = a$. Also in this case, we will denote such an element by $a^{\frac{1}{2}}$ or $\sqrt{a}$. The principal ideal generated by e is the set

$$A_e = \{a \in A : \exists \alpha > 0 \text{ s.t. } |a| \leq \alpha e\}.$$

It is immediate to see that A_e is a subalgebra of A with unit e. If A is an Arens algebra, then $A_e = A$. If there exists a linear and strictly positive functional $\bar{\varphi} : A_e \to \mathbb{R}$, then we can define $d : A \times A \to [0, \infty)$ by

$$d(a, b) = \bar{\varphi}(|b - a| \wedge e) \qquad \forall a, b \in A.$$

As in the case of an Arens algebra, d is an invariant metric. As already noted, an Arens algebra, in particular one of $\mathcal{L}^\infty$ type, is an Archimedean f-algebra with unit. In this paper, other than algebras of $\mathcal{L}^\infty$ type, we focus also on another particular class of f-algebras:

Definition 6 *Let A be an Archimedean f-algebra with unit e. We say that A is an f-algebra of $\mathcal{L}^0$ type if and only if A_e is an* Arens algebra of $\mathcal{L}^\infty$ type *and A is* Dedekind complete *and d* complete.

By [6, Theorems 2.28 and 4.7], if A is an f-algebra of $\mathcal{L}^0$ type, d is generated by the Riesz pseudonorm $c \mapsto \bar{\varphi}(|c| \wedge e)$, then it is easy to prove that the topology generated by d is linear, locally solid, and Fatou. Moreover, it can be shown that A is universally complete and such that:

1. If $a_n \downarrow 0$ and $b \geq 0$, then $a_n b \downarrow 0$ and $a_n b \xrightarrow{d} 0$;
2. If $b \geq 0$ and $a_n \xrightarrow{d} a$, then $ba_n \xrightarrow{d} ba$.

3 The vector space structure of H

In this section, we will first show that a pre-Hilbert A-module has a natural structure of vector space. Next, we will show that the A valued inner product $\langle \ , \ \rangle_H$ shares most of the properties of standard real valued inner products. In particular, under mild assumptions on A, we will show that it also induces a real valued inner product on H, thus making H into a pre-Hilbert space.

We use the outer product $\cdot$ to define a scalar product:

$$\begin{array}{rccc} \cdot^e : & \mathbb{R} \times H & \to & H \\ & (\alpha, x) & \mapsto & (\alpha e) \cdot x \end{array}.$$

We next show that $\cdot^e$ makes the abelian group H into a real vector space.

Proposition 2 *Let A be an Archimedean f-algebra with unit e and H an A-module. $(H, +, \cdot^e)$ is a real vector space.*

Proof. By assumption, H is an abelian group. For each $\alpha, \beta \in \mathbb{R}$ and each $x, y \in H$, we have that

(1) $\alpha \cdot^e (x+y) = \alpha e \cdot (x+y) = (\alpha e) \cdot x + (\alpha e) \cdot y = \alpha \cdot^e x + \alpha \cdot^e y$;

(2) $(\alpha+\beta) \cdot^e x = ((\alpha+\beta) e) \cdot x = (\alpha e + \beta e) \cdot x = (\alpha e) \cdot x + (\beta e) \cdot x = \alpha \cdot^e x + \beta \cdot^e x$;

(3) $\alpha \cdot^e (\beta \cdot^e x) = (\alpha e) \cdot ((\beta e) \cdot x) = ((\alpha e)(\beta e)) \cdot x = ((\alpha\beta) e) \cdot x = (\alpha\beta) \cdot^e x$;

(4) $1 \cdot^e x = (1e) \cdot x = e \cdot x = x$. ■

From now on, we will often write αx in place of $\alpha \cdot^e x$.

Corollary 1 *Let A be an Archimedean f-algebra with unit e and H an A-module. If $f : H \to A$ is an A-linear operator, then f is linear.*

If A is an Arens algebra, given a finite probability measure m on the Borel σ-algebra of K we can also define $\langle\ ,\ \rangle_m : H \times H \to \mathbb{R}$ by

$$\langle x, y\rangle_m = \int_K \langle \langle x, y\rangle_H, \varphi\rangle \, dm(\varphi) \quad \forall x, y \in H.$$

For each $\varphi \in K$, we also define and study the functionals $\langle\ ,\ \rangle_\varphi : H \times H \to \mathbb{R}$ defined by

$$\langle x, y\rangle_\varphi = \langle \langle x, y\rangle_H, \varphi\rangle \qquad \forall x, y \in H.$$

Note that $\langle\ ,\ \rangle_\varphi = \langle\ ,\ \rangle_{\delta_\varphi}$ for all $\varphi \in K$ where δ_φ is the Dirac measure at φ. We next show that $\langle\ ,\ \rangle_m$ is a symmetric bilinear form which is positive semidefinite on $H \times H$.

Proposition 3 *Let A be an Arens algebra and H a pre-Hilbert A-module. The following statements are true:*

1. *$\langle\ ,\ \rangle_m$ is a positive semidefinite symmetric bilinear form;*
2. *$\langle x, x\rangle_m = 0$ implies $x = 0$, provided m separates points;*
3. *$\langle x, y\rangle_m^2 \le \langle x, x\rangle_m \langle y, y\rangle_m$ for all $x, y \in H$;*
4. *$\langle x, a \cdot y\rangle_m = \langle a \cdot x, y\rangle_m$ for all $a \in A$ and for all $x, y \in H$;*
5. *$\langle x, y\rangle_m = \bar{\varphi}(\langle x, y\rangle_H)$ for all $x, y \in H$, provided m separates points.*

Proof. We here prove points 2. and 5. and leave the remaining easy ones to the reader. Assume that m separates points. Define $\bar{\varphi}$ as in (1). By Proposition 1, it follows that

$$\bar{\varphi}(a) = \int_K \langle a, \varphi\rangle \, dm(\varphi) \qquad \forall a \in A \tag{2}$$

is a strictly positive and linear functional.

5. By definition of $\langle\ ,\ \rangle_m$ and (2), observe that for each $x, y \in H$

$$\langle x, y\rangle_m = \int_K \langle \langle x, y\rangle_H, \varphi\rangle \, dm(\varphi) = \bar{\varphi}(\langle x, y\rangle_H).$$

2. By assumption, $\langle x, x\rangle_H \geq 0$ for all $x \in H$. By point 5., if $\langle x, x\rangle_m = 0$, then $\bar{\varphi}(\langle x, x\rangle_H) = 0$. Since $\bar{\varphi}$ is strictly positive, we have that $\langle x, x\rangle_H = 0$, proving that $x = 0$. ■

Corollary 2 *Let A be an Arens algebra and H a pre-Hilbert A-module. If A admits a measure m that separates points, then $(H, +, \cdot^e, \langle\ ,\ \rangle_m)$ is a pre-Hilbert space.*

Proposition 4 *Let A be an Arens algebra and H a pre-Hilbert A-module. The following statements are true:*

1. $\langle x, y\rangle_H^2 \leq \langle x, x\rangle_H \langle y, y\rangle_H$ *for all* $x, y \in H$*;*
2. $|\langle x, y\rangle_H| \leq \langle x, x\rangle_H^{\frac{1}{2}} \langle y, y\rangle_H^{\frac{1}{2}}$ *for all* $x, y \in H$*;*
3. $\left\|\langle x, y\rangle_H^2\right\|_A \leq \|\langle x, x\rangle_H\|_A \|\langle y, y\rangle_H\|_A$ *for all* $x, y \in H$*;*
4. $\|\langle x, y\rangle_H\|_A \leq \|\langle x, x\rangle_H\|_A^{\frac{1}{2}} \|\langle y, y\rangle_H\|_A^{\frac{1}{2}}$ *for all* $x, y \in H$*.*

Proof. By Proposition 2 and Corollary 1 and since A is, in particular, an Archimedean f-algebra with unit, point 1. is an easy consequence of [23, Theorem 3.4]. Since A is an Arens algebra, each positive element admits a unique square root and point 2. also follows. Since A is an Arens algebra and $\|\ \|_A$ is also a lattice norm, we have that for each $x, y \in H$

$$\|\langle x, y\rangle_H\|_A^2 = \left\|\langle x, y\rangle_H^2\right\|_A \leq \|\langle x, x\rangle_H \langle y, y\rangle_H\|_A \leq \|\langle x, x\rangle_H\|_A \|\langle y, y\rangle_H\|_A,$$

proving points 3. and 4. ■

Remark 2 *If A is a Dedekind complete Archimedean f-algebra with unit e, then points 1. and 2. are still true and their proofs remain the same.*

3.1 Topological structure

Since a pre-Hilbert A-module H is also a vector space, we can try to endow H with a topology induced by either a norm or an invariant metric. In fact, given the structure of A and H, we have several different competing norms and topologies. The next subsections are devoted to the study of these norms and metric and their relations. Before starting, note that if A is an Arens algebra or a Dedekind complete Archimedean

f-algebra with unit, then $\langle\ ,\ \rangle_H$ defines a vector-valued norm,[6] $N : H \to A_+$, via the formula

$$N(x) = \langle x, x \rangle_H^{\frac{1}{2}} \qquad \forall x \in H.$$

If A were equal to $\mathbb{R}$, then N would be a standard norm and we would say that

$$x_n \to x \overset{\text{def}}{\Longleftrightarrow} N(x - x_n) \to 0.$$

Since $\mathbb{R}$ is always endowed with the usual topology, this definition would be unambiguous. When $A \neq \mathbb{R}$, such a statement is not true anymore, since we could endow A with different linear topologies changing the meaning of $N(x - x_n) \to 0$. In other words, by combining the topological structure of A with N we are able to induce different topologies on H (see point 5. of Proposition 5, point 3. of Proposition 6, equation (5), and point 2. of Proposition 7).

3.1.1 The $\|\ \|_H$ norm

Assume A is an Arens algebra. Define $\|\ \|_H : H \to [0, \infty)$ by

$$\|x\|_H = \sqrt{\|\langle x, x \rangle_H\|_A} \qquad \forall x \in H.$$

For each $\varphi \in K$, define also $\|\ \|_\varphi : H \to [0, \infty)$ by

$$\|x\|_\varphi = \sqrt{\langle x, x \rangle_\varphi} \qquad \forall x \in H.$$

Proposition 5 *Let A be an Arens algebra and H a pre-Hilbert A-module. The following statements are true:*

1. *$\|\ \|_\varphi$ is a seminorm for all $\varphi \in K$;*
2. *$\|\ \|_H$ is a norm;*
3. *$\|x\|_H = \max_{\varphi \in K} \|x\|_\varphi = \sqrt{\max_{\varphi \in K} \langle x, x \rangle_\varphi}$ for all $x \in H$;*
4. *$\|a \cdot x\|_H \leq \|a\|_A \|x\|_H$ for all $a \in A$ and all $x \in H$;*
5. *$\|x\|_H = \|N(x)\|_A$ for all $x \in H$.*

---

[6] In particular, N is such that

1. $N(x) = 0$ if and only if $x = 0$;
2. $N(a \cdot x) = |a| N(x)$ for all $a \in A$ and for all $x \in H$;
3. $N(x + y) \leq N(x) + N(y)$ for all $x, y \in H$.

Proof. Points 1. and 2. follow from routine arguments.

3. Recall that $T : A \to C(K)$ is a linear isometry. Thus, we have that

$$\|x\|_H^2 = \|\langle x,x\rangle_H\|_A = \|T(\langle x,x\rangle_H)\|_\infty = \max_{\varphi\in K} |T(\langle x,x\rangle_H)(\varphi)| = \max_{\varphi\in K} |\langle\langle x,x\rangle_H, \varphi\rangle|$$
$$= \max_{\varphi\in K} \left|\langle x,x\rangle_\varphi\right| = \max_{\varphi\in K} \langle x,x\rangle_\varphi = \max_{\varphi\in K} \|x\|_\varphi^2 \qquad \forall x \in H,$$

proving the statement.

4. Given any $a \in A$ and $x \in H$, it holds

$$\|a\cdot x\|_H^2 = \|\langle a\cdot x, a\cdot x\rangle_H\|_A = \left\|a^2 \langle x,x\rangle_H\right\|_A \le \left\|a^2\right\|_A \|\langle x,x\rangle_H\|_A \le \|a\|_A^2 \|\langle x,x\rangle_H\|_A$$

proving the statement.

5. Since A is an Arens algebra, it follows that $\|\sqrt{a}\|_A = \sqrt{\|a\|_A}$ for all $a \in A_+$.[7] This implies that

$$\|x\|_H = \sqrt{\|\langle x,x\rangle_H\|_A} = \left\|\sqrt{\langle x,x\rangle_H}\right\|_A = \|N(x)\|_A \qquad \forall x \in H,$$

proving the statement. ■

By Proposition 4, it readily follows that

$$\|\langle x,y\rangle_H\|_A \le \|x\|_H \|y\|_H \qquad \forall x,y \in H. \tag{3}$$

Corollary 3 *Let A be an Arens algebra and H a pre-Hilbert A-module. For each $y \in H$, the functional $\langle\cdot,y\rangle_H : H \to A$ is A-linear, $\|\ \|_H - \|\ \|_A$ continuous, and has norm $\|y\|_H$.*

Proof. Fix $y \in H$. It is immediate to see that the operator induced by y is A-linear, thus, linear. Continuity easily follows from (3). Since the norm of the linear operator is given by

$$\sup\left\{\|\langle x,y\rangle_H\|_A / \|x\|_H : x \ne 0\right\},$$

the statement easily follows from (3) and the definition of $\|\ \|_H$. ■

In light of these observations and since $\|\ \|_H$ can be defined for any pre-Hilbert A-module when A is an Arens algebra, we propose the following definition:

Definition 7 *Let A be an Arens algebra and H a pre-Hilbert A-module. We say that H is an Hilbert A-module if and only if H is $\|\ \|_H$ complete.*

---

[7] Recall that if $a \ge 0$, then $a^{\frac{1}{2}} = b$ is the unique positive element such that $b^2 = a$. Since A is an Arens algebra, it follows that $\|a\|_A = \left\|b^2\right\|_A = \|b\|_A^2 = \|\sqrt{a}\|_A^2$, proving that

$$\|\sqrt{a}\|_A = \sqrt{\|a\|_A} \qquad \forall a \in A_+.$$

3.1.2 The $\| \|_p$ norm

Assume A is an Arens algebra that admits a strictly positive linear functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$. Define $\| \|_p : H \to [0, \infty)$ by

$$\|x\|_p = \bar{\varphi}\left(\langle x, x\rangle_H^{\frac{1}{2}}\right) \qquad \forall x \in H.$$

Proposition 6 *Let A be an Arens algebra and H a pre-Hilbert A-module. If A admits a strictly positive functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$, then the following statements are true:*

1. *$\| \|_p$ is a norm;*
2. *$\|a \cdot x\|_p \leq \|a\|_A \|x\|_p$ for all $a \in A$ and for all $x \in H$;*
3. *$\|x\|_p = \bar{\varphi}(N(x)) = \|N(x)\|_1$ for all $x \in H$.*

Proof. Since $\bar{\varphi}$ is strictly positive, note that

$$x = 0 \iff \langle x, x\rangle_H = 0 \iff \langle x, x\rangle_H^{\frac{1}{2}} = 0 \iff \bar{\varphi}\left(\langle x, x\rangle_H^{\frac{1}{2}}\right) = 0 \iff \|x\|_p = 0.$$

Second, since $\bar{\varphi}$ is linear, observe that for each $\alpha \in \mathbb{R}$ and for each $x \in H$

$$\|\alpha x\|_p = \bar{\varphi}\left(\langle \alpha x, \alpha x\rangle_H^{\frac{1}{2}}\right) = \bar{\varphi}\left(|\alpha| \langle x, x\rangle_H^{\frac{1}{2}}\right) = |\alpha| \bar{\varphi}\left(\langle x, x\rangle_H^{\frac{1}{2}}\right) = |\alpha| \|x\|_p .$$

Third, by Proposition 4, note that for each $x, y \in H$

$$\begin{aligned}\langle x+y, x+y\rangle_H &= \langle x, x\rangle_H + 2\langle x, y\rangle_H + \langle y, y\rangle_H \leq \langle x, x\rangle_H + 2\langle x, x\rangle_H^{\frac{1}{2}} \langle y, y\rangle_H^{\frac{1}{2}} + \langle y, y\rangle_H \\ &= \left(\langle x, x\rangle_H^{\frac{1}{2}} + \langle y, y\rangle_H^{\frac{1}{2}}\right)^2 .\end{aligned}$$

We can conclude that $\langle x+y, x+y\rangle_H^{\frac{1}{2}} \leq \langle x, x\rangle_H^{\frac{1}{2}} + \langle y, y\rangle_H^{\frac{1}{2}}$ for all $x, y \in H$. Since $\bar{\varphi}$ is positive and linear, the triangular inequality for $\| \|_p$ follows.

2. Given any $a \in A$ and $x \in H$, since $\bar{\varphi}$ is positive and linear, it holds

$$\begin{aligned}\|a \cdot x\|_p &= \bar{\varphi}\left(\langle a \cdot x, a \cdot x\rangle_H^{\frac{1}{2}}\right) = \bar{\varphi}\left(|a| \langle x, x\rangle_H^{\frac{1}{2}}\right) \leq \bar{\varphi}\left((\|a\|_A e) \langle x, x\rangle_H^{\frac{1}{2}}\right) \\ &\leq \|a\|_A \bar{\varphi}\left(\langle x, x\rangle_H^{\frac{1}{2}}\right) = \|a\|_A \|x\|_p ,\end{aligned}$$

proving the statement.

3. The statement follows by definition of $\| \|_p$ and $\| \|_1$. ■

3.1.3 The $\| \ \|_m$ norm

Assume A is an Arens algebra that admits a strictly positive linear functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$ or, equivalently, a probability measure m that separates points (see Proposition 1). Define $\|\cdot\|_m : H \to [0, \infty)$ by

$$\|x\|_m = \sqrt{\langle x, x\rangle_m} = \sqrt{\int_K \langle x, x\rangle_\varphi \, dm(\varphi)} \qquad \forall x \in H. \tag{4}$$

By Propositions 1 and 3, $\langle \ , \ \rangle_m$ is an inner product on H and it is immediate to see that $\| \ \|_m$ is a norm and

$$\|x\|_m = \sqrt{\bar{\varphi}(\langle x, x\rangle_H)} = \sqrt{\bar{\varphi}\left(N(x)^2\right)} \qquad \forall x \in H. \tag{5}$$

3.1.4 The d_H metric

Assume that A is either a Dedekind complete Archimedean f-algebra with unit that admits a strictly positive linear functional $\bar{\varphi} : A_e \to \mathbb{R}$ or A is an Arens algebra that admits a strictly positive linear functional $\bar{\varphi} : A \to \mathbb{R}$. In the second case, assume also that $\|\bar{\varphi}\|_{A^*} = 1$. Define $d_H : H \times H \to [0, \infty)$ by

$$d_H(x, y) = \bar{\varphi}(N(x - y) \wedge e) \qquad \forall x, y \in H. \tag{6}$$

Recall that the hypothesis of Dedekind completeness is needed to define $N(x) = \langle x, x\rangle_H^{\frac{1}{2}}$ when A is an Archimedean f-algebra with unit.

Proposition 7 *Let A be either a Dedekind complete Archimedean f-algebra with unit e or an Arens algebra and H a pre-Hilbert A-module. If A admits a strictly positive functional $\bar{\varphi} : A_e \to \mathbb{R}$, then the following statements are true:*

1. *d_H is an invariant metric;*

2. *$d_H(x, y) = d(0, N(x - y))$ for all $x, y \in H$.*

Proof. 1. Since $\bar{\varphi}$ is strictly positive, we have that

$$\begin{aligned} d_H(x, y) = 0 \iff \bar{\varphi}(N(x - y) \wedge e) = 0 \iff N(x - y) \wedge e = 0 \\ \iff N(x - y) = 0 \iff x = y. \end{aligned}$$

It is immediate to see that $d_H(x, y) = d_H(y, x)$ for all $x, y \in H$ as well as

$$d_H(x + z, y + z) = d_H(x, y) \qquad \forall x, y, z \in H.$$

Finally, by [6, Lemma 1.4] and since $N(x+y) \leq N(x) + N(y)$ for all $x, y \in H$, we can conclude that

$$\begin{aligned} d_H(x,y) &= \bar{\varphi}(N(x-y) \wedge e) = \bar{\varphi}(N((x-z)+(z-y)) \wedge e) \\ &\leq \bar{\varphi}((N(x-z) + N(z-y)) \wedge e) \leq \bar{\varphi}(N(x-z) \wedge e + N(z-y) \wedge e) \\ &= d_H(x,z) + d_H(z,y) \qquad \forall x,y,z \in H, \end{aligned}$$

proving the statement.

2. By definition of d_H, d, and N, we have that

$$d_H(x,y) = \bar{\varphi}(N(x-y) \wedge e) = \bar{\varphi}(|N(x-y) - 0| \wedge e) = d(0, N(x-y)) \quad \forall x,y \in H,$$

proving the statement. ■

3.1.5 Relations among norms

Assume A is an Arens Algebra that admits a strictly positive linear functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$. First, by Propositions 1 and 5 and by equation (4), we have that

$$\|x\|_m \leq \|x\|_H \qquad \forall x \in H.$$

We can conclude that

$$x_n \overset{\|\ \|_H}{\to} 0 \implies x_n \overset{\|\ \|_m}{\to} 0.$$

The $\|\ \|_H$ norm topology τ_H is thus finer than the $\|\ \|_m$ norm topology τ_m; i.e., $\tau_m \subseteq \tau_H$. Similarly, by Proposition 1, we have that

$$\begin{aligned} \|x\|_p &= \bar{\varphi}\left(\langle x,x\rangle_H^{\frac{1}{2}}\right) = \int_K \left\langle \langle x,x\rangle_H^{\frac{1}{2}}, \varphi\right\rangle dm(\varphi) = \int_K \langle\langle x,x\rangle_H, \varphi\rangle^{\frac{1}{2}} dm(\varphi) \\ &\leq \sqrt{\int_K \langle\langle x,x\rangle_H, \varphi\rangle\, dm(\varphi)} = \|x\|_m \qquad \forall x \in H. \end{aligned}$$

We can conclude that

$$x_n \overset{\|\ \|_m}{\to} 0 \implies x_n \overset{\|\ \|_p}{\to} 0. \tag{7}$$

The $\|\ \|_m$ norm topology τ_m is thus finer than the $\|\ \|_p$ norm topology τ_p; i.e., $\tau_p \subseteq \tau_m$. Summing up, we have that

$$\|x\|_p \leq \|x\|_m \leq \|x\|_H \qquad \forall x \in H. \tag{8}$$

Finally, note that

$$d_H(x,y) \leq \|x-y\|_p \qquad \forall x,y \in H. \tag{9}$$

The $\|\ \|_p$ norm topology τ_p is thus finer than the d_H topology τ_{d_H}; i.e., $\tau_{d_H} \subseteq \tau_p$.

We next explore the continuity\boundedness properties of A-linear operators: $f : H \to A$. We then conclude by showing that our three norms are equivalent when A is finite dimensional.

Proposition 8 *Let A be a Dedekind complete Arens algebra and H a pre-Hilbert A-module. If A admits a strictly positive linear functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$, the following statements are true:*

1. *An A-linear operator $f : H \to A$ is bounded if and only if f is $\| \|_H - \| \|_A$ continuous;*

2. *If $f : H \to A$ is A-linear and $\| \|_H - \| \|_A$ continuous, then $f : H \to A$ is $\| \|_p - \| \|_1$ continuous;*

3. *If $f : H \to A$ is A-linear and $\| \|_H - \| \|_A$ continuous, then $f : H \to A$ is $\| \|_m - \| \|_1$ continuous;*

4. *For each $x, y \in H$,*

$$|\bar{\varphi}(\langle x, y\rangle_H)| \leq \bar{\varphi}(|\langle x, y\rangle_H|) = \|\langle x, y\rangle_H\|_1 \leq \|x\|_p \|y\|_H ;$$

5. *For each $x, y \in H$,*

$$|\bar{\varphi}(\langle x, y\rangle_H)| \leq \bar{\varphi}(|\langle x, y\rangle_H|) = \|\langle x, y\rangle_H\|_1 \leq \|x\|_m \|y\|_H .$$

Proof. 1. By Corollary 1 and since f is A-linear, f is linear. If f is bounded, then there exists $c \in A_+$ such that

$$f^2(x) \leq c\langle x, x\rangle_H \qquad \forall x \in H.$$

Since $\| \|_A$ is a lattice norm and A is an Arens algebra, this implies that

$$\|f(x)\|_A^2 = \left\|f^2(x)\right\|_A \leq \|c\langle x, x\rangle_H\|_A \leq \|c\|_A \|\langle x, x\rangle_H\|_A \qquad \forall x \in H,$$

that is, $\|f(x)\|_A \leq \sqrt{\|c\|_A}\,\|x\|_H$ for all $x \in H$. We can conclude that f is $\| \|_H - \| \|_A$ continuous. Viceversa, assume that f is $\| \|_H - \| \|_A$ continuous. It follows that there exists $k \geq 0$ such that $\|f(x)\|_A \leq k\|x\|_H$ for all $x \in H$. Fix $x \in H$. Since f is A-linear, it follows that for each $a \in A$

$$\left\|a^2 f^2(x)\right\|_A = \|af(x)\|_A^2 = \|f(a \cdot x)\|_A^2 \leq k^2 \|\langle a \cdot x, a \cdot x\rangle_H\|_A = k^2 \left\|a^2\langle x, x\rangle_H\right\|_A .$$

Since A is a Dedekind complete Arens algebra, this implies that $f^2(x) \leq (k^2 e)\langle x, x\rangle_H$.[8] Since x was arbitrarily chosen, it follows that f is bounded.

[8]For a Dedekind complete Arens algebra A, it is true that if $c, d \in A_+$ are such that

$$\|bc\|_A \leq \|bd\|_A \qquad \forall b \in A_+,$$

then $c \leq d$. In our case, $c = f^2(x)$, $d = k^2\langle x, x\rangle_H$, and, given Subsection 2.1, it is enough to observe that $A_+ = \left\{a^2 : a \in A\right\}$.

2. By point 1. and since A is a Dedekind complete Arens algebra, if $f : H \to A$ is $\| \|_H - \| \|_A$ continuous, then there exists $c \in A_+$ such that

$$|f(x)| \le c \langle x, x \rangle_H^{\frac{1}{2}} \qquad \forall x \in H.$$

Since $\bar{\varphi}$ is (strictly) positive, we have that

$$\begin{aligned} 0 &\le \|f(x)\|_1 = \bar{\varphi}(|f(x)|) \le \bar{\varphi}\left(c \langle x, x \rangle_H^{\frac{1}{2}}\right) \le \bar{\varphi}\left((\|c\|_A e) \langle x, x \rangle_H^{\frac{1}{2}}\right) \\ &= \|c\|_A \bar{\varphi}\left(\langle x, x \rangle_H^{\frac{1}{2}}\right) = \|c\|_A \|x\|_p \qquad \forall x \in H. \end{aligned}$$

3. By the proof of point 2. and (7), the statement follows.

4. By Proposition 4 and since $\bar{\varphi}$ is (strictly) positive and linear, we have that

$$\begin{aligned} |\bar{\varphi}(\langle x, y \rangle_H)| &\le \bar{\varphi}(|\langle x, y \rangle_H|) \le \bar{\varphi}\left(\langle x, x \rangle_H^{\frac{1}{2}} \langle y, y \rangle_H^{\frac{1}{2}}\right) \le \bar{\varphi}\left(\langle x, x \rangle_H^{\frac{1}{2}} \left(\left\|\langle y, y \rangle_H^{\frac{1}{2}}\right\|_A e\right)\right) \\ &= \sqrt{\|\langle y, y \rangle_H\|_A} \bar{\varphi}\left(\langle x, x \rangle_H^{\frac{1}{2}}\right) = \|y\|_H \|x\|_p \qquad \forall x, y \in H, \end{aligned}$$

proving the statement, since $\bar{\varphi}(|\langle x, y \rangle_H|) = \|\langle x, y \rangle_H\|_1$ for all $x, y \in H$.

5. By the proof of point 4. and (8), the statement follows. ■

Remark 3 *It is important to note that in the proof of point 1. the existence of a strictly positive functional $\bar{\varphi}$ did not play any role. Similarly, in the proof of points 4. and 5. the assumption of Dedekind completeness was not used.*

Proposition 9 *Let A be an Arens algebra and H a pre-Hilbert A-module. If A is finite dimensional, then A admits a strictly positive linear functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$ and the norms $\| \|_p$, $\| \|_m$, and $\| \|_H$ are equivalent.*

Proof. Since A is finite dimensional, we have that K is finite (see Semadeni [36, Corollary 6.4.9 and Propositions 6.2.10 and 7.1.4]). Since K is finite, consider $m = m_{\bar{\varphi}}$ such that $m(\{\varphi\}) = \frac{1}{|K|}$ for all $\varphi \in K$. By Proposition 1, $\bar{\varphi}$ is strictly positive, linear, and such that $\|\bar{\varphi}\|_{A^*} = 1$. Consider now a generic strictly positive linear functional $\bar{\varphi}$. By Proposition 1 and since K is finite, it is immediate to see that $m(\{\varphi\}) > 0$ for all $\varphi \in K$. Consider a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq H$. By Proposition 1, Proposition 6, point 3. of Proposition 5, and since K is finite, it follows that

$$\begin{aligned} x_n \overset{\| \|_p}{\to} 0 &\iff \|x_n\|_p \to 0 \iff \int_K \left\langle \langle x_n, x_n \rangle_H^{\frac{1}{2}}, \varphi \right\rangle dm(\varphi) \to 0 \\ &\iff \|x_n\|_\varphi \to 0 \quad \forall \varphi \in K \iff \|x_n\|_H \to 0 \iff x_n \overset{\| \|_H}{\to} 0, \end{aligned}$$

proving that $\| \|_p$ and $\| \|_H$ are equivalent. By (8), the statement follows. ■

3.2 The norm duals of H

Assume A is an Arens algebra that admits a strictly positive functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$. Recall that on H we have at least two competing norms for H: $\| \, \|_H$ and $\| \, \|_m$. We denote by B_H the closed unit ball with respect to $\| \, \|_H$ and by B_m the closed unit ball with respect to $\| \, \|_m$.

In this subsection, we study the norm duals these norms induce. We denote them, respectively, H^* and H'. Since $\|x\|_m \leq \|x\|_H$ for all $x \in H$, we have that $H' \subseteq H^*$ and, given $l \in H'$, that

$$\|l\|_{H'} = \sup\{|l(x)| : \|x\|_m \leq 1\} \geq \sup\{|l(x)| : \|x\|_H \leq 1\} = \|l\|_{H^*}. \qquad (10)$$

Consider $\varphi \in A^*$ and $y \in H$. By Corollary 1, note that $\varphi \circ \langle \, , y\rangle_H$ is a linear functional. By Corollary 3, the $\| \, \|_H$ continuity of $\varphi \circ \langle \, , y\rangle_H$ follows, since the operator $\langle \, , y\rangle_H$ is $\| \, \|_H - \| \, \|_A$ continuous and φ is $\| \, \|_A$ continuous. On the other hand, by Proposition 8 (see also Remark 3), if $\varphi \in A' \subseteq A^*$, $\| \, \|_m$ continuity of $\varphi \circ \langle \, , y\rangle_H$ follows since the operator $\langle \, , y\rangle_H$ is $\| \, \|_m - \| \, \|_1$ continuous and φ is $\| \, \|_1$ continuous.

Lemma 1 *Let A be an Arens algebra and H a pre-Hilbert A-module. If A admits a strictly positive functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$ and $S : H \to H' \subseteq H^*$ is defined by*

$$S(y) = \bar{\varphi}(\langle \, , y\rangle_H) \qquad \forall y \in H,$$

then:

1. *S is well defined and linear;*
2. *$\ker S = \{0\}$;*
3. *S is such that $\|S(y)\|_{H'} = \|y\|_m$ for all $y \in H$. In particular, S is $\| \, \|_m - \| \, \|_{H'}$ continuous.*

Proof. 1. Note that $\bar{\varphi} \in A' \subseteq A^*$. By the arguments preceding the proof, we have that for each $y \in H$ the functional $S(y)$ is linear and $\| \, \|_H$ and $\| \, \|_m$ continuous, so that, S is well defined. For each $\alpha_1, \alpha_2 \in \mathbb{R}$ and $y_1, y_2 \in H$, we also have that

$$\begin{aligned} S(\alpha_1 y_1 + \alpha_2 y_2)(x) &= \bar{\varphi}(\langle x, \alpha_1 y_1 + \alpha_2 y_2\rangle_H) = \bar{\varphi}(\langle x, \alpha_1 y_1\rangle_H + \langle x, \alpha_2 y_2\rangle_H) \\ &= \bar{\varphi}(\langle x, (\alpha_1 e) \cdot y_1\rangle_H + \langle x, (\alpha_2 e) \cdot y_2\rangle_H) \\ &= \bar{\varphi}((\alpha_1 e)\langle x, y_1\rangle_H + (\alpha_2 e)\langle x, y_2\rangle_H) \\ &= \bar{\varphi}(\alpha_1 \langle x, y_1\rangle_H + \alpha_2 \langle x, y_2\rangle_H) \\ &= \alpha_1 \bar{\varphi}(\langle x, y_1\rangle_H) + \alpha_2 \bar{\varphi}(\langle x, y_2\rangle_H) \\ &= \alpha_1 S(y_1)(x) + \alpha_2 S(y_2)(x) \qquad \forall x \in H, \end{aligned}$$

proving S is linear.

2. Consider $y \in H$. Assume that $S(y) = 0$. It follows that

$$\bar{\varphi}\left(\langle x, y\rangle_H\right) = S(y)(x) = 0 \qquad \forall x \in H.$$

By choosing $x = y$ and since $\bar{\varphi}$ is strictly positive, we have that

$$\bar{\varphi}\left(\langle y, y\rangle_H\right) = 0 \implies \langle y, y\rangle_H = 0 \implies y = 0.$$

This yields that $\ker S = \{0\}$.

3. By Corollary 2 (see also Proposition 3), recall that H with the inner product $\langle\ ,\ \rangle_m$ is a pre-Hilbert space. It follows that

$$|S(y)(x)| = |\bar{\varphi}\left(\langle x, y\rangle_H\right)| = |\langle x, y\rangle_m| \leq \|x\|_m \|y\|_m \qquad \forall x, y \in H.$$

We can conclude that $\|S(y)\|_{H'} = \|y\|_m$ for all $y \in H$. ■

Next proposition shows that $S : H \to H^*$ is an isometry when H is endowed with the norm $\|\ \|_p$ and H^* is endowed with the norm $\|\ \|_{H^*}$.

Proposition 10 *Let A be a Dedekind complete Arens algebra and H a pre-Hilbert A-module. If A admits a strictly positive linear functional $\bar{\varphi}$ such that $\|\bar{\varphi}\|_{A^*} = 1$, then $\|S(y)\|_{H^*} = \|y\|_p$ for all $y \in H$.*

Proof. Consider $l \in S(H)$. It follows that there exists $y \in H$ such that $l(x) = \bar{\varphi}\left(\langle x, y\rangle_H\right)$ for all $x \in H$ and

$$\|l\|_{H^*} = \sup_{x \in B_H} \bar{\varphi}\left(\langle x, y\rangle_H\right).$$

First, consider the problem $\sup_{x \in B_H} \langle x, y\rangle_H$. If $x \in B_H$, then $\|\langle x, x\rangle_H\|_A \leq 1$ which yields $\langle x, x\rangle_H^{\frac{1}{2}} \leq e$. By Proposition 4, this implies that

$$\langle x, y\rangle_H \leq \langle x, x\rangle_H^{\frac{1}{2}} \langle y, y\rangle_H^{\frac{1}{2}} \leq \langle y, y\rangle_H^{\frac{1}{2}} \qquad \forall x \in B_H.$$

Since A is Dedekind complete, it follows that $A \ni \sup_{x \in B_H} \langle x, y\rangle_H \leq \langle y, y\rangle_H^{\frac{1}{2}}$. Next, we show that $\|y\|_p = \bar{\varphi}\left(\langle y, y\rangle_H^{\frac{1}{2}}\right) = \sup_{x \in B_H} \bar{\varphi}\left(\langle x, y\rangle_H\right) = \|l\|_{H^*}$. Consider $a_n = \langle y, y\rangle_H^{\frac{1}{2}} + \frac{1}{n}e \geq 0$ for all $n \in \mathbb{N}$. Since $\langle y, y\rangle_H^{\frac{1}{2}} \in A_+$, it is immediate to see that a_n is invertible for all $n \in \mathbb{N}$. Define $y_n = a_n^{-1} \cdot y$ for all $n \in \mathbb{N}$.

Claim: For each $n \in \mathbb{N}$,

$$\langle y_n, y_n\rangle_H = \left\langle a_n^{-1} \cdot y, a_n^{-1} \cdot y\right\rangle_H \leq e.$$

Moreover, $0 \leq \langle y_n, y\rangle_H \leq \langle y, y\rangle_H^{\frac{1}{2}}$ *for all* $n \in \mathbb{N}$, $\langle y_n, y\rangle_H \uparrow$, *and* $\langle y_n, y\rangle_H \overset{\| \|_A}{\to} \langle y, y\rangle_H^{\frac{1}{2}}$.

Proof of the Claim. See the Appendix. □

We can conclude that $\|y_n\|_H \leq 1$ for all $n \in \mathbb{N}$. Since $\bar{\varphi} \in A^*$ is positive and $\langle y, y\rangle_H^{\frac{1}{2}} \geq \sup_{x\in B_H} \langle x, y\rangle_H \geq \langle z, y\rangle_H$ for all $z \in B_H$, it follows that

$$\bar{\varphi}\left(\langle y, y\rangle_H^{\frac{1}{2}}\right) \geq \bar{\varphi}\left(\sup_{x\in B_H} \langle x, y\rangle_H\right) \geq \bar{\varphi}\left(\langle z, y\rangle_H\right) \qquad \forall z \in B_H,$$

that is, $\bar{\varphi}\left(\langle y, y\rangle_H^{\frac{1}{2}}\right) \geq \sup_{x\in B_H} \bar{\varphi}\left(\langle x, y\rangle_H\right)$. Viceversa, since $\bar{\varphi} \in A^*$ is positive, note that $\sup_{x\in B_H} \bar{\varphi}\left(\langle x, y\rangle_H\right) \geq \sup_{n\in\mathbb{N}} \bar{\varphi}\left(\langle y_n, y\rangle_H\right) = \lim_n \bar{\varphi}\left(\langle y_n, y\rangle_H\right) = \bar{\varphi}\left(\langle y, y\rangle_H^{\frac{1}{2}}\right)$, proving the statement. ■

Remark 4 *Observe that, under the assumptions of previous proposition, the completion of* H *with respect to* $\| \|_p$, *denoted by* H_p, *can be identified with the* $\| \|_{H^*}$ *closure of* $S(H)$ *in* H^*. *We will always adopt this identification.*

We conclude the section with an ancillary lemma which will be instrumental in proving one of our main results on self-duality.

Lemma 2 *Let* A *be a Dedekind complete Arens algebra and* H *a pre-Hilbert* A*-module. If* A *admits a strictly positive functional* $\bar{\varphi}$ *such that* $\|\bar{\varphi}\|_{A^*} = 1$, *then* $\langle\ ,\ \rangle_m = \bar{\varphi}\circ\langle\ ,\ \rangle_H$ *admits a unique bilinear extension from* $H \times H_p$ *to* $\mathbb{R}$, *denoted* $\langle\ ,\ \rangle_m^-$, *such that:*

1. $\left|\langle x, y\rangle_m^-\right| \leq \|x\|_H \|y\|_p$ *for all* $x \in H$ *and* $y \in H_p$;
2. *If* $\langle x, y\rangle_m^- = \langle x, y'\rangle_m^-$ *for all* $x \in H$, *then* $y = y'$.

Proof. Denote the dual pairing of H and H^* by $\langle\ ,\ \rangle_{H,H^*}$. Note that for each $x, y \in H$

$$\langle x, S(y)\rangle_{H,H^*} = S(y)(x) = \langle x, y\rangle_m.$$

Moreover, by Proposition 10, $\left|\langle x, S(y)\rangle_{H,H^*}\right| \leq \|x\|_H \|S(y)\|_{H^*} = \|x\|_H \|y\|_p$ for all $x, y \in H$. Since H_p can be identified with $\left(cl_{\| \|_{H^*}}(S(H)), \| \|_{H^*}\right)$, by defining $\langle\ ,\ \rangle_m^- = \langle\ ,\ \rangle_{H,H^*}$, the main statement and point 1. follow. We next prove uniqueness. Assume that $\langle\ ,\ \rangle_m^\circ$ is a bilinear extension of $\langle\ ,\ \rangle_m$ satisfying 1. Consider $y \in H_p$ and $x \in H$. There exists $\{y_n\}_{n\in\mathbb{N}} \subseteq H$ such that $y_n \overset{\| \|_p}{\to} y$. Thus,

$$\begin{aligned}\left|\langle x, y\rangle_m^- - \langle x, y\rangle_m^\circ\right| &\leq \left|\langle x, y\rangle_m^- - \langle x, y_n\rangle_m^\circ\right| + \left|\langle x, y_n\rangle_m^\circ - \langle x, y\rangle_m^\circ\right| \\ &= \left|\langle x, y\rangle_m^- - \langle x, y_n\rangle_m^-\right| + \left|\langle x, y_n\rangle_m^\circ - \langle x, y\rangle_m^\circ\right| \to 0,\end{aligned}$$

proving that $\langle x, y\rangle_m^- = \langle x, y\rangle_m^\circ$. Since x and y were arbitrarily chosen, uniqueness follows. Finally, we have that if $y, y' \in H_p$ are such that for each $x \in H$

$$\langle x, y\rangle_m^- = \langle x, y'\rangle_m^-,$$

then, $\langle x, y\rangle_{H,H^*} = \langle x, y'\rangle_{H,H^*}$, yielding $y = y'$. ■

4 Dual module

Given an Archimedean f-algebra A with unit e and a pre-Hilbert A-module H, we define

$$H^{\sim} = \left\{ f \in A^H : f \text{ is } A\text{-linear and bounded} \right\}.$$

Proposition 11 *If A is a Dedekind complete Archimedean f-algebra with unit e and H a pre-Hilbert A-module, then $H^{\sim}$ is an A-module.*

Proof. Define $+ : H^{\sim} \times H^{\sim} \to H^{\sim}$ to be such that for each $f, g \in H^{\sim}$

$$(f+g)(x) = f(x) + g(x) \qquad \forall x \in H.$$

In other words, $+$ is the usual pointwise sum of operators. Define $\cdot : A \times H^{\sim} \to H^{\sim}$ to be such that for each $a \in A$ and for each $f \in H^{\sim}$

$$(a \cdot f)(x) = af(x) \qquad \forall x \in H.$$

It is immediate to verify that $H^{\sim}$ is closed under $+$ and $\cdot$.[9] In particular, $(H, +)$ is an abelian group. Note that for each $a, b \in A$ and each $f, g \in H^{\sim}$:

1. $(a \cdot (f+g))(x) = a((f+g)(x)) = a(f(x) + g(x)) = af(x) + ag(x) = (a \cdot f)(x) + (a \cdot g)(x) = (a \cdot f + a \cdot g)(x)$ for all $x \in H$, that is, $a \cdot (f+g) = a \cdot f + a \cdot g$.

2. $((a+b) \cdot f)(x) = (a+b) f(x) = af(x) + bf(x) = (a \cdot f)(x) + (b \cdot f)(x) = (a \cdot f + b \cdot f)(x)$ for all $x \in H$, that is, $(a+b) \cdot f = a \cdot f + b \cdot f$.

3. $(a \cdot (b \cdot f))(x) = a((b \cdot f)(x)) = a(bf(x)) = (ab) f(x) = ((ab) \cdot f)(x)$ for all $x \in H$, that is, $a \cdot (b \cdot f) = (ab) \cdot f$.

4. $(e \cdot f)(x) = ef(x) = f(x)$ for all $x \in H$, that is, $e \cdot f = f$. ■

If A is Dedekind complete, then define $S^{\sim} : H \to H^{\sim}$ by

$$S^{\sim}(y) = \langle\ , y \rangle_H \qquad \forall y \in H.$$

Given Remark 2 and the properties of $\langle\ ,\ \rangle_H$, the map $S^{\sim}$ is well defined. We next study the (topological) properties of this map and its connection to the self-duality problem. Since the topologies involved are different, we split this study in two cases: Dedekind complete Arens algebras and f-algebras of $\mathcal{L}^0$ type. Before starting, we need one more definition:

[9] Dedekind completeness yields the existence of the square root for positive elements of A. This property is used to show that the sum of two bounded A-linear operators is bounded.

Definition 8 *Let H_1 and H_2 be two A-modules and $S^{\sim}: H_1 \to H_2$. We say that $S^{\sim}$ is a module homomorphism if and only if*

$$S^{\sim}(a \cdot x + b \cdot y) = a \cdot S^{\sim}(x) + b \cdot S^{\sim}(y) \qquad \forall a, b \in A, \forall x, y \in H_1.$$

We say that $S^{\sim}$ is a module isomorphism if and only if it is a bijective module homomorphism.

4.1 Dual module: Arens algebras

If A is a Dedekind complete Arens algebra (see Proposition 8 and Remark 3), we have that

$$H^{\sim} = \left\{ f \in A^H : f \text{ is } A\text{-linear and } \| \ \|_H - \| \ \|_A \text{ continuous} \right\}.$$

In this case, we define $\| \ \|_{H^{\sim}} : H^{\sim} \to [0, \infty)$ by

$$\|f\|_{H^{\sim}} = \sup_{x \in B_H} \|f(x)\|_A \qquad \forall f \in H^{\sim}.$$

Recall that if $f \in H^{\sim}$, then f is linear. Thus, in this case, we have that $H^{\sim} \subseteq B(H, A)$, where the latter is the set of all *norm* bounded linear operators from H to A when H is endowed with $\| \ \|_H$ and A is endowed with $\| \ \|_A$.

Proposition 12 *Let A be a Dedekind complete Arens algebra and H a pre-Hilbert A-module. The following statements are true:*

1. $H^{\sim} = \left\{ f \in A^H : f \text{ is } A\text{-linear and } \| \ \|_H - \| \ \|_A \text{ continuous} \right\}$.
2. *$H^{\sim}$ is a $\| \ \|_{H^{\sim}}$ complete A-module.*
3. *$S^{\sim}$ is a well defined module homomorphism and $\|S^{\sim}(y)\|_{H^{\sim}} = \|y\|_H$ for all $y \in H$.*
4. *If H is self-dual, then $S^{\sim}$ is onto and H is $\| \ \|_H$ complete.*

Proof. 1. It follows from point 1. of Proposition 8 (see also Remark 3).

2. By Proposition 11, $H^{\sim}$ is an A-module. In particular, $H^{\sim}$ is a vector subspace of $B(H, A)$. Consider a $\| \ \|_{H^{\sim}}$ Cauchy sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq H^{\sim} \subseteq B(H, A)$. By [4, Theorem 6.6], we have that there exists $f \in B(H, A)$ such that $f_n \overset{\| \ \|_{H^{\sim}}}{\to} f$. We are left to show that f is A-linear. First, observe that $f : H \to A$ is such that

$$f(x) = \lim_n f_n(x) \qquad \forall x \in H$$

where the limit is in $\| \|_A$ norm. We can conclude that for each $a, b \in A$ and $x, y \in H$

$$f_n(x) \overset{\| \|_A}{\to} f(x), \; f_n(y) \overset{\| \|_A}{\to} f(y) \implies af_n(x) \overset{\| \|_A}{\to} af(x), \; bf_n(y) \overset{\| \|_A}{\to} bf(y)$$
$$\implies af_n(x) + bf_n(y) \overset{\| \|_A}{\to} af(x) + bf(y).$$

At the same time, $af_n(x)+bf_n(y) = f_n(a \cdot x + b \cdot y) \overset{\| \|_A}{\to} f(a \cdot x + b \cdot y)$ for all $a, b \in A$ and $x, y \in H$. By the uniqueness of the limit, we can conclude that $f(a \cdot x + b \cdot y) = af(x) + bf(y)$ for all $a, b \in A$ and $x, y \in H$, proving the statement.

3. Define $S^{\sim} : H \to H^{\sim}$ by

$$S^{\sim}(y)(x) = \langle x, y \rangle_H \qquad \forall x \in H.$$

By Corollary 3, it follows that $S^{\sim}$ is well defined and such that $\|S^{\sim}(y)\|_{H^{\sim}} = \|y\|_H$ for all $y \in H$. Note also that for each $a, b \in A$ and for each $y, z \in H$

$$\begin{aligned} S^{\sim}(a \cdot y + b \cdot z)(x) &= \langle x, a \cdot y + b \cdot z \rangle_H = a \langle x, y \rangle_H + b \langle x, z \rangle_H \\ &= aS^{\sim}(y)(x) + bS^{\sim}(z)(x) \\ &= (a \cdot S^{\sim}(y))(x) + (b \cdot S^{\sim}(z))(x) \qquad \forall x \in H, \end{aligned}$$

in other words, we have that $S^{\sim}(a \cdot y + b \cdot z) = a \cdot S^{\sim}(y) + b \cdot S^{\sim}(z)$, that is, $S^{\sim}$ is a module homomorphism. We can also conclude that

$$\|S^{\sim}(y) - S^{\sim}(z)\|_{H^{\sim}} = \|S^{\sim}(y - z)\|_{H^{\sim}} = \|y - z\|_H,$$

that is, $S^{\sim}$ is an isometry.

4. If H is self-dual, it is immediate to see that $S^{\sim}$ is onto. Consider a $\| \|_H$ Cauchy sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq H$. Since $S^{\sim}$ is an isometry, it follows that $\{S^{\sim}(x_n)\}_{n \in \mathbb{N}}$ is a $\| \|_{H^{\sim}}$ Cauchy sequence in $H^{\sim}$. Since $H^{\sim}$ is $\| \|_{H^{\sim}}$ complete and $S^{\sim}$ is onto, it follows that there exists $f \in H^{\sim}$ such that $S^{\sim}(x_n) \overset{\| \|_{H^{\sim}}}{\to} f = S^{\sim}(x)$ for some $x \in H^{\sim}$. Since $S^{\sim}$ is an isometry, we have that $x_n \overset{\| \|_H}{\to} x$, proving that H is $\| \|_H$ complete. ■

4.2 Dual module: f-algebras of $\mathcal{L}^0$ type

In order to discuss the continuity properties of the map $S^{\sim}$ we need to endow $H^{\sim}$ with a topology. We saw that if A is an Arens algebra, then the choice of topology for $H^{\sim}$ is rather natural: the one induced by the standard operator norm $\| \|_{H^{\sim}}$. The same choice cannot be made when A is an f-algebra of $\mathcal{L}^0$ type. Thus, first we define the operator vector-valued norm $N_* : H^{\sim} \to A_+$ by

$$N_*(f) = \sup_{x \in H} \left(\sup_{n \in \mathbb{N}} \frac{|f(x)|}{N(x) + \frac{1}{n}e} \right) \qquad \forall f \in H^{\sim}.$$

Then, we define the metric $d_{H^\sim} : H^\sim \times H^\sim \to [0, \infty)$

$$d_{H^\sim}(f, g) = d(0, N_*(f - g)) \qquad \forall f, g \in H^\sim.$$

Lemma 3 *If A is an f-algebra of $\mathcal{L}^0$ type and H a pre-Hilbert A-module, then $d_{H^\sim}$ is an invariant metric.*

Proposition 13 *Let A be an f-algebra of $\mathcal{L}^0$ type and H a pre-Hilbert A-module. The following statements are true:*

1. *$H^\sim = \{f \in A^H : f$ is A-linear and $d_H - d$ continuous$\}$.*
2. *$S^\sim$ is a well defined module homomorphism and $d_{H^\sim}(S^\sim(x), S^\sim(y)) = d_H(x, y)$ for all $x, y \in H$.*
3. *If H is self-dual, then $S^\sim$ is onto and H is d_H complete.*

Proof. 1. We first show that

$$H^\sim \subseteq \left\{f \in A^H : f \text{ is } A\text{-linear and } d_H - d \text{ continuous}\right\}.$$

Consider $f \in H^\sim$. We only need to show that f is $d_H - d$ continuous. Since f is bounded, there exists $c \in A_+$ such that

$$|f(x)| \leq cN(x) \qquad \forall x \in H. \tag{11}$$

By [35, Theorem 1.32] and since f is A-linear, f is linear and we only need to show continuity at 0. Consider $\{x_n\}_{n\in\mathbb{N}} \subseteq H$ such that $x_n \stackrel{d_H}{\to} 0$. By definition of d_H, it follows that $N(x_n) \stackrel{d}{\to} 0$, thus $cN(x_n) \stackrel{d}{\to} 0$. By (11) and definition of d, it follows that $d(0, |f(x_n)|) = \bar{\varphi}(|f(x_n)| \wedge e) \leq \bar{\varphi}((cN(x_n)) \wedge e) = d(0, cN(x_n)) \to 0$, proving continuity at 0.

As for the opposite inclusion, we consider an A-linear and $d_H - d$ continuous function f and show it is bounded. First, define $B_{H_e} = \{x \in H : N(x) \leq e\}$. Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq B_{H_e}$ and $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $\alpha_n \to 0$. It follows that there exists $\bar{n} \in \mathbb{N}$ such that $|\alpha_n| < 1$ for all $n \geq \bar{n}$. This implies that for $n \geq \bar{n}$

$$\begin{aligned} d_H(0, \alpha_n x_n) &= \bar{\varphi}(N(\alpha_n x_n) \wedge e) = \bar{\varphi}((|\alpha_n| N(x_n)) \wedge e) \\ &\leq \bar{\varphi}(|\alpha_n| e \wedge e) = |\alpha_n| \bar{\varphi}(e) \to 0, \end{aligned}$$

proving that $\alpha_n x_n \stackrel{d_H}{\to} 0$. By [35, Theorem 1.30] and [35, Theorem 1.32], we have that B_{H_e} is topologically bounded and so are $f(B_{H_e})$ and $\{|f(x)|\}_{x\in B_{H_e}}$. Consider the following binary relation on B_{H_e}:

$$x \succeq y \iff |f(x)| \geq |f(y)|.$$

It is immediate to see that $\succeq$ is reflexive and transitive. Next, consider $x, y \in B_{H_e}$. Since A is an f-algebra of $\mathcal{L}^0$ type, it follows that there exists $c_1, c_2 \in A_+$ such that $c_1 \wedge c_2 = 0$, $c_1, c_2 \leq e$, and

$$c_1 |f(x)| + c_2 |f(y)| = |f(x)| \vee |f(y)| .^{10}$$

Define $z = c_1 \cdot x + c_2 \cdot y$. Note that $N(c_1 \cdot x + c_2 \cdot y) \leq c_1 N(x) + c_2 N(y) \leq c_1 e + c_2 e \leq e$, proving that $z \in B_{H_e}$. At the same time, since $c_1 \wedge c_2 = 0$, it follows that $c_1 c_2 = 0$. This implies that $(c_1 |f(x)|)(c_2 |f(y)|) = 0$, yielding that $(c_1 |f(x)|) \wedge (c_2 |f(y)|) = 0$. By [4, Theorem 8.12] and since f is A-linear, we can conclude that

$$|f(z)| = |f(c_1 \cdot x + c_2 \cdot y)| = |c_1 f(x) + c_2 f(y)| = c_1 |f(x)| + c_2 |f(y)| = |f(x)| \vee |f(y)| .$$

Thus, for each $x, y \in B_{H_e}$ there exists $z \in B_{H_e}$ such that $z \succeq x$ and $z \succeq y$. It follows that $(B_{H_e}, \succeq)$ is a directed set and $\{|f(x)|\}_{x \in B_{H_e}}$ is an increasing net. By [6, Theorem 7.14 and Theorem 7.50], we can conclude that $\{|f(x)|\}_{x \in B_{H_e}}$ is order bounded. Therefore, there exists $c \in A_+$ such that $|f(x)| \leq c$ for all $x \in B_{H_e}$. Next, consider $x \in H_e$. Define $a_n = \left(N(x) + \frac{1}{n} e\right)^{-1}$ and $x_n = a_n \cdot x$ for all $n \in \mathbb{N}$. It is immediate to see that $x_n \in B_{H_e}$. By the previous part of the proof, this implies that $|f(a_n \cdot x)| \leq c$ for all $n \in \mathbb{N}$. Since f is A-linear, we can conclude that

$$|f(x)| \leq c \left(N(x) + \frac{1}{n} e \right) \qquad \forall n \in \mathbb{N}.$$

By taking the limit and since the topology induced by d is solid and x was arbitrarily chosen, it follows that

$$|f(x)| \leq c N(x) \qquad \forall x \in H, \tag{12}$$

proving the statement.

2. For each $a, b \in A$ and for each $y, z \in H$

$$\begin{aligned} S^{\sim}(a \cdot y + b \cdot z)(x) &= \langle x, a \cdot y + b \cdot z \rangle_H = a \langle x, y \rangle_H + b \langle x, z \rangle_H \\ &= a S^{\sim}(y)(x) + b S^{\sim}(z)(x) \\ &= (a \cdot S^{\sim}(y))(x) + (b \cdot S^{\sim}(z))(x) \qquad \forall x \in H, \end{aligned}$$

in other words, we have that $S^{\sim}(a \cdot y + b \cdot z) = a \cdot S^{\sim}(y) + b \cdot S^{\sim}(z)$, that is, $S^{\sim}$ is a module homomorphism. We next show that $S^{\sim}$ is an isometry. Consider $z \in H$. By

[10] Consider $b_1, b_2 \in A_+$. By [11, Lemma 3], there exists $c_1 \in A$ such that $0 \leq c_1 \leq e$, $c_1 \wedge (e - c_1) = 0$, and $c_1 (b_1 - b_2) = (b_1 - b_2)^+$. Define $c_2 = e - c_1$. It follows that

$$c_1 b_1 + c_2 b_2 = b_2 + c_1 (b_1 - b_2) = b_2 + (b_1 - b_2)^+ = b_2 + (b_1 - b_2) \vee 0 = b_1 \vee b_2.$$

It is enough to set $b_1 = |f(x)|$ and $b_2 = |f(y)|$.

Proposition 4 (see also Remark 2), we have that for each $x \in H$ and for each $n \in \mathbb{N}$

$$|S^{\sim}(z)(x)| = |\langle x, z\rangle_H| \leq \langle x, x\rangle_H^{\frac{1}{2}} \langle z, z\rangle_H^{\frac{1}{2}} = N(x) N(z) \leq \left(N(x) + \frac{1}{n}e\right) N(z).$$

This implies that

$$\frac{|S^{\sim}(z)(x)|}{N(x) + \frac{1}{n}e} \leq N(z) \qquad \forall x \in H, \forall n \in \mathbb{N},$$

which yields that

$$N_*(S^{\sim}(z)) = \sup_{x \in H}\left(\sup_{n \in \mathbb{N}} \frac{|S^{\sim}(z)(x)|}{N(x) + \frac{1}{n}e}\right) \leq N(z).$$

Consider now $x = z$. It follows that

$$\sup_{n \in \mathbb{N}} \frac{|S^{\sim}(z)(z)|}{N(z) + \frac{1}{n}e} = \sup_{n \in \mathbb{N}} \frac{N^2(z)}{N(z) + \frac{1}{n}e} = N(z), \text{[11]}$$

yielding that

$$N_*(S^{\sim}(z)) = \sup_{x \in H}\left(\sup_{n \in \mathbb{N}} \frac{|S^{\sim}(z)(x)|}{N(x) + \frac{1}{n}e}\right) \geq \sup_{n \in \mathbb{N}} \frac{|S^{\sim}(z)(z)|}{N(z) + \frac{1}{n}e} = N(z)$$

and proving that $N_*(S^{\sim}(z)) = N(z)$. Since z was arbitrarily chosen, we have that for each $x, y \in H$

$$\begin{aligned} d_{H^{\sim}}(S^{\sim}(x), S^{\sim}(y)) &= d(0, N_*(S^{\sim}(x) - S^{\sim}(y))) = d(0, N_*(S^{\sim}(x - y))) \\ &= d(0, N(x - y)) = d_H(x, y), \end{aligned}$$

proving the statement.

3. If H is self-dual, it is immediate to see that $S^{\sim}$ is onto. As for d_H completeness of H, we postpone the proof. It will be the implication "(ii) implies (i)" in Theorem 5. ■

[11] Consider $a \in A_+$. Define $a_n = a + \frac{1}{n}e$ for all $n \in \mathbb{N}$. It follows that $\{a_n^{-1}\}_{n \in \mathbb{N}}$ is a well defined and increasing sequence, and so is $\{a_n^{-1}a^2\}_{n \in \mathbb{N}}$. Note that $a \leq a_n$, thus $aa_n^{-1} \leq e$ for all $n \in \mathbb{N}$. Note also that

$$a^2 + \frac{1}{n}a = a\left(a + \frac{1}{n}e\right) = aa_n \quad \forall n \in \mathbb{N},$$

that is,

$$a - a_n^{-1}a^2 = \frac{1}{n}aa_n^{-1} \leq \frac{1}{n}e \quad \forall n \in \mathbb{N}.$$

It follows that

$$d\left(a_n^{-1}a^2, a\right) = \bar{\varphi}\left(\left|a - a_n^{-1}a^2\right| \wedge e\right) \leq \bar{\varphi}\left(\frac{1}{n}e\right) \to 0.$$

By [4, Theorem 8.43] and since the topology generated by d is locally solid and Hausdorff, we can conclude that $\sup_n \left(a_n^{-1}a^2\right) = a$. It is enough to set $a = N(z)$.

4.3 Dual module as a congruent space

In this subsection, we consider an Arens algebra of $\mathcal{L}^\infty$ type. We show that the dual module unit ball, $B_{H^\sim}$, is compact with respect to a weak topology, thus providing a sort of Banach-Alaoglu theorem for the dual module. As a corollary, we obtain that $H^\sim$ can be seen as the norm dual of a specific Banach space.

Fix $x \in H$. Define $\ell : H^\sim \to \mathbb{R}$ by

$$\ell(f) = \bar{\varphi}(f(x)) \qquad \forall f \in H^\sim. \tag{13}$$

Note that for each $f, g \in H^\sim$ and $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned}\ell(\alpha f + \beta g) &= \bar{\varphi}((\alpha f + \beta g)(x)) = \bar{\varphi}(\alpha f(x) + \beta g(x)) \\ &= \alpha\bar{\varphi}(f(x)) + \beta\bar{\varphi}(g(x)) = \alpha\ell(f) + \beta\ell(g),\end{aligned}$$

that is, ℓ is linear. Similarly, we have that

$$|\ell(f)| = |\bar{\varphi}(f(x))| \leq \|\bar{\varphi}\|_{A^*} \|f(x)\|_A \leq \|\bar{\varphi}\|_{A^*} \|x\|_H \|f\|_{H^\sim} \qquad \forall f \in H^\sim,$$

that is, $\ell \in (H^\sim)^*$ where $(H^\sim)^*$ is the $\| \ \|_{H^\sim}$ norm dual of $H^\sim$. Define

$$V = \{\ell \in (H^\sim)^* : \ell \text{ is defined as in (13)}\}$$

and $\bar{V}$ as the $\| \ \|_{(H^\sim)^*}$ norm closure of V in $(H^\sim)^*$.

Theorem 1 *If A is an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module, then $B_{H^\sim}$ is $\sigma(H^\sim, V)$ compact.*

Proof. By definition of V, note that we need to show that for each net $\{f_i\}_{i\in I} \subseteq B_{H^\sim}$ there exists $f \in B_{H^\sim}$ and a subnet $\{f_{i_j}\}_{j\in J} \subseteq B_{H^\sim}$ such that $\ell(f_{i_j}) \to \ell(f)$ for all $\ell \in V$. This is equivalent to show that

$$\bar{\varphi}(f_{i_j}(x)) \to \bar{\varphi}(f(x)) \qquad \forall x \in H.$$

Before starting, recall that $\| \ \|_1 : A \to [0, \infty)$ is defined by $\|a\|_1 = \bar{\varphi}(|a|)$ for all $a \in A$. Recall also that A' denotes the $\| \ \|_1$ dual of A. It follows that $\langle A, A'\rangle$ is a Riesz dual system. Since $\bar{\varphi}$ is order continuous, it is immediate to verify that $\| \ \|_1$ is indeed a well defined order continuous norm. This implies that $\sigma(A, A')$ is order continuous. Moreover, by [6, Theorem 3.57] and since A is Dedekind complete, we have that order intervals are $\sigma(A, A')$ compact. We conclude the first part of the proof by fixing $b \in A$ and proving that the map $L : A \to A$, defined by $L(a) = ba$ for all $a \in A$, is $\sigma(A, A') - \sigma(A, A')$ continuous. It is immediate to check that L is linear. Next, we show that L is norm bounded. In fact, $\|L(a)\|_1 = \bar{\varphi}(|ba|) = \bar{\varphi}(|b|\,|a|) \leq$

$\bar{\varphi}\left(\|b\|_A e\,|a|\right) \leq \|b\|_A \bar{\varphi}\left(|a|\right) \leq \|b\|_A \|a\|_1$ for all $a \in A$. By [4, Theorem 6.17], we can conclude that L is $\sigma\left(A, A'\right) - \sigma\left(A, A'\right)$ continuous.

In this proof, we consider A endowed with the $\sigma\left(A, A'\right)$ topology and A^H with the product topology.

Consider a net $\{f_i\}_{i\in I} \subseteq B_{H^\sim}$. It follows that $\{f_i\}_{i\in I} \subseteq A^H$. For each $x \in H$ define

$$A_x = \{a \in A : \exists i \in I \text{ s.t. } f_i\left(x\right) = a\}.$$

Note that $\|f_i\left(x\right)\|_A \leq \|f_i\|_{H^\sim} \|x\|_H$ for all $i \in I$ and for all $x \in H$. Since $\{f_i\}_{i\in I} \subseteq B_{H^\sim}$, we can conclude that

$$A_x \subseteq \left[-\|x\|_H e, \|x\|_H e\right] \qquad \forall x \in H.$$

Thus, we have that

$$\{f_i\}_{i\in I} \subseteq \Pi_{x\in H} A_x \subseteq \Pi_{x\in H}\left[-\|x\|_H e, \|x\|_H e\right] \subseteq A^H$$

where the last by one set is compact by Tychonoff's theorem. We can thus extract a subnet $\left\{f_{i_j}\right\}_{j\in J} \subseteq \{f_i\}_{i\in I}$ such that $f_{i_j}\left(x\right) \overset{\sigma(A,A')}{\rightarrow} a_x \in \left[-\|x\|_H e, \|x\|_H e\right]$ for all $x \in H$. Define $f : H \to A$ by $f\left(x\right) = a_x$ for all $x \in H$. Note that f is well defined and A-linear. For, consider $a, b \in A$ and $y, z \in H$

$$f_{i_j}\left(y\right) \overset{\sigma(A,A')}{\rightarrow} f\left(y\right),\ f_{i_j}\left(z\right) \overset{\sigma(A,A')}{\rightarrow} f\left(z\right) \implies af_{i_j}\left(y\right) \overset{\sigma(A,A')}{\rightarrow} af\left(y\right),\ bf_{i_j}\left(z\right) \overset{\sigma(A,A')}{\rightarrow} bf\left(z\right)$$
$$\implies af_{i_j}\left(y\right) + bf_{i_j}\left(z\right) \overset{\sigma(A,A')}{\rightarrow} af\left(y\right) + bf\left(z\right).$$

At the same time, $af_{i_j}\left(y\right) + bf_{i_j}\left(z\right) = f_{i_j}\left(a \cdot y + b \cdot z\right) \overset{\sigma(A,A')}{\rightarrow} f\left(a \cdot y + b \cdot z\right)$. By the uniqueness of the limit, we can conclude that $f\left(a \cdot y + b \cdot z\right) = af\left(y\right) + bf\left(z\right)$. Since $a, b \in A$ and $y, z \in H$ were arbitrarily chosen, it follows that f is A-linear. In particular, we have that f is linear. We next show that f is norm bounded. In fact, recall that $f\left(x\right) \in \left[-\|x\|_H e, \|x\|_H e\right]$ for all $x \in H$. This implies that

$$\|f\left(x\right)\|_A \leq \|x\|_H \qquad \forall x \in H, \tag{14}$$

that is, f is $\|\ \|_H - \|\ \|_A$ continuous and $\|f\|_{H^\sim} \leq 1$. Thus, f belongs to $B_{H^\sim}$. Since $\bar{\varphi} \in A'$, we can conclude that $\left\{f_{i_j}\right\}_{j\in J} \subseteq \{f_i\}_{i\in I}$

$$\bar{\varphi}\left(f_{i_j}\left(x\right)\right) \to \bar{\varphi}\left(f\left(x\right)\right) \qquad \forall x \in H$$

and $f \in B_{H^\sim}$. ■

Next, we show that the norm dual of $\left(V, \|\ \|_{(H^\sim)^*}\right)$, V^*, is congruent to $H^\sim$. Define $J : H^\sim \to V^*$ by $f \mapsto J\left(f\right)$ where

$$J\left(f\right)\left(\ell\right) = \ell\left(f\right) \qquad \forall f \in H^\sim, \forall \ell \in V.$$

Lemma 4 *If A is an Arens algebra of $\mathcal{L}^{\infty}$ type and H a pre-Hilbert A-module, then J is a well defined onto linear isometry. Moreover, J is $\sigma(H^{\sim}, V) - \sigma(V^{*}, V)$ continuous.*

Proof. The statement follows by replicating the arguments contained in [32] (see also [24] or [21, p. 211]). ■

5 Self-duality

5.1 Arens algebras of $\mathcal{L}^{\infty}$ type

Using the results derived in Section 4, we start by observing that a necessary condition for self-duality is the compactness of B_H in the $\sigma(H, S(H))$ topology. In what follows, we provide the results that will be instrumental in showing that such a condition is also sufficient. We then move to state our first result on self-duality (Theorem 3). We conclude by giving a sufficient condition and a different necessary one for self-duality (Propositions 16 and 17).

Proposition 14 *Let A be an Arens algebra of $\mathcal{L}^{\infty}$ type and H a pre-Hilbert A-module. If H is self-dual, then B_H is $\sigma(H, S(H))$ compact.*

Proof. Since $S^{\sim}$ is an isometry and H is self-dual, it follows that $S^{\sim}(B_H) = B_{H^{\sim}}$. Consider a net $\{y_i\}_{i\in I} \subseteq B_H$ and define $\{f_i\}_{i\in I} \subseteq B_{H^{\sim}}$ to be such that $f_i = S^{\sim}(y_i)$ for all $i \in I$. By Theorem 1, we have that $B_{H^{\sim}}$ is $\sigma(H^{\sim}, V)$ compact. This implies that there exists a subnet $\left\{f_{i_j}\right\}_{j\in J} \subseteq \{f_i\}_{i\in I}$ and $f \in B_{H^{\sim}}$ such that $\ell\left(f_{i_j}\right) \to \ell(f)$ for all $\ell \in V$, that is, $\bar{\varphi}\left(f_{i_j}(x)\right) \to \bar{\varphi}(f(x))$ for all $x \in H$. Since $S^{\sim}(B_H) = B_{H^{\sim}}$, there exists $y \in B_H$ such that $f = S^{\sim}(y)$. We can conclude that $\bar{\varphi}\left(\left\langle y_{i_j}, x\right\rangle_H\right) = \bar{\varphi}\left(\left\langle x, y_{i_j}\right\rangle_H\right) = \bar{\varphi}\left(f_{i_j}(x)\right) \to \bar{\varphi}(f(x)) = \bar{\varphi}(\langle x, y\rangle_H) = \bar{\varphi}(\langle y, x\rangle_H)$ for all $x \in H$, that is, $y_{i_j} \overset{\sigma(H,S(H))}{\to} y \in B_H$. ■

Corollary 4 *Let A be an Arens algebra of $\mathcal{L}^{\infty}$ type and H a pre-Hilbert A-module. If H is self-dual, then B_H is $\sigma\left(H, cl_{\|\,\|_{H^*}}(S(H))\right)$ compact. In particular, B_H is $\sigma(H, H')$ compact.*

Proof. Consider $\{y_i\}_{i\in I} \subseteq B_H$. By Proposition 14, there exists $\left\{y_{i_j}\right\}_{j\in J} \subseteq \{y_i\}_{i\in I}$ and $y \in B_H$ such that $y_{i_j} \overset{\sigma(H,S(H))}{\to} y$. Since B_H is $\|\,\|_H$ bounded, this implies that $l\left(y_{i_j}\right) \to l(y)$ for all $l \in cl_{\|\,\|_{H^*}}(S(H))$. By Lemma 1 and since H is a pre-Hilbert space, $cl_{\|\,\|_{H'}}(S(H)) = H'$. By (10), we can conclude that $H' \subseteq cl_{\|\,\|_{H^*}}(S(H))$, proving $\sigma(H, H')$ compactness. ■

Proposition 15 *Let A be an Arens algebra of $\mathcal{L}^{\infty}$ type and H a pre-Hilbert A-module. If B_H is $\sigma(H, S(H))$ compact, then $S^{\sim}(H) \cap B_{H^{\sim}}$ is $\sigma(H^{\sim}, V)$ and $\sigma\left(H^{\sim}, \bar{V}\right)$ compact.*

Proof. Consider a net $\{f_i\}_{i\in I} \subseteq S^{\sim}(H) \cap B_{H^{\sim}}$. By assumption and since $S^{\sim}$ is an isometry, it follows that there exists $\{y_i\}_{i\in I} \subseteq B_H$ such that $S^{\sim}(y_i) = f_i$ for all $i \in I$. Since B_H is $\sigma(H, S(H))$ compact, we have that there exists a subnet $\left\{y_{i_j}\right\}_{j\in J}$ and $y \in B_H$ such that

$$\bar{\varphi}\left(\left\langle y_{i_j}, z\right\rangle_H\right) \to \bar{\varphi}\left(\langle y, z\rangle_H\right) \qquad \forall z \in H.$$

Define $f = \langle\ , y\rangle_H \in S^{\sim}(H) \cap B_{H^{\sim}}$. We can conclude that $\left\{f_{i_j}\right\}_{j\in J} \subseteq \{f_i\}_{i\in I} \subseteq S^{\sim}(H) \cap B_{H^{\sim}}$ and $f \in S^{\sim}(H) \cap B_{H^{\sim}}$ are such that

$$\bar{\varphi}\left(f_{i_j}(z)\right) \to \bar{\varphi}\left(f(z)\right) \qquad \forall z \in H,$$

proving that $S^{\sim}(H) \cap B_{H^{\sim}}$ is $\sigma(H^{\sim}, V)$ compact. Next, consider $\ell \in \bar{V}$. It follows that there exists $\{\ell_n\}_{n\in\mathbb{N}} \subseteq V$ such that $\ell_n \overset{\|\ \|_{(H^{\sim})^*}}{\to} \ell$. Note that

$$\begin{aligned}
\left|\ell(f) - \ell\left(f_{i_j}\right)\right| &\leq \left|\ell(f) - \ell_n(f)\right| + \left|\ell_n(f) - \ell_n\left(f_{i_j}\right)\right| + \left|\ell_n\left(f_{i_j}\right) - \ell\left(f_{i_j}\right)\right| \\
&\leq \|\ell - \ell_n\|_{(H^{\sim})^*} \|f\|_{H^{\sim}} + \left|\ell_n(f) - \ell_n\left(f_{i_j}\right)\right| + \|\ell - \ell_n\|_{(H^{\sim})^*} \left\|f_{i_j}\right\|_{H^{\sim}} \\
&\leq 2\|\ell - \ell_n\|_{(H^{\sim})^*} + \left|\ell_n(f) - \ell_n\left(f_{i_j}\right)\right|.
\end{aligned}$$

By taking the limit in j, we have that

$$0 \leq \limsup_j \left|\ell(f) - \ell\left(f_{i_j}\right)\right| \leq 2\|\ell - \ell_n\|_{(H^{\sim})^*} \qquad \forall n \in \mathbb{N}.$$

By taking the limit in n, we can conclude that

$$0 \leq \liminf_j \left|\ell(f) - \ell\left(f_{i_j}\right)\right| \leq \limsup_j \left|\ell(f) - \ell\left(f_{i_j}\right)\right| \leq 0,$$

proving that $\ell\left(f_{i_j}\right) \to \ell(f)$. It follows that $S^{\sim}(H) \cap B_{H^{\sim}}$ is $\sigma\left(H^{\sim}, \bar{V}\right)$ compact. ■

Corollary 5 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If B_H is $\sigma(H, S(H))$ compact, then $S^{\sim}(H)$ is $\sigma\left(H^{\sim}, \bar{V}\right)$ closed.*

Proof. By Lemma 4, $(H^{\sim}, \|\ \|_{H^{\sim}})$ can be identified with the dual of $\left(V, \|\ \|_{(H^{\sim})^*}\right)$. It is then immediate to see that $(H^{\sim}, \|\ \|_{H^{\sim}})$ can be identified with the dual of $\left(\bar{V}, \|\ \|_{(H^{\sim})^*}\right)$. By Proposition 15, $S^{\sim}(H) \cap B_{H^{\sim}}$ is $\sigma\left(H^{\sim}, \bar{V}\right)$ closed. By the Krein-Smulian Theorem (see [31, Corollary 2.7.12]) and since $S^{\sim}(H)$ is a vector subspace of $H^{\sim}$, it follows that $S^{\sim}(H)$ is $\sigma\left(H^{\sim}, \bar{V}\right)$ closed. ■

Theorem 2 *If A is an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module, then $S^{\sim}(H)$ separates the points of $\bar{V}$.*

Proof. We proceed by steps. Before starting, we denote by H_p the norm completion of H with respect to $\| \|_p$ and A_1 the norm completion of A with respect to $\| \|_1$ (see also Remark 4). We still denote by $\bar{\varphi}$ the extension of $\bar{\varphi}$ from A to A_1.

Step 1. If f is an A-linear and $\| \|_H - \| \|_A$ continuous operator, then f admits a unique $\| \|_p - \| \|_1$ continuous linear extension $\bar{f} : H_p \to A_1$.

Proof of the Step. By Proposition 8 and since f is linear, the statement trivially follows. □

Step 2. Let $\{\ell_n\}_{n\in\mathbb{N}} \subseteq V$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq H$ such that $\ell_n(f) = \bar{\varphi}(f(x_n))$ for all $f \in H^\sim$. If $\ell_n \overset{\| \|_{(H^\sim)^*}}{\to} \ell \in (H^\sim)^*$, then $x_n \overset{\| \|_p}{\to} x \in H_p$. Moreover, we have that

$$\ell(f) = \bar{\varphi}\left(\bar{f}(x)\right) \qquad \forall f \in H^\sim.$$

Proof of the Step. Consider $x \in H$. Define $\ell_x : H^\sim \to \mathbb{R}$ by $\ell_x(f) = \bar{\varphi}(f(x))$ for all $f \in H^\sim$. By definition of $\| \|_{(H^\sim)^*}$ and the proof of Proposition 10, it follows that

$$\|\ell_x\|_{(H^\sim)^*} = \sup_{f\in B_{H^\sim}} |\bar{\varphi}(f(x))| \geq \sup_{f\in B_{H^\sim}\cap S^\sim(H)} |\bar{\varphi}(f(x))| = \sup_{y\in B_H} |\bar{\varphi}(\langle x,y\rangle_H)| = \|x\|_p .$$

Since $\ell_n \overset{\| \|_{(H^\sim)^*}}{\to} \ell$, then $\{x_n\}_{n\in\mathbb{N}} \subseteq H$ is a $\| \|_p$ Cauchy sequence which thus converges to an element in the completion. Finally, fix $f \in H^\sim$. Since $\ell_n \overset{\| \|_{(H^\sim)^*}}{\to} \ell$, it follows that

$$\bar{\varphi}(f(x_n)) = \ell_n(f) \to \ell(f).$$

At the same time, by Step 1 and since $x_n \overset{\| \|_p}{\to} x \in H_p$, we have that $f(x_n) = \bar{f}(x_n) \overset{\| \|_1}{\to} \bar{f}(x)$. This implies that

$$\ell_n(f) = \bar{\varphi}(f(x_n)) \to \bar{\varphi}\left(\bar{f}(x)\right).$$

Since f was arbitrarily chosen, we can conclude that $\ell(f) = \bar{\varphi}\left(\bar{f}(x)\right)$ for all $f \in H^\sim$. □

Step 3. For each $y \in H$ and $x \in H_p$

$$\bar{\varphi}\left(\overline{S^\sim(y)}(x)\right) = \langle y,x\rangle_m^- .$$

Proof of the Step. Consider $y \in H$ and $x \in H_p$. There exists $\{x_n\}_{n\in\mathbb{N}} \subseteq H$ such that $x_n \overset{\| \|_p}{\to} x$. It follows that

$$S^\sim(y)(x_n) = \overline{S^\sim(y)}(x_n) \overset{\| \|_1}{\to} \overline{S^\sim(y)}(x).$$

This implies that

$$\langle y,x_n\rangle_m = \langle x_n,y\rangle_m = \bar{\varphi}(\langle x_n,y\rangle_H) = \bar{\varphi}(S^\sim(y)(x_n)) \to \bar{\varphi}\left(\overline{S^\sim(y)}(x)\right).$$

At the same time, by Lemma 2, $\langle y, x_n \rangle_m = \langle y, x_n \rangle_m^- \to \langle y, x \rangle_m^-$, proving the statement. □

We are ready to prove the main statement. Consider $\ell, \ell' \in \bar{V}$. By Step 2, it follows that there exists $x, x' \in H_p$ such that

$$\ell(f) = \bar{\varphi}\left(\bar{f}(x)\right) \text{ and } \ell'(f) = \bar{\varphi}\left(\bar{f}(x')\right) \qquad \forall f \in H^{\sim}. \tag{15}$$

Assume that

$$\ell(f) = \ell'(f) \qquad \forall f \in S^{\sim}(H).$$

It follows that

$$\bar{\varphi}\left(\overline{S^{\sim}(y)}(x)\right) = \bar{\varphi}\left(\overline{S^{\sim}(y)}(x')\right) \qquad \forall y \in H.$$

By Step 3, this implies that

$$\langle y, x \rangle_m^- = \langle y, x' \rangle_m^- \qquad \forall y \in H.$$

By Lemma 2, $x = x'$. By (15), this proves that $\ell = \ell'$. ■

We are now ready to state our first result on self-dual modules.

Theorem 3 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. The following statements are equivalent:*

(i) H is self-dual;

(ii) B_H is $\sigma(H, S(H))$ compact;

(iii) B_H is $\sigma\left(H, cl_{\| \|_{H^*}}(S(H))\right)$ compact;

(iv) H is $\sigma\left(H, cl_{\| \|_{H^*}}(S(H))\right)$ sequentially complete;

(v) B_H is $\| \|_m$ complete.

Proof. Before starting recall that H' (resp., H^*) denotes the norm dual of H when H is endowed with the norm $\| \|_m$ (resp., $\| \|_H$). We denote by H'' the second dual of H (that is, the norm dual of H' when H' is endowed with the norm $\| \|_{H'}$). Given Corollary 2, it is well known that H' is an Hilbert space (see [9] or [10, Exercise pp. 149-150]). By Lemma 1, we have that $S : H \to H'$ is a $\| \|_m - \| \|_{H'}$ linear isometry.

(i) implies (ii). It is Proposition 14. (ii) implies (iii). It is Corollary 4.

(iii) implies (i). Since $S(H) \subseteq cl_{\| \|_{H^*}}(S(H))$, B_H is $\sigma(H, S(H))$ compact. Recall that $(H^{\sim}, \| \|_{H^{\sim}})$ can be identified with the norm dual of $\left(\bar{V}, \| \|_{(H^{\sim})^*}\right)$ and $S^{\sim}(H)$ is a vector subspace of $H^{\sim}$. By Theorem 2, $S^{\sim}(H)$ separates the points of $\bar{V}$. By [4, Corollary 5.108], it follows that $S^{\sim}(H)$ is $\sigma\left(H^{\sim},\bar{V}\right)$ dense in $H^{\sim}$. By Corollary 5, $S^{\sim}(H)$ is $\sigma\left(H^{\sim},\bar{V}\right)$ closed, proving $S^{\sim}(H)=H^{\sim}$.

Thus, we have showed that (i), (ii), and (iii) are equivalent.

(iii) implies (iv). By point 4. of Proposition 12 and since H is self-dual, $(H,\|\ \|_H)$ is a Banach space. By [24], we have that the Banach space $\left(cl_{\|\ \|_{H^*}}(S(H)),\|\ \|_{H^*}\right)$ has a dual that can be identified with $(H,\|\ \|_H)$. If we consider a $\sigma\left(H,cl_{\|\ \|_{H^*}}(S(H))\right)$ Cauchy sequence in H, then it is a weakly* Cauchy sequence. By [31, Corollary 2.6.21], we can conclude that H is $\sigma\left(H,cl_{\|\ \|_{H^*}}(S(H))\right)$ sequentially complete.

(iv) implies (v). Consider a $\|\ \|_m$ Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq B_H$. It follows that

$$|l(x_n)-l(x_k)|\leq\|l\|_{H'}\|x_n-x_k\|_m\qquad\forall n,k\in\mathbb{N},\forall l\in H'.\tag{16}$$

Since $S(H)\subseteq H'$, this implies that $\{l(x_n)\}_{n\in\mathbb{N}}\subseteq\mathbb{R}$ is a Cauchy sequence for each $l\in S(H)$. Next, consider $\bar{l}\in cl_{\|\ \|_{H^*}}(S(H))$. Consider $\varepsilon>0$. It follows that there exists $l\in S(H)$ such that $\left\|\bar{l}-l\right\|_{H^*}<\varepsilon/4$. Since $\{x_n\}_{n\in\mathbb{N}}\subseteq B_H$, this implies that for each $n,k\in\mathbb{N}$

$$\left|\bar{l}(x_n-x_k)-l(x_n-x_k)\right|\leq\left\|\bar{l}-l\right\|_{H^*}\|x_n-x_k\|_H\leq 2\left\|\bar{l}-l\right\|_{H^*}\leq\frac{\varepsilon}{2}.\tag{17}$$

At the same time, since $\{l(x_n)\}_{n\in\mathbb{N}}\subseteq\mathbb{R}$ is a Cauchy sequence, there exists $n_{\varepsilon,l}\in\mathbb{N}$ such that

$$|l(x_n)-l(x_k)|<\frac{\varepsilon}{4}\qquad\forall n,k\geq n_{\varepsilon,l}.\tag{18}$$

By (17) and (18), we can conclude that

$$\left|\bar{l}(x_n-x_k)\right|\leq\left|\bar{l}(x_n-x_k)-l(x_n-x_k)\right|+|l(x_n)-l(x_k)|\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{4}<\varepsilon\qquad\forall n,k\geq n_{\varepsilon,l}.$$

Since ε was arbitrarily chosen, it follows that $\left\{\bar{l}(x_n)\right\}_{n\in\mathbb{N}}\subseteq\mathbb{R}$ is a Cauchy sequence. Since $\bar{l}$ was arbitrarily chosen, it follows that $\{l(x_n)\}_{n\in\mathbb{N}}\subseteq\mathbb{R}$ is a Cauchy sequence for all $l\in cl_{\|\ \|_{H^*}}(S(H))$. Since H is $\sigma\left(H,cl_{\|\ \|_{H^*}}(S(H))\right)$ sequentially complete, we have that there exists $x\in H$ such that $l(x_n)\to l(x)$ for all $l\in cl_{\|\ \|_{H^*}}(S(H))$. By (16) and since $\{x_n\}_{n\in\mathbb{N}}$ is a $\|\ \|_m$ Cauchy sequence, we also have that for each $\varepsilon>0$ and for each $l\in B_{H'}$ there exists $n_\varepsilon\in\mathbb{N}$ such that

$$|l(x_n)-l(x_k)|<\frac{\varepsilon}{2}\qquad\forall n,k\geq n_\varepsilon.$$

By taking the limit in k and since $H'\subseteq cl_{\|\ \|_{H^*}}(S(H))$, this implies that $|l(x_n-x)|=|l(x_n)-l(x)|\leq\varepsilon/2$ for all $l\in B_{H'}$ and for all $n\geq n_\varepsilon$. We thus have that $\|x-x_n\|_m=\sup_{l\in B_{H'}}|l(x_n-x)|<\varepsilon$ for all $n\geq n_\varepsilon$. It follows that $x_n\overset{\|\ \|_m}{\to}x$. We are left to show that $x\in B_H$. Note that

$$\bar{\varphi}(|N(x)-N(x_n)|)\leq\bar{\varphi}(|N(x-x_n)|)=\|x-x_n\|_p\leq\|x-x_n\|_m\to 0,$$

proving that $N(x_n) \overset{\| \|_1}{\to} N(x)$. Since $N(x_n) \leq e$ for all $n \in \mathbb{N}$ and the topology induced by $\| \|_1$ is locally solid and Hausdorff, it follows that $N(x) \leq e$, proving that $x \in B_H$.

(v) implies (ii). Since S is a linear isometry, if B_H is $\| \|_m$ complete, then $S(B_H)$ is convex and $\| \|_{H'}$ closed. It follows that $S(B_H)$ is $\sigma(H', H'')$ closed. Moreover, since $B_H \subseteq B_m$, $S(B_H)$ turns out to be contained in the unit ball induced by $\| \|_{H'}$. Since H' is an Hilbert space (thus, reflexive), this latter set is $\sigma(H', H'')$ compact. We can conclude that $S(B_H)$ is $\sigma(H', H'')$ compact. By definition of S, it is then immediate to see that this yields the desired compactness of B_H in the $\sigma(H, S(H))$ topology. ■

We conclude by providing a sufficient condition for the self-duality of H (Proposition 16) and a necessary one (Proposition 17).

Proposition 16 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. If B_H is d_H complete, then H is self-dual.*

Proof. By Theorem 3, it is sufficient to show that B_H is $\| \|_m$ complete. Consider a $\| \|_m$ Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq B_H$. By (8) and (9), it follows that $\{x_n\}_{n\in\mathbb{N}}$ is a d_H Cauchy sequence. Since B_H is d_H complete, it follows that there exists $x \in B_H$ such that $x_n \overset{d_H}{\to} x$. Define $\{y_n\}_{n\in\mathbb{N}}$ by $y_n = \frac{1}{2}x_n$ for all $n \in \mathbb{N}$ and $y = \frac{1}{2}x$. Since $\{x_n\}_{n\in\mathbb{N}} \subseteq B_H$ and $x \in B_H$, it follows that $0 \leq N(y_n - y) \leq e$ for all $n \in \mathbb{N}$. Moreover, $y_n \overset{d_H}{\to} y$. Next, note that

$$\begin{aligned} 0 &\leq \|y_n - y\|_m^2 = \bar{\varphi}\left(N(y_n - y)^2\right) = \bar{\varphi}\left(N(y_n - y) N(y_n - y)\right) \leq \bar{\varphi}\left(N(y_n - y)\right) \\ &= \bar{\varphi}\left(N(y_n - y) \wedge e\right) = d_H(y_n, y) \to 0. \end{aligned}$$

By (5), this implies that $0 = \lim_n 2\|y_n - y\|_m = \lim_n \|x_n - x\|_m$, proving that $x_n \overset{\| \|_m}{\to} x$ and that B_H is $\| \|_m$ complete. ■

As mentioned in the introduction, Kaplansky [25] first studied self-duality for pre-Hilbert A-modules H where A was a commutative AW^*-algebra.[12] In such a context, he proved that self-duality is equivalent to H satisfying certain algebraic property which we summarize in the next definition (see also [16]). Proposition 17 shows that these algebraic conditions are necessary for self-duality also in the real case. It has eluded us whether they are also sufficient.

Definition 9 *Let A be an Arens algebra of $\mathcal{L}^\infty$ type and H a pre-Hilbert A-module. We say that*

1. *$\{e_i\}_{i\in I}$ is an orthogonal partition of the unit e if and only if $e = \sup_{i\in I} e_i$ and $e_i \wedge e_j = 0$ for all $i \neq j$.*

[12] Recall that the the main distinction with the current work is the focus on complex valued algebras A.

2. *H is an Hilbert-Kaplansky module if and only if for each orthogonal partition of the unit* $\{e_i\}_{i\in I} \subseteq A$

   (a) *for each* $x \in H$*,* $e_i \cdot x = 0$ *for all* $i \in I$ *implies* $x = 0$*;*

   (b) *for each* $\| \ \|_H$ *bounded collection* $\{x_i\}_{i\in I} \subseteq H$ *there exists* $x \in H$ *such that* $e_i \cdot x_i = e_i \cdot x$*.*

Proposition 17 *Let* A *be an Arens algebra of* $\mathcal{L}^\infty$ *type and* H *a pre-Hilbert* A*-module. If* H *is self-dual, then* H *is an Hilbert-Kaplansky module.*

Proof. Consider $\{e_i\}_{i\in I} \subseteq A_+$ such that $\|e_i\|_A \leq 1$ for all $i \in I$ and such that $\sup_{i\in I} e_i = e$ and $e_i \wedge e_{i'} = 0$ for all $i \neq i'$. Note that $e_i e_{i'} = 0$ for all $i \neq i'$. Moreover, we can also conclude that $t_B = \sum_{i\in B} e_i = \vee_{i\in B} e_i \uparrow e$ where B is a finite subset of I and this finite subsets are directed by using the inclusion relation. This latter observation yields that $e_i e_i = e_i$ for all $i \in I$.[13] Finally, since $\bar{\varphi}$ is order continuous, observe that $0 \leq \bar{\varphi}(e - t_B) \downarrow 0$.

We start by providing a construction which will be instrumental in proving the statement. Assume that $\{x_i\}_{i\in I} \subseteq H$ is $\| \ \|_H$ bounded. Call 2_0^I the collection of all finite subsets of I. Direct 2_0^I with the inclusion relation. Define $\{s_B\}_{B\in 2_0^I} \subseteq H$ by

$$s_B = \sum_{i\in B} e_i \cdot x_i \qquad \forall B \in 2_0^I.$$

Note that $\langle s_B, s_B\rangle_H = \sum_{i\in B} e_i \langle x_i, x_i\rangle_H$ for all B.[14] Since $\{x_i\}_{i\in I} \subseteq H$ is $\| \ \|_H$ bounded, we have that there exists $M > 0$ such that $\|x_i\|_H \leq M$ for all $i \in I$. By Proposition 5

[13] Note that for each $i \in I$ and B

$$|e_i - e_i t_B| = |e_i e - e_i t_B| = |e_i| \, |e - t_B| = e_i (e - t_B) \leq (e - t_B).$$

By passing to the order limit, we have that $e_i t_B \uparrow e_i$ for all $i \in I$. Since $e_i t_B = e_i \cdot e_i$ for all B such that $\{i\} \subseteq B$, the statement follows.

[14] First, consider B and $i' \notin B$. Observe that

$$\begin{aligned} e_{i'} \langle s_B, x_{i'}\rangle_H &= e_{i'} \left\langle \sum_{i\in B} e_i \cdot x_i, x_{i'} \right\rangle_H = e_{i'} \sum_{i\in B} \langle e_i \cdot x_i, x_{i'}\rangle_H \\ &= e_{i'} \sum_{i\in B} e_i \langle x_i, x_{i'}\rangle_H = \sum_{i\in B} (e_{i'} e_i) \langle x_i, x_{i'}\rangle_H = 0. \end{aligned}$$

Thus, we have that:
- if $|B| = 1$, then

$$\langle s_B, s_B\rangle_H = \langle e_i \cdot x_i, e_i \cdot x_i\rangle_H = e_i^2 \langle x_i, x_i\rangle_H = e_i \langle x_i, x_i\rangle_H ;$$

- if the statement is true for $|B'| = m$, then consider B such that $|B| = m + 1$. It follows that

$$B = B' \cup \{i'\}.$$

and since each φ in K is an algebra homomorphism, it follows that for each $\varphi \in K$

$$\begin{aligned} 0 &\leq \varphi\left(\langle s_B, s_B\rangle_H\right) = \varphi\left(\sum_{i\in B} e_i \langle x_i, x_i\rangle_H\right) = \sum_{i\in B} \varphi\left(e_i \langle x_i, x_i\rangle_H\right) \\ &= \sum_{i\in B} \varphi(e_i)\, \varphi\left(\langle x_i, x_i\rangle_H\right) \leq \sum_{i\in B} \varphi(e_i) \|x_i\|_H^2 \leq \sum_{i\in B} \varphi(e_i) M^2 \\ &= M^2 \varphi\left(\sum_{i\in B} e_i\right) \leq M^2 \varphi(e) = M^2 \qquad \forall B \in 2^I_0. \end{aligned}$$

By Proposition 5, this implies that

$$\|s_B\|_H = \sqrt{\max_{\varphi\in K} \varphi\left(\langle s_B, s_B\rangle_H\right)} \leq M \qquad \forall i \in I.$$

Since B_H is $\sigma(H, S(H))$ compact, it follows that there exist a subnet $\left\{s_{B_j}\right\}_{j\in J}$ and $\bar{x} \in H$ such that $s_{B_j} \overset{\sigma(H,S(H))}{\to} \bar{x} \in H$, that is,

$$\bar{\varphi}\left(\left\langle s_{B_j}, y\right\rangle_H\right) \to \bar{\varphi}\left(\langle \bar{x}, y\rangle_H\right) \qquad \forall y \in H.$$

1. Consider $x_i = x$ for all $i \in I$. It is immediate to see that $\{x_i\}_{i\in I}$ is $\|\ \|_H$ bounded. Next, we show that $x = \bar{x}$. Note that for each $j \in J$ and for each $y \in H$

$$\begin{aligned} \left|\bar{\varphi}\left(\langle x, y\rangle_H\right) - \bar{\varphi}\left(\left\langle s_{B_j}, y\right\rangle_H\right)\right| &= \left|\bar{\varphi}\left(\left\langle e\cdot x - s_{B_j}, y\right\rangle_H\right)\right| \\ &= \left|\bar{\varphi}\left(\left\langle\left(e - \sum_{i\in B_j} e_i\right)\cdot x, y\right\rangle_H\right)\right| \\ &= \left|\bar{\varphi}\left(\left(e - t_{B_j}\right)\langle x, y\rangle_H\right)\right| \leq \bar{\varphi}\left(\left|\langle x, y\rangle_H \left(e - t_{B_j}\right)\right|\right) \\ &= \bar{\varphi}\left(\left|\langle x, y\rangle_H\right| \left|e - t_{B_j}\right|\right) \leq \bar{\varphi}\left(\left\|\langle x, y\rangle_H\right\|_A e\left(e - t_{B_j}\right)\right) \\ &= \left\|\langle x, y\rangle_H\right\|_A \bar{\varphi}\left(e - t_{B_j}\right). \end{aligned}$$

By passing to the limit with respect to j, we obtain that $\bar{\varphi}\left(\left\langle s_{B_j}, y\right\rangle_H\right) \to \bar{\varphi}\left(\langle x, y\rangle_H\right)$ for all $y \in H$. By the uniqueness of the limit, we obtain that $x = \bar{x}$. Thus, if $e_i \cdot x = 0$ for all $i \in I$, then $s_{B_j} = 0$ for all $j \in J$, implying that $x = \bar{x} = 0$.

We thus have that

$$\begin{aligned} \langle s_B, s_B\rangle_H &= \langle s_{B'} + e_{i'}\cdot x_{i'}, s_{B'} + e_{i'}\cdot x_{i'}\rangle_H \\ &= \langle s_{B'}, s_{B'} + e_{i'}\cdot x_{i'}\rangle_H + \langle e_{i'}\cdot x_{i'}, s_{B'} + e_{i'}\cdot x_{i'}\rangle_H \\ &= \langle s_{B'}, s_{B'}\rangle_H + 2e_{i'}\langle s_{B'}, x_{i'}\rangle_H + \langle e_{i'}\cdot x_{i'}, e_{i'}\cdot x_{i'}\rangle_H \\ &= \langle s_{B'}, s_{B'}\rangle_H + e_{i'}\langle x_{i'}, x_{i'}\rangle_H \\ &= \sum_{i\in B'} e_i\langle x_i, x_i\rangle_H + e_{i'}\langle x_{i'}, x_{i'}\rangle_H = \sum_{i\in B} e_i\langle x_i, x_i\rangle_H, \end{aligned}$$

proving that the statement is true for $m+1$. The whole statement follows by induction.

2. Assume that $\{x_i\}_{i\in I} \subseteq H$ is $\| \|_H$ bounded. Consider $\bar{\imath} \in I$. Consider $j \in J$ such that $B_j \supseteq \{\bar{\imath}\}$. It follows that

$$e_{\bar{\imath}} \cdot s_{B_j} = e_{\bar{\imath}} \cdot \left(\sum_{i\in B_j} e_i \cdot x_i \right) = \left(\sum_{i\in B_j} (e_{\bar{\imath}} e_i) \cdot x_i \right) = e_{\bar{\imath}} \cdot x_{\bar{\imath}}.$$

We can conclude that, for each $y \in H$, eventually

$$\bar{\varphi}\left(\left\langle e_{\bar{\imath}} \cdot s_{B_j}, y \right\rangle_H \right) = \bar{\varphi}\left(\langle e_{\bar{\imath}} \cdot x_{\bar{\imath}}, y \rangle_H \right)$$

thus, proving that

$$e_{\bar{\imath}} \cdot s_{B_j} \overset{\sigma(H,S(H))}{\to} e_{\bar{\imath}} \cdot x_{\bar{\imath}}.$$

At the same time, since $\bar{\varphi}\left(\left\langle s_{B_j}, y \right\rangle_H \right) \to \bar{\varphi}\left(\langle \bar{x}, y \rangle_H \right)$ for all $y \in H$, we have that for each $y \in H$

$$\lim_j \bar{\varphi}\left(\left\langle e_{\bar{\imath}} \cdot s_{B_j}, y \right\rangle_H \right) = \lim_j \bar{\varphi}\left(\left\langle s_{B_j}, e_{\bar{\imath}} \cdot y \right\rangle_H \right) = \bar{\varphi}\left(\langle \bar{x}, e_{\bar{\imath}} \cdot y \rangle_H \right) = \bar{\varphi}\left(\langle e_{\bar{\imath}} \cdot \bar{x}, y \rangle_H \right),$$

that is, $e_{\bar{\imath}} \cdot s_{B_j} \overset{\sigma(H,S(H))}{\to} e_{\bar{\imath}} \cdot \bar{x}$. By the uniqueness of the limit, we can conclude that $e_{\bar{\imath}} \cdot x_{\bar{\imath}} = e_{\bar{\imath}} \cdot \bar{x}$. Since $\bar{\imath}$ was arbitrarily chosen, the statement follows. ■

5.2 Finite dimensional case

In this subsection, we discuss separately the case in which A is a finite dimensional Arens algebra, that is, A is isomorphic to some $\mathbb{R}^n$. It is immediate to see that if A is finite dimensional, then it is of $\mathcal{L}^\infty$ type. At the same time, the finite dimensional case merits to be discussed separately. First, the result of self-duality can be obtained via more direct methods. Second, it is the only case in which we can show that Hilbert A-modules are indeed self-dual. In other words, we can show that $\| \|_H$ completeness is necessary *and sufficient* for self-duality.

Theorem 4 *Let A be a finite dimensional Arens algebra and H a pre-Hilbert A-module. The following statements are equivalent:*

(i) H is $\| \|_H$ complete, that is, H is an Hilbert A-module;

(ii) H is $\| \|_m$ complete;

(iii) H is self-dual.

Proof. Before starting, observe that, since A is finite dimensional, it is Dedekind complete and admits a strictly positive functional $\bar{\varphi}$ (see Proposition 9).

(i) implies (ii). By Proposition 9 and since A is finite dimensional, $\| \|_m$ and $\| \|_H$ are equivalent. It follows that H is $\| \|_m$ complete.

(ii) implies (iii). By Corollary 2 and since H is $\| \|_m$ complete, it follows that H is an Hilbert space with inner product $\langle \, , \, \rangle_m$. Consider $f : H \to A$ which is A-linear and bounded. By Proposition 8, it follows that $f : H \to A$ is A-linear and $\| \|_m - \| \|_1$ continuous. Consider the linear functional $l = \bar{\varphi} \circ f$. Since $\bar{\varphi}$ is $\| \|_1$ continuous and f is $\| \|_m - \| \|_1$ continuous, we have that l is $\| \|_m$ continuous. By the standard Riesz representation theorem, there exists (a unique) $y \in H$ such that $l(x) = \langle x, y \rangle_m$ for all $x \in H$. It follows that

$$\bar{\varphi}\left(f(x) - \langle x, y \rangle_H\right) = \bar{\varphi}\left(f(x)\right) - \bar{\varphi}\left(\langle x, y \rangle_H\right) = l(x) - \langle x, y \rangle_m = 0 \qquad \forall x \in H. \quad (19)$$

Fix $\bar{x} \in H$. Define $a = f(\bar{x}) - \langle \bar{x}, y \rangle_H \in A$. By (19), we have that

$$\begin{aligned} 0 &= \bar{\varphi}\left(f(a \cdot \bar{x}) - \langle a \cdot \bar{x}, y \rangle_H\right) = \bar{\varphi}\left(af(\bar{x}) - a \langle \bar{x}, y \rangle_H\right) \\ &= \bar{\varphi}\left(a\left(f(\bar{x}) - \langle \bar{x}, y \rangle_H\right)\right) = \bar{\varphi}(aa) = \bar{\varphi}\left(a^2\right). \end{aligned}$$

Since $\bar{\varphi}$ is strictly positive, this implies that $a^2 = 0$. Since A is an Arens algebra, we can conclude that $f(\bar{x}) - \langle \bar{x}, y \rangle_H = a = 0$. Since $\bar{x}$ was arbitrarily chosen, it follows that $f(x) = \langle x, y \rangle_H$ for all $x \in H$, proving that H is self-dual.

(iii) implies (i). By point 4. of Proposition 12, it follows that H is $\| \|_H$ complete. ■

5.3 f-algebras of $\mathcal{L}^0$ type

If A is an f-algebra of $\mathcal{L}^0$ type and H is a pre-Hilbert A-module, then the inner product $\langle \, , \, \rangle_H$ still satisfies the Cauchy-Schwarz inequality as in points 1. and 2. of Proposition 4 (see also Remark 2). By Proposition 7, we can endow H with the invariant metric d_H. We define

$$H_e = \left\{ x \in H : \langle x, x \rangle_H^{\frac{1}{2}} \in A_e \right\}.$$

Finally, we define

$$H_e^\sim = \left\{ f \in A_e^{H_e} : f \text{ is } A_e\text{-linear and bounded} \right\}.$$

Observe that $f \in H_e^\sim$ only if there exists $0 \leq c \in A_e$ such that $f^2(x) \leq c \langle x, x \rangle_H$ for all $x \in H_e$.

Proposition 18 *Let A be an f-algebra of $\mathcal{L}^0$ type and H a pre-Hilbert A-module. The following statements are true:*

1. *A_e is an Arens algebra of $\mathcal{L}^\infty$ type;*

2. H_e *is a pre-Hilbert* A_e*-module;*

3. H_e *is* d_H *dense in* H*;*

4. $H_e^{\sim}$ *is the dual module of* H_e*.*

Proof. 1. and 2. By definition, A_e is an Arens algebra of $\mathcal{L}^\infty$ type. Next, we show that H_e is closed under $+$. Consider $x, y \in H_e$. By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}\langle x+y, x+y\rangle_H &= \langle x,x\rangle_H + 2\langle x,y\rangle_H + \langle y,y\rangle_H \\ &\leq \langle x,x\rangle_H + 2\langle x,x\rangle_H^{\frac{1}{2}}\langle y,y\rangle_H^{\frac{1}{2}} + \langle y,y\rangle_H \in A_e.\end{aligned}$$

Next, we show that if $a \in A_e$ and $x \in H_e$, then $a \cdot x \in H_e$. Since $a, \langle x,x\rangle_H^{\frac{1}{2}} \in A_e$ and A_e is an Arens algebra, it follows that $\langle a\cdot x, a\cdot x\rangle_H = a^2\langle x,x\rangle_H \in A_e$, proving that $\langle a\cdot x, a\cdot x\rangle_H^{\frac{1}{2}} \in A_e$. The closure of H_e with respect to $+$ and $\cdot$ yields that $(H_e, +)$ is an abelian (sub)group and properties (1)-(4) of Definition 1 automatically follow. Finally, note that if $x, y \in H_e$, then $\langle x,x\rangle_H^{\frac{1}{2}}, \langle y,y\rangle_H^{\frac{1}{2}} \in A_e$ and $|\langle x,y\rangle_H| \leq \langle x,x\rangle_H^{\frac{1}{2}}\langle y,y\rangle_H^{\frac{1}{2}} \in A_e$, proving that $\langle x,y\rangle_H \in A_e$. Thus, $\langle\ ,\ \rangle_{H_e}$ is $\langle\ ,\ \rangle_H$ restricted to $H_e \times H_e$. Then, properties (5)-(8) of Definition 1 automatically follow.

3. Since A is an f-algebra of $\mathcal{L}^0$ type, it follows that for each $c \in A_+$ there exists $\{e_n\}_{n\in\mathbb{N}} \subseteq A_e$ such that $0 \leq e_n \uparrow e$, $e_n^2 = e_n$, and $e_n c \in A_e$ for all $n \in \mathbb{N}$. Consider $c = \langle x,x\rangle_H^{\frac{1}{2}} = N(x)$ for some $x \in H$. Define $x_n = e_n \cdot x$ for all $n \in \mathbb{N}$. It follows that $N(x_n) = e_n N(x) \in A_e$, that is, $x_n \in H_e$. Moreover, note that $((e - e_n)N(x)) \wedge e \downarrow 0$. Since $\bar{\varphi}$ is order continuous, it follows that

$$\begin{aligned}d_H(x, x_n) &= \bar{\varphi}(N(x - x_n)\wedge e) = \bar{\varphi}(N(e\cdot x - e_n\cdot x)\wedge e) = \bar{\varphi}(N((e-e_n)\cdot x)\wedge e) \\ &= \bar{\varphi}(((e-e_n)N(x))\wedge e) \to 0,\end{aligned}$$

proving that $x_n \overset{d_H}{\to} x$. Since x was arbitrarily chosen, it follows that H_e is d_H dense in H.

4. It follows by definition of dual module. ■

Our proof of self-duality for a pre-Hilbert A-module H where A is an f-algebra of $\mathcal{L}^0$ type hinges on our self-duality result for a pre-Hilbert A-module H where A is an algebra of $\mathcal{L}^\infty$ type. This latter pre-Hilbert module will be H_e and the algebra will be the subalgebra A_e.

Theorem 5 *Let* A *be an* f*-algebra of* $\mathcal{L}^0$ *type and* H *a pre-Hilbert* A*-module. The following statements are equivalent:*

(i) H *is* d_H *complete;*

(ii) H is self-dual.

Proof. (i) implies (ii). By Proposition 18, we have that H_e is a pre-Hilbert A_e-module and A_e is an Arens algebra of $\mathcal{L}^\infty$ type. We can thus define $\| \|_{H_e}$ and B_{H_e}. It follows that $x \in B_{H_e}$ if and only if $\langle x, x\rangle_H \leq e$. We proceed by steps.

Step 1. If $f \in H^\sim$, then f is $d_H - d$ continuous.

Proof of the Step. By point 1. of Proposition 13, the statement follows. □

Step 2. B_{H_e} is d_H complete.

Proof of the Step. Consider a d_H Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq B_{H_e}$. Since H is d_H complete, it follows that there exists $x \in H$ such that $x_n \overset{d_H}{\to} x$. We next show that $x \in B_{H_e}$. By one of Birkhoff's inequalities (see [6, Theorem 1.9]) and since $\langle x_n, x_n\rangle_H \leq e$ for all $n \in \mathbb{N}$, we also have that

$$|e - N(x) \vee e| = |N(x_n) \vee e - N(x) \vee e| \leq |N(x_n) - N(x)| \leq N(x_n - x) \quad \forall n \in \mathbb{N}.$$

It follows that

$$|e - N(x) \vee e| \wedge e \leq N(x_n - x) \wedge e \qquad \forall n \in \mathbb{N}.$$

We have that

$$0 \leq \bar{\varphi}(|e - N(x) \vee e| \wedge e) \leq \bar{\varphi}(N(x_n - x) \wedge e) = d_H(x_n, x) \to 0.$$

This implies that

$$\bar{\varphi}(|e - N(x) \vee e| \wedge e) = 0 \implies |e - N(x) \vee e| \wedge e = 0 \implies |e - N(x) \vee e| = 0$$
$$\implies e = N(x) \vee e \implies N(x) \leq e,$$

that is, $\langle x, x\rangle_H \leq e$ and $x \in B_{H_e}$. □

Step 3. The pre-Hilbert A_e-module H_e is self-dual, that is, for each $f \in H_e^\sim$ there exists $z \in H_e$ such that $f(x) = \langle x, z\rangle_H$ for all $x \in H_e$.

Proof of the Step. By points 1. and 2. of Proposition 18, Step 2, and Proposition 16, the statement follows. □

Step 4. For each $f \in H^\sim$ there exists $z \in H$ such that $f(x) = \langle x, z\rangle_H$ for all $x \in H_e$.

Proof of the Step. Consider $f \in H^\sim$. It follows that there exists $c \in A_+$ such that $f^2(x) \leq c\langle x, x\rangle_H$ for all $x \in H$. Since A is an f-algebra of $\mathcal{L}^0$ type, it follows that there exists a sequence $\{e_n\}_{n\in\mathbb{N}} \subseteq A_e$ such that $0 \leq e_n \uparrow e$, $e_n^2 = e_n$, and $e_n c \in A_e$ for all $n \in \mathbb{N}$. Define $f_n = e_n \cdot f$ for all $n \in \mathbb{N}$. Note that for each $x \in H$

$$d(f_n(x), f(x)) = \bar{\varphi}(|f(x) - f_n(x)| \wedge e) = \bar{\varphi}(((e - e_n)|f(x)|) \wedge e) \to 0. \qquad (20)$$

Fix $n \in \mathbb{N}$. By Proposition 11, we have that f_n is A-linear. In particular, $f_n^2(x) \leq e_n^2 c \langle x, x \rangle_H = c_n \langle x, x \rangle_H$ for all $x \in H$ where $c_n = e_n c \in A_e$. Thus, f_n restricted to H_e belongs to $H_e^{\sim}$. By Step 3, there exists a unique $z_n \in H_e$ such that

$$(e_n \cdot f)(x) = e_n f(x) = f_n(x) = \langle x, z_n \rangle_H \qquad \forall x \in H_e.$$

Note that for each $n > k$ and for each $x \in H_e$

$$((f_n - f_k)(x))^2 = (e_n f(x) - e_k f(x))^2 = (e_n - e_k)^2 f^2(x) \leq (e_n - e_k)^2 c \langle x, x \rangle_H .$$

It follows that

$$\langle x, z_n - z_k \rangle_H \leq |(f_n - f_k)(x)| \leq (e_n - e_k) c^{\frac{1}{2}} \langle x, x \rangle_H^{\frac{1}{2}} \qquad \forall x \in H_e. \tag{21}$$

Since A is an f-algebra of $\mathcal{L}^0$ type, we have that

$$A_e \ni \sup_{x \in B_{H_e}} \langle x, z_n - z_k \rangle_H \leq (e_n - e_k) c^{\frac{1}{2}}.$$

By the claim contained in the proof of Proposition 10, there exists a sequence $\{x_l\}_{l \in \mathbb{N}} \subseteq B_{H_e}$ such that $\langle x_l, z_n - z_k \rangle_H \uparrow$ and $\langle x_l, z_n - z_k \rangle_H \overset{\| \|_{A_e}}{\to} \langle z_n - z_k, z_n - z_k \rangle_H^{\frac{1}{2}}$. It follows that

$$\langle x_l, z_n - z_k \rangle_H \wedge e \overset{\| \|_{A_e}}{\to} \langle z_n - z_k, z_n - z_k \rangle_H^{\frac{1}{2}} \wedge e.$$

By Theorem [6, Theorem 1.8] and (21), we have that

$$\begin{aligned} \langle x_l, z_n - z_k \rangle_H \wedge e &\leq \sup_{x \in B_{H_e}} (\langle x, z_n - z_k \rangle_H \wedge e) = \left(\sup_{x \in B_{H_e}} \langle x, z_n - z_k \rangle_H \right) \wedge e \\ &\leq (e_n - e_k) c^{\frac{1}{2}} \wedge e \qquad \forall l \in \mathbb{N}. \end{aligned}$$

We can conclude that

$$\begin{aligned} d_H(z_n, z_k) &= \bar{\varphi}\left(\langle z_n - z_k, z_n - z_k \rangle_H^{\frac{1}{2}} \wedge e \right) = \lim_l \bar{\varphi}(\langle x_l, z_n - z_k \rangle_H \wedge e) \\ &\leq \bar{\varphi}\left((e_n - e_k) c^{\frac{1}{2}} \wedge e \right) = d\left(e_k c^{\frac{1}{2}}, e_n c^{\frac{1}{2}} \right). \end{aligned}$$

It follows that $\{z_n\}_{n \in \mathbb{N}}$ is a d_H Cauchy sequence. Since H is d_H complete, it follows that there exists $z \in H$ such that $z_n \overset{d_H}{\to} z$. At the same time, for each $x \in B_{H_e}$ we have that

$$\begin{aligned} d(\langle x, z_n \rangle_H, \langle x, z \rangle_H) &= \bar{\varphi}(|\langle x, z \rangle_H - \langle x, z_n \rangle_H| \wedge e) = \bar{\varphi}(|\langle x, z - z_n \rangle_H| \wedge e) \\ &\leq \bar{\varphi}\left(\left(\langle x, x \rangle_H^{\frac{1}{2}} \langle z - z_n, z - z_n \rangle_H^{\frac{1}{2}} \right) \wedge e \right) \leq \bar{\varphi}\left(\langle z - z_n, z - z_n \rangle_H^{\frac{1}{2}} \wedge e \right) \\ &= d_H(z, z_n) \to 0. \end{aligned}$$

It follows that $f(x) = \langle x, z\rangle_H$ for all $x \in B_{H_e}$. This implies that $f(x) = \langle x, z\rangle_H$ for all $x \in H_e$. $\square$

By Step 4, if $f \in H^\sim$, then there exists $z \in H$ such that $f(x) = \langle x, z\rangle_H$ for all $x \in H_e$. By Proposition 18, H_e is d_H dense in H. By Step 1, f is $d_H - d$ continuous. This implies that $f(x) = \langle x, z\rangle_H$ for all $x \in H$, proving that H is self-dual.

(ii) implies (i). Consider a d_H Cauchy sequence $\{y_n\}_{n\in\mathbb{N}}$. Define $\{f_n\}_{n\in\mathbb{N}}$ as $f_n = S^\sim(y_n)$ for all $n \in \mathbb{N}$.

Step 1. There exists $f : H_e \to A$ *such that* $f_n(x) \xrightarrow{d} f(x)$ *for all* $x \in H_e$.

Proof of the Step. By Proposition 4, we have that

$$|f_n(x) - f_m(x)| \leq N(x)\, N(y_n - y_m) \qquad \forall n, m \in \mathbb{N}, \forall x \in H. \tag{22}$$

Fix $x \in B_{H_e}$. We can conclude that

$$|f_n(x) - f_m(x)| \wedge e \leq (N(x)\, N(y_n - y_m)) \wedge e \leq N(y_n - y_m) \wedge e,$$

yielding that

$$\begin{aligned} d(f_n(x), f_m(x)) &= \bar{\varphi}(|f_n(x) - f_m(x)| \wedge e) \\ &\leq \bar{\varphi}(N(y_n - y_m) \wedge e) = d_H(y_n, y_m) \quad \forall n, m \in \mathbb{N}. \end{aligned}$$

We thus have that $\{f_n(x)\}_{n\in\mathbb{N}} \subseteq A$ is a d Cauchy sequence. Since A is complete, this yields that $f_n(x) \xrightarrow{d} a_x$. Next, note that if $x \in H_e \backslash B_{H_e}$, then $\bar{x} = \frac{x}{\|x\|_{H_e}} \in B_{H_e}$. We have that $f_n(\bar{x}) \xrightarrow{d} a_{\bar{x}}$. Thus, we can conclude that there exists $a_x \in A$ such that

$$f_n(x) = \|x\|_{H_e} f_n(\bar{x}) \xrightarrow{d} \|x\|_{H_e} a_{\bar{x}} = a_x.$$

By the uniqueness of the limit and since x was arbitrarily chosen, we can define a map $f : H_e \to A$ such that $f(x) = a_x$ for all $x \in H_e$. $\square$

Step 2. The map f *is such that*

$$f(a \cdot x + b \cdot y) = af(x) + bf(y) \qquad \forall a, b \in A_e, \forall x, y \in H_e.$$

Proof of the Step. Consider $a, b \in A_e$ and $x, y \in H_e$. We have that $a \cdot x + b \cdot y \in H_e$. By Step 1 and since each f_n is A-linear, this implies that

$$af_n(x) + bf_n(y) = f_n(a \cdot x + b \cdot y) \xrightarrow{d} f(a \cdot x + b \cdot y).$$

At the same time, since $f_n(x) \xrightarrow{d} f(x)$ and $f_n(y) \xrightarrow{d} f(y)$, we have that

$$af_n(x) + bf_n(y) \xrightarrow{d} af(x) + bf(y).$$

By the uniqueness of the limit, we can conclude that $f(a \cdot x + b \cdot y) = af(x) + bf(y)$, proving the statement. $\square$

Step 3. There exists $c \in A_+$ such that

$$|f(x)| \leq cN(x) \qquad \forall x \in H_e. \tag{23}$$

In particular, f is uniformly continuous on H_e.

Proof of the Step. Since N is a vector-valued norm, we have that

$$|N(y_n) - N(y_m)| \leq N(y_n - y_m) \qquad \forall n, m \in \mathbb{N}.$$

It follows that $\{N(y_n)\}_{n\in\mathbb{N}} \subseteq A_+$ is a d Cauchy sequence. Since A is d complete, it follows that there exists $c \in A_+$ such that $N(y_n) \xrightarrow{d} c \in A_+$. By Proposition 4, we also have that

$$|f_n(x)| \leq N(y_n) N(x) \qquad \forall x \in H_e, \forall n \in \mathbb{N}.$$

By passing to the limit and since the topology induced by d is linear and locally solid, we can conclude that

$$|f(x)| \leq cN(x) \qquad \forall x \in H_e, \tag{24}$$

proving the statement. Finally, by [4, Lemma 5.17], it is enough to show continuity at 0. Consider $x_n \xrightarrow{d_H} 0$. It follows that $N(x_n) \xrightarrow{d} 0$. This implies that $cN(x_n) \xrightarrow{d} 0$. By (24), we have that

$$d(0, f(x_n)) = \bar{\varphi}(|f(x_n)| \wedge e) \leq \bar{\varphi}((cN(x_n)) \wedge e) = d(0, cN(x_n)) \to 0,$$

proving continuity at 0. $\square$

Step 4. f admits a unique extension to H which is A-linear and bounded. In particular, there exists $y \in H$ such that $f = S^{\sim}(y)$.

Proof of the Step. Since f is uniformly continuous and H_e is d_H dense in H, it is well known that f admits a unique uniform continuous extension to H. For the moment, we denote such extension by $\bar{f}$. Then, we will simply denote it f. Since f was additive, it is a routine argument to show that so is $\bar{f}$. Next, consider $a \in A_e$ and $x \in H$. There exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq H_e$ such that $x_n \xrightarrow{d_H} x$. It follows that $a \cdot x_n \xrightarrow{d_H} a \cdot x$. Since $\bar{f}$ is a continuous extension of f and by Step 2, we can conclude that

$$af(x_n) = f(a \cdot x_n) = \bar{f}(a \cdot x_n) \xrightarrow{d} \bar{f}(a \cdot x).$$

At the same time, we have that $f(x_n) = \bar{f}(x_n) \xrightarrow{d} \bar{f}(x)$, thus, $af(x_n) = a\bar{f}(x_n) \xrightarrow{d} a\bar{f}(x)$. By the uniqueness of the limit, we can conclude that $\bar{f}(a \cdot x) = a\bar{f}(x)$. Next, consider $a \in A$ and $x \in H$. There exists $\{a_n\}_{n\in\mathbb{N}} \subseteq A_e$ such that $a_n \xrightarrow{d} a$. We have that $a_n \cdot x \xrightarrow{d_H} a \cdot x$. Since $\bar{f}$ is continuous, we can conclude that

$$a_n\bar{f}(x) = \bar{f}(a_n \cdot x) \xrightarrow{d} \bar{f}(a \cdot x).$$

At the same time, we have that $a_n \bar{f}(x) \xrightarrow{d} a\bar{f}(x)$. By the uniqueness of the limit, we can conclude that $\bar{f}(a \cdot x) = a\bar{f}(x)$, proving A-linearity. Finally, consider $x \in H$. There exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq H_e$ such that $x_n \xrightarrow{d_H} x$. In particular, $N(x_n) \xrightarrow{d} N(x)$. By (23) and since $\bar{f}$ is a uniform continuous extension of f and the topology induced by d is locally solid, we have that

$$\left|\bar{f}(x)\right| \xleftarrow{d} \left|\bar{f}(x_n)\right| = |f(x_n)| \leq cN(x_n) \xrightarrow{d} cN(x),$$

The last part of the statement follows since H is self-dual. □

Step 5. $f_n \xrightarrow{d_{H^\sim}} f$.

Proof of the Step. By (22), we have that

$$\frac{|f_n(x) - f_m(x)|}{N(x) + \frac{1}{k}e} \leq N(y_n - y_m) \qquad \forall k, m, n \in \mathbb{N}, \forall x \in H.$$

This yields that

$$d\left(0, \frac{|f_n(x) - f_m(x)|}{N(x) + \frac{1}{k}e}\right) \leq d_H(y_n, y_m) \qquad \forall k, m, n \in \mathbb{N}, \forall x \in H.$$

Consider $\varepsilon > 0$. Since $\{y_n\}_{n\in\mathbb{N}}$ is a d_H Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d\left(0, \frac{|f_n(x) - f_m(x)|}{N(x) + \frac{1}{k}e}\right) \leq \varepsilon \qquad \forall k \in \mathbb{N}, \forall m, n \geq n_\varepsilon, \forall x \in H.$$

By taking the limit in n, we have that

$$d\left(0, \frac{|f(x) - f_m(x)|}{N(x) + \frac{1}{k}e}\right) \leq \varepsilon \qquad \forall k \in \mathbb{N}, \forall m \geq n_\varepsilon, \forall x \in H.$$

Next, consider the sequence $\left\{\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e} \wedge e\right\}_{k\in\mathbb{N}}$. This sequence is bounded by e and increasing. Thus, $\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e} \wedge e \uparrow \sup_{k\in\mathbb{N}}\left(\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e} \wedge e\right) = \sup_{k\in\mathbb{N}}\left(\frac{|f(x)-f_m(x)|}{N(x)+\frac{1}{k}e}\right) \wedge e$. Since $\bar{\varphi}$ is order continuous, we can conclude that

$$\begin{aligned} d\left(0, \sup_{k\in\mathbb{N}}\left(\frac{|f(x) - f_m(x)|}{N(x) + \frac{1}{k}e}\right)\right) &= \bar{\varphi}\left(\sup_{k\in\mathbb{N}}\left(\frac{|f(x) - f_m(x)|}{N(x) + \frac{1}{k}e}\right) \wedge e\right) \\ &= \bar{\varphi}\left(\sup_{k\in\mathbb{N}}\left(\frac{|f(x) - f_m(x)|}{N(x) + \frac{1}{k}e} \wedge e\right)\right) \\ &= \sup_{k\in\mathbb{N}} \bar{\varphi}\left(\frac{|f(x) - f_m(x)|}{N(x) + \frac{1}{k}e} \wedge e\right) \leq \varepsilon \qquad \forall m \geq n_\varepsilon, \forall x \in H. \end{aligned}$$

By the proof of point 2 of Proposition 13, we also have that

$$N_*(f - f_m) = N_*(S^\sim(y) - S^\sim(y_m)) = N_*(S^\sim(y - y_m)) = \sup_{k\in\mathbb{N}} \frac{|\langle y - y_m, y - y_m\rangle_H|}{N(y - y_m) + \frac{1}{k}e}.$$

We can conclude that

$$\begin{aligned} d_{H^{\sim}}(f, f_m) &= d\left(0, N_*(f - f_m)\right) \\ &= d\left(0, \sup_{k \in \mathbb{N}} \left(\frac{|f(y - y_m) - f_m(y - y_m)|}{N(y - y_m) + \frac{1}{k}e}\right)\right) \leq \varepsilon \quad \forall m \geq n_\varepsilon, \end{aligned}$$

proving the statement. □

By point 2 of Proposition 13 and Steps 4 and 5, we have that

$$d_H(y, y_n) = d_{H^{\sim}}(f, f_n) \to 0,$$

proving completeness of H. ■

6 Examples

Consider a nonempty set Ω, a σ-algebra of subsets of Ω denoted by $\mathcal{F}$, a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, and a probability measure $P : \mathcal{F} \to [0, 1]$. Two $\mathcal{F}$-measurable random variables are defined to be equivalent if and only if they coincide almost surely. Define:

1. $A = \mathcal{L}^0(\mathcal{G})$, that is, A is the space (of equivalence classes) of real valued and $\mathcal{G}$-measurable functions;[15]

2. $b \geq a$ if and only if $b(\omega) \geq a(\omega)$ almost surely;

3. $e = 1_\Omega$, that is, e is the function that takes constant value 1;

4. It follows that $A_e = \mathcal{L}^\infty(\mathcal{G})$, that is, A_e is the space of all essentially bounded and $\mathcal{G}$-measurable functions;

5. $\bar{\varphi} : \mathcal{L}^\infty(\mathcal{G}) \to \mathbb{R}$ as
$$\bar{\varphi}(a) = \int a dP \qquad \forall a \in \mathcal{L}^\infty(\mathcal{G});$$

6. $d : \mathcal{L}^0(\mathcal{G}) \times \mathcal{L}^0(\mathcal{G}) \to \mathbb{R}$ as
$$d(a, b) = \bar{\varphi}(|b - a| \wedge e) = \int (|b - a| \wedge e)\, dP \qquad \forall a, b \in \mathcal{L}^0(\mathcal{G}).$$

   Note that the topology induced by d is the one of convergence in probability P.

It is immediate to verify that $\mathcal{L}^\infty(\mathcal{G})$ is an Arens algebra of $\mathcal{L}^\infty$ type and $A = \mathcal{L}^0(\mathcal{G})$ is an f-algebra of $\mathcal{L}^0$ type.

[15] As usual, we view the equivalence classes as functions. This convention will apply throughout the rest of the paper.

6.1 The module $\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$

We denote by $\mathcal{L}^0(\mathcal{F}) = \mathcal{L}^0(\Omega, \mathcal{F}, P)$ the space of real valued and $\mathcal{F}$-measurable functions. We call x, y, and z the elements of $\mathcal{L}^0(\mathcal{F})$. Given an $\mathcal{F}$-measurable function $x : \Omega \to \mathbb{R}$ such that $x \geq 0$, we denote by $\mathbb{E}(x||\mathcal{G})$ its conditional expected value with respect to P given $\mathcal{G}$ (see Loeve [27, Section 27]). Denote by

$$H = \mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P) = \left\{x \in \mathcal{L}^0(\mathcal{F}) : \mathbb{E}\left(x^2||\mathcal{G}\right) \in \mathcal{L}^0(\mathcal{G})\right\}$$
$$= \left\{x \in \mathcal{L}^0(\mathcal{F}) : \sqrt{\mathbb{E}(x^2||\mathcal{G})} \in \mathcal{L}^0(\mathcal{G})\right\}.$$

We endow H with two operations:

1. $+ : H \times H \to H$ which is the usual pointwise sum operation;

2. $\cdot : A \times H \to H$ such that $a \cdot x = ax$ where ax is the usual pointwise product.

The space $\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$ was introduced by Hansen and Richard [19]. Finally, we also define an inner product, namely, $\langle\ ,\ \rangle_H : H \times H \to \mathcal{L}^0(\mathcal{G})$ by

$$\langle x, y\rangle_H = \mathbb{E}(xy||\mathcal{G}) \qquad \forall x, y \in H.$$

Hansen and Richard [19, p. 592] show that $\langle\ ,\ \rangle_H$ is well defined. They also prove the next result.

Proposition 19 *$(H, +, \cdot, \langle\ ,\ \rangle_H)$ is a pre-Hilbert $\mathcal{L}^0(\mathcal{G})$-module.*

Note that $d_H : H \times H \to \mathbb{R}$

$$d_H(x, y) = \int \left(\sqrt{\mathbb{E}\left((x-y)^2||\mathcal{G}\right)} \wedge e\right) dP \qquad \forall x, y \in H.$$

Theorem 6 *H is self-dual.*

Proof. By Theorem 5, it is enough to check that H is d_H complete. This follows from [19, Theorem A.1]. ■

6.2 The module $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$

We denote by $\mathcal{L}^2(\mathcal{F}) = \mathcal{L}^2(\Omega, \mathcal{F}, P)$ the space of $\mathcal{F}$-measurable and square integrable functions. Denote by

$$H_e = \mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P) = \left\{x \in \mathcal{L}^0(\mathcal{F}) : \mathbb{E}\left(x^2||\mathcal{G}\right) \in \mathcal{L}^\infty(\mathcal{G})\right\}$$
$$= \left\{x \in \mathcal{L}^2(\mathcal{F}) : \sqrt{\mathbb{E}(x^2||\mathcal{G})} \in \mathcal{L}^\infty(\mathcal{G})\right\} \subseteq \mathcal{L}^2(\mathcal{F}).$$

Since $\mathcal{L}^{2,\infty}(\Omega,\mathcal{G},\mathcal{F},P) \subseteq \mathcal{L}^{2,0}(\Omega,\mathcal{G},\mathcal{F},P)$, we endow H_e with the two operations $+$ and $\cdot$ of Subsection 6.1. We also restrict $(f,g) \mapsto \mathbb{E}(fg||\mathcal{G})$ to H_e. By Proposition 18, it follows that $\mathcal{L}^{2,\infty}(\Omega,\mathcal{G},\mathcal{F},P)$ is a pre-Hilbert $\mathcal{L}^{\infty}(\mathcal{G})$-module.

Note that $\| \ \|_{H_e} : H_e \to \mathbb{R}$ is such that

$$\|x\|_{H_e} = \sqrt{\|\mathbb{E}(x^2||\mathcal{G})\|_{\mathcal{L}^{\infty}(\mathcal{G})}} \qquad \forall x \in H_e.$$

Similarly, we have that

$$\|x\|_m = \sqrt{\int \mathbb{E}(x^2||\mathcal{G})\, dP} = \sqrt{\int x^2 dP} = \|x\|_{\mathcal{L}^2(\mathcal{F})} \qquad \forall x \in H_e.$$

Theorem 7 *H_e is self-dual.*

Proof. By Theorem 3, we only need to show that B_{H_e} is $\| \ \|_m = \| \ \|_{\mathcal{L}^2(\mathcal{F})}$ complete. To this end, it is enough to show that B_{H_e} is $\| \ \|_{\mathcal{L}^2(\mathcal{F})}$ closed. Thus, consider $\{x_n\}_{n\in\mathbb{N}} \subseteq B_{H_e}$ such that $x_n \overset{\| \ \|_{\mathcal{L}^2(\mathcal{F})}}{\to} x \in \mathcal{L}^2(\mathcal{F})$. This implies that there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k} \overset{a.s}{\to} x$. By the conditional Fatou's lemma (see [13, p. 340]), we have that

$$\mathbb{E}\left(x^2||\mathcal{G}\right) = \mathbb{E}\left(\liminf_k x_{n_k}^2||\mathcal{G}\right) \le \liminf_k \mathbb{E}\left(x_{n_k}^2||\mathcal{G}\right) \le 1 < \infty,$$

proving that $x \in B_{H_e}$. ■

6.3 The module $\mathcal{L}^{2,\infty}(\Omega,\mathcal{G},\mathcal{F},P;E)$

Consider an infinite dimensional separable Hilbert space E (for example, $l^2(\mathbb{N})$).[16] Define $\langle \ , \ \rangle_E$ to be the inner product of E. Let us denote by d and d' generic elements of E. Since E is separable, it admits a countable orthonormal basis: $\{d_n\}_{n\in\mathbb{N}}$. Given a function, $x : \Omega \to E$ we say that x is weakly measurable if and only if the real valued function

$$\omega \mapsto \langle x(\omega), d\rangle_E$$

is $\mathcal{F}$-measurable for all $d \in E$. By [7, Theorems 34.2 and 34.4], if $x, y : \Omega \to E$ are weakly measurable, then

$$\omega \mapsto \langle x(\omega), y(\omega)\rangle_E = \sum_{n=1}^{\infty} \langle x(\omega), d_n\rangle_E \langle y(\omega), d_n\rangle_E$$

is real valued and $\mathcal{F}$-measurable. In particular, by Perseval's identity,

$$\omega \mapsto \langle x(\omega), x(\omega)\rangle_E = \sum_{n=1}^{\infty} |\langle x(\omega), d_n\rangle_E|^2 \qquad \forall \omega \in \Omega$$

[16] A similar analysis can be carried over when E is finite dimensional.

is real valued and $\mathcal{F}$-measurable. Given this observation, we denote by $\mathcal{L}^2_E(\mathcal{F}) = \mathcal{L}^2(\Omega, \mathcal{F}, P; E)$ the space

$$\mathcal{L}^2_E(\mathcal{F}) = \left\{ x \in E^\Omega : x \text{ is weakly measurable and } \int \langle x, x \rangle_E \, dP < \infty \right\}.$$

As before we identify the elements of $\mathcal{L}^2_E(\mathcal{F})$ whenever they coincide almost surely. At the same time, as usual, we view the equivalence classes as functions. Thus, $x \in \mathcal{L}^2_E(\mathcal{F})$ if and only if x is weakly measurable and $\omega \mapsto \langle x(\omega), x(\omega) \rangle_E^{\frac{1}{2}}$ belongs to $\mathcal{L}^2(\mathcal{F})$. Consider $x, y \in \mathcal{L}^2_E(\mathcal{F})$, we showed that $\omega \mapsto \langle x(\omega), y(\omega) \rangle_E$ is $\mathcal{F}$-measurable, we next show it is also integrable. Since E is an Hilbert space, observe that

$$|\langle x(\omega), y(\omega) \rangle_E| \leq \langle x(\omega), x(\omega) \rangle_E^{\frac{1}{2}} \langle y(\omega), y(\omega) \rangle_E^{\frac{1}{2}} \qquad \forall \omega \in \Omega. \tag{25}$$

Since $x, y \in \mathcal{L}^2_E(\mathcal{F})$, it follows that

$$\int |\langle x, y \rangle_E| \, dP \leq \int \langle x, x \rangle_E^{\frac{1}{2}} \langle y, y \rangle_E^{\frac{1}{2}} \, dP \leq \sqrt{\int \langle x, x \rangle_E \, dP \int \langle y, y \rangle_E \, dP} < \infty. \tag{26}$$

In this way, for each $x, y \in \mathcal{L}^2_E(\mathcal{F})$, we can define the $\mathcal{G}$-conditional expectation of $\omega \mapsto \langle x(\omega), y(\omega) \rangle_E$, that is, $\mathbb{E}(\langle x, y \rangle_E \| \mathcal{G})$. We thus define

$$H = \mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P; E) = \left\{ x \in \mathcal{L}^2_E(\mathcal{F}) : \sqrt{\mathbb{E}(\langle x, x \rangle_E \| \mathcal{G})} \in \mathcal{L}^\infty(\mathcal{G}) \right\}.$$

Define A to be the Arens algebra $\mathcal{L}^\infty(\mathcal{G}) = \mathcal{L}^\infty(\Omega, \mathcal{G}, P)$, we endow H with two operations:

1. $+ : H \times H \to H$ which is the usual pointwise sum operation, that is, $(x + y)(\omega) = x(\omega) + y(\omega)$ for all $\omega \in \Omega$.

2. $\cdot : A \times H \to H$ such that $(a \cdot x)(\omega) = a(\omega) x(\omega)$ for all $\omega \in \Omega$.[17]

Finally, we also define an inner product, namely, $\langle \ , \ \rangle_H : H \times H \to \mathcal{L}^\infty(\mathcal{G})$ by

$$\langle x, y \rangle_H = \mathbb{E}(\langle x, y \rangle_E \| \mathcal{G}) \qquad \forall (x, y) \in H \times H.$$

Proposition 20 *$(H, +, \cdot, \langle \ , \ \rangle_H)$ is a pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module.*

Proof. Consider $x, y \in H$. It is straightforward to verify that $x + y$ is weakly measurable. By (26), it follows that $x + y \in \mathcal{L}^2_E(\mathcal{F})$. Similarly, by (25) and the conditional Cauchy-Schwarz inequality, we have that

$$\begin{aligned} \mathbb{E}(\langle x + y, x + y \rangle_E \| \mathcal{G}) &= \mathbb{E}(\langle x, x \rangle_E \| \mathcal{G}) + 2\mathbb{E}(\langle x, y \rangle_E \| \mathcal{G}) + \mathbb{E}(\langle y, y \rangle_E \| \mathcal{G}) \\ &\leq \mathbb{E}(\langle x, x \rangle_E \| \mathcal{G}) + 2\mathbb{E}\left(\langle x, x \rangle_E^{\frac{1}{2}} \langle y, y \rangle_E^{\frac{1}{2}} \| \mathcal{G}\right) + \mathbb{E}(\langle y, y \rangle_E \| \mathcal{G}) \\ &\leq \mathbb{E}(\langle x, x \rangle_E \| \mathcal{G}) + 2\sqrt{\mathbb{E}(\langle x, x \rangle_E \| \mathcal{G}) \, \mathbb{E}(\langle y, y \rangle_E \| \mathcal{G})} + \mathbb{E}(\langle y, y \rangle_E \| \mathcal{G}). \end{aligned}$$

[17] Observe that for each $\omega \in \Omega$, $a(\omega) \in \mathbb{R}$ and $x(\omega) \in E$. Thus, $a(\omega) x(\omega)$ is the scalar product of $a(\omega)$ with $x(\omega)$.

This implies that H is closed under $+$. It is then easy to prove that $(H,+)$ is an abelian group. Similarly, if $a \in A$ and $x \in H$, then

$$\mathbb{E}\left(\langle a\cdot x, a\cdot x\rangle_E \|\mathcal{G}\right) = \mathbb{E}\left(a^2\langle x,x\rangle_E \|\mathcal{G}\right) = a^2\mathbb{E}\left(\langle x,x\rangle_E \|\mathcal{G}\right) \in \mathcal{L}^\infty(\mathcal{G}),$$

proving that H is closed under $\cdot$. Properties (1)-(4) of Definition 1 are then easily verified. This shows that H is an $\mathcal{L}^\infty(\mathcal{G})$-module.

Next, by (25) and the conditional Cauchy-Schwarz inequality, we have that $\langle\ ,\ \rangle_H$ is well defined, and:

5. Consider $x \in H$. Since $\langle x(\omega), x(\omega)\rangle_E \geq 0$ for all $\omega \in \Omega$, we have that $\langle x,x\rangle_H = \mathbb{E}\left(\langle x,x\rangle_E \|\mathcal{G}\right) \geq 0$. At the same time, we can conclude that

$$\langle x,x\rangle_H = 0 \iff \mathbb{E}\left(\langle x,x\rangle_E \|\mathcal{G}\right) = 0 \iff \langle x,x\rangle_E = 0 \iff x = 0.$$

6. Consider $x, y \in H$. We have that

$$\langle x,y\rangle_H = \mathbb{E}\left(\langle x,y\rangle_E \|\mathcal{G}\right) = \mathbb{E}\left(\langle y,x\rangle_E \|\mathcal{G}\right) = \langle y,x\rangle_H.$$

7. Consider $x, y, z \in H$. We have that

$$\begin{aligned}\langle x+y,z\rangle_H &= \mathbb{E}\left(\langle x+y,z\rangle_E \|\mathcal{G}\right) = \mathbb{E}\left(\langle x,z\rangle_E + \langle y,z\rangle_E \|\mathcal{G}\right)\\ &= \mathbb{E}\left(\langle x,z\rangle_E \|\mathcal{G}\right) + \mathbb{E}\left(\langle y,z\rangle_E \|\mathcal{G}\right) = \langle x,z\rangle_H + \langle y,z\rangle_H.\end{aligned}$$

8. Consider $a \in A$ and $x, y \in H$. We have that

$$\langle a\cdot x,y\rangle_H = \mathbb{E}\left(\langle a\cdot x,y\rangle_E \|\mathcal{G}\right) = \mathbb{E}\left(a\langle x,y\rangle_E \|\mathcal{G}\right) = a\mathbb{E}\left(\langle x,y\rangle_E \|\mathcal{G}\right) = a\langle x,y\rangle_H.$$

We can conclude that H is a pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module. ■

Note that for each $x \in H$

$$\|x\|_m = \sqrt{\bar{\varphi}\left(\langle x,x\rangle_H\right)} = \sqrt{\int \mathbb{E}\left(\langle x,x\rangle_E \|\mathcal{G}\right) dP} = \sqrt{\int \langle x,x\rangle_E\, dP} = \|x\|_{\mathcal{L}^2_E(\mathcal{F})}.$$

Theorem 8 *If $\mathcal{G}$ is generated by a finite partition where each atom has strict positive probability, then H is self-dual.*

Proof. Since $\mathcal{G}$ is generated by a finite partition, $\mathcal{L}^\infty(\mathcal{G})$ is finite dimensional. Moreover, we have that $H = \mathcal{L}^2_E(\mathcal{F})$. By Theorem 4, we only need to show that H is $\|\ \|_m$ complete. Since $H = \mathcal{L}^2_E(\mathcal{F})$ and $\|\ \|_m = \|\ \|_{\mathcal{L}^2_E(\mathcal{F})}$, the statement follows by Diestel and Uhl [12, p. 50]. ■

6.3.1 A Finance illustration

Let $\delta \in (0,1]$ and define $E = \{d \in \mathbb{R}^{\mathbb{N}} : \sum_{t=1}^{\infty} \delta^t d_t^2 < \infty\}$. The space E is an infinite dimensional separable Hilbert space where

$$\langle d, d' \rangle_E = \sum_{t=1}^{\infty} \delta^t d_t d'_t \qquad \forall d, d' \in E.$$

For each $t \in \mathbb{N}$, define also $\pi_t : E \to \mathbb{R}$ such that $\pi_t(d) = d_t$ for all $d \in E$.

Consider a filtration space $\left(\Omega, \{\mathcal{F}_t\}_{t\in\mathbb{N}}, \mathcal{F}, P\right)$ where $\mathcal{F} = \sigma(\cup_{t\in\mathbb{N}} \mathcal{F}_t)$. Call $\mathcal{G} = \mathcal{F}_1$. Define

$$H = \mathcal{L}^{2,\infty}\left(\Omega, \{\mathcal{F}_t\}_{t\in\mathbb{N}}, P; E\right) = \left\{x \in \mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P; E) : \pi_t \circ x \in \mathcal{L}^0(\mathcal{F}_t)\right\}.$$

In words, H is the space of processes in $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P; E)$ that are adapted to the filtration. Since H is a subset of $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P; E)$, we endow H with the same $+$ and $\cdot$ operations of Subsection 6.3. Similarly, we consider $(x, y) \mapsto \mathbb{E}(\langle x, y\rangle_E \| \mathcal{G})$ restricted to H. It is standard to show that H is a pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module. By Theorem 8, it easily follows that if $\mathcal{F}_1$ is generated by a finite partition where each atom has strict positive probability, then H is self-dual.

Remark 5 *In Finance, this module can be used to price infinite streams of payoffs.*

6.4 The module $M^{2,\infty}(\mathcal{G})$

Consider a filtration space $\left(\Omega, \{\mathcal{F}_t\}_{t\in\mathbb{N}}, \mathcal{F}, P\right)$ where $\mathcal{F} = \sigma(\cup_{t\in\mathbb{N}} \mathcal{F}_t)$. Assume that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. We denote by M^2 the space of $\mathcal{L}^2$ bounded martingales. Recall that $x \in M^2$ (see, e.g., [28, p. 209]) if and only if there exists a unique (terminal variable/final value) $x_\infty \in \mathcal{L}^2(\mathcal{F})$ such that

$$x_t = \mathbb{E}(x_\infty \| \mathcal{F}_t) \qquad \forall t \in \mathbb{N}.$$

Define $A = \mathcal{L}^\infty(\mathcal{G})$ and

$$H = M^{2,\infty}(\mathcal{G}) = \left\{x \in M^2 : \mathbb{E}\left(x_\infty^2 \| \mathcal{G}\right) \in \mathcal{L}^\infty(\mathcal{G})\right\}.$$

We endow H with two operations:

1. $+ : H \times H \to H$ which is the usual pointwise sum operation;

2. $\cdot : A \times H \to H$ such that $a \cdot x$ is defined by

$$(a \cdot x)_t = \mathbb{E}(a x_\infty \| \mathcal{F}_t) \qquad \forall t \in \mathbb{N}. \tag{27}$$

We also define $\langle\ ,\ \rangle_H : H \times H \to A$ as

$$\langle x, y\rangle_H = \mathbb{E}(x_\infty y_\infty \| \mathcal{G}) \qquad \forall x, y \in H.$$

Proposition 21 $(H, +, \cdot, \langle\ ,\ \rangle_H)$ *is a pre-Hilbert* $\mathcal{L}^\infty(\mathcal{G})$*-module.*

Proof. By the conditional Cauchy-Schwarz inequality, it is immediate to see that H is closed under $+$ and $(H, +)$ is an abelian group. The outer product $\cdot$ is also well defined. In fact, by [28, p. 209] and (27), $a \cdot x$ is an element of M^2. At the same time, since the terminal value of $a \cdot x$ is unique, $(a \cdot x)_\infty = ax_\infty \in \mathcal{L}^2(\mathcal{F})$ and $\mathbb{E}\left((a \cdot x)_\infty^2 \,||\mathcal{G}\right) = \mathbb{E}\left(a^2 x_\infty^2||\mathcal{G}\right) = a^2 \mathbb{E}\left(x_\infty^2||\mathcal{G}\right) \in \mathcal{L}^\infty(\mathcal{G})$. Observe also that:

1. Consider $a \in A$, $x, y \in H$, and $a \cdot (x + y)$. It follows that

$$\begin{aligned}(a \cdot (x+y))_t &= \mathbb{E}\left(a(x+y)_\infty ||\mathcal{F}_t\right) = \mathbb{E}\left(a(x_\infty + y_\infty)||\mathcal{F}_t\right) \\ &= \mathbb{E}\left(ax_\infty + ay_\infty||\mathcal{F}_t\right) = \mathbb{E}\left(ax_\infty||\mathcal{F}_t\right) + \mathbb{E}\left(ay_\infty||\mathcal{F}_t\right) \\ &= (a \cdot x)_t + (a \cdot y)_t \qquad \forall t \in \mathbb{N},\end{aligned}$$

   proving that $a \cdot (x + y) = a \cdot x + a \cdot y$.

2. Consider $a, b \in A$, $x \in H$, and $(a + b) \cdot x$. It follows that

$$\begin{aligned}((a+b) \cdot x)_t &= \mathbb{E}\left((a+b)x_\infty||\mathcal{F}_t\right) = \mathbb{E}\left(ax_\infty + bx_\infty||\mathcal{F}_t\right) \\ &= \mathbb{E}\left(ax_\infty||\mathcal{F}_t\right) + \mathbb{E}\left(bx_\infty||\mathcal{F}_t\right) = (a \cdot x)_t + (b \cdot x)_t \qquad \forall t \in \mathbb{N},\end{aligned}$$

   proving that $(a + b) \cdot x = a \cdot x + b \cdot x$.

3. Consider $a, b \in A$, $x \in H$, and $a \cdot (b \cdot x)$. It follows that

$$\begin{aligned}(a \cdot (b \cdot x))_t &= \mathbb{E}\left(a(b \cdot x)_\infty ||\mathcal{F}_t\right) = \mathbb{E}\left(a(bx_\infty)||\mathcal{F}_t\right) \\ &= \mathbb{E}\left((ab)x_\infty||\mathcal{F}_t\right) = ((ab) \cdot x)_t \qquad \forall t \in \mathbb{N},\end{aligned}$$

   proving that $a \cdot (b \cdot x) = (ab) \cdot x$.

4. Consider $e \in A$ and $x \in H$. It follows that

$$(e \cdot x)_t = \mathbb{E}\left(ex_\infty||\mathcal{F}_t\right) = \mathbb{E}\left(x_\infty||\mathcal{F}_t\right) = x_t \qquad \forall t \in \mathbb{N},$$

   proving that $e \cdot x = x$.

This shows that H is an $\mathcal{L}^\infty(\mathcal{G})$-module. Next, by the conditional Cauchy-Schwarz inequality, we have that $\langle\ ,\ \rangle_H$ is well defined, and:

5. Consider $x \in H$. We have that $\langle x, x \rangle_H = \mathbb{E}\left(x_\infty^2||\mathcal{G}\right) \geq 0$. At the same time, we can conclude that

$$\begin{aligned}\langle x, x \rangle_H = 0 &\iff \mathbb{E}\left(x_\infty^2||\mathcal{G}\right) = 0 \iff x_\infty = 0 \\ &\iff x_t = \mathbb{E}\left(x_\infty||\mathcal{F}_t\right) = 0 \quad \forall t \in \mathbb{N} \\ &\iff x = 0\end{aligned}$$

6. Consider $x, y \in H$. We have that

$$\langle x, y\rangle_H = \mathbb{E}\left(x_\infty y_\infty || \mathcal{G}\right) = \mathbb{E}\left(y_\infty x_\infty || \mathcal{G}\right) = \langle y, x\rangle_H .$$

7. Consider $x, y, z \in H$. We have that

$$\begin{aligned}\langle x + y, z\rangle_H &= \mathbb{E}\left((x+y)_\infty z_\infty || \mathcal{G}\right) = \mathbb{E}\left((x_\infty + y_\infty) z_\infty || \mathcal{G}\right) \\ &= \mathbb{E}\left(x_\infty z_\infty + y_\infty z_\infty || \mathcal{G}\right) = \langle x, z\rangle_H + \langle y, z\rangle_H .\end{aligned}$$

8. Consider $a \in A$ and $x, y \in H$. We have that

$$\langle a \cdot x, y\rangle_H = \mathbb{E}\left((a \cdot x)_\infty y_\infty || \mathcal{G}\right) = \mathbb{E}\left((a x_\infty) y_\infty || \mathcal{G}\right) = \mathbb{E}\left(a\left(x_\infty y_\infty\right) || \mathcal{G}\right) = a \langle x, y\rangle_H .$$

We can conclude that H is a pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module. ■

Note that $\| \ \|_H : H \to \mathbb{R}$ is such that

$$\|x\|_H = \sqrt{\|\langle x, x\rangle_H\|_{\mathcal{L}^\infty(\mathcal{G})}} = \sqrt{\|\mathbb{E}\left(x_\infty^2 || \mathcal{G}\right)\|_{\mathcal{L}^\infty(\mathcal{G})}} \qquad \forall x \in H.$$

Similarly, we have that

$$\|x\|_m = \sqrt{\bar{\varphi}\left(\langle x, x\rangle_H\right)} = \sqrt{\int \mathbb{E}\left(x_\infty^2 || \mathcal{G}\right) dP} = \|x_\infty\|_{\mathcal{L}^2(\mathcal{F})} \qquad \forall x \in H.$$

Theorem 9 *H is self-dual.*

Proof. By Theorem 3, we only need to show that B_H is $\| \ \|_m$ complete. Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq B_H$ which is $\| \ \|_m$ Cauchy. It follows that $\{(x_n)_\infty\}_{n\in\mathbb{N}}$ is a $\| \ \|_{\mathcal{L}^2(\mathcal{F})}$ Cauchy sequence and

$$\left|\mathbb{E}\left((x_n)_\infty^2 || \mathcal{G}\right)\right| \le e = 1_\Omega \qquad \forall n \in \mathbb{N}. \tag{28}$$

It follows that there exists an element $x_\infty \in \mathcal{L}^2(\mathcal{F})$ such that $(x_n)_\infty \overset{\| \ \|_{\mathcal{L}^2(\mathcal{F})}}{\to} x_\infty$. This implies that there exists also a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $(x_{n_k})_\infty \overset{a.s}{\to} x_\infty$. By (28) and the conditional Fatou's lemma (see [13, p. 340]), we have that

$$\mathbb{E}\left(x_\infty^2 || \mathcal{G}\right) = \mathbb{E}\left(\liminf_k (x_{n_k})_\infty^2 || \mathcal{G}\right) \le \liminf_k \mathbb{E}\left((x_{n_k})_\infty^2 || \mathcal{G}\right) \le 1 < \infty.$$

If we define x as the element in $M^{2,\infty}(\mathcal{G})$ such that

$$x_t = \mathbb{E}\left(x_\infty || \mathcal{F}_t\right) \qquad \forall t \in [0, \infty),$$

then $x_n \overset{\| \ \|_m}{\to} x$ as well as $x \in B_H$, proving the statement. ■

Remark 6 *An important example of* $M^{2,\infty}(\mathcal{G})$ *is when* $\mathcal{G}$ *is a stopping time* σ*-algebra, that is,*

$$\mathcal{G} = \{E \in \mathcal{F} : E \cap \{\tau = t\} \in \mathcal{F}_t \quad \forall t \in \mathbb{N}\}$$

where $\tau : \Omega \to \mathbb{N}$ *is a (finite valued) stopping time. An important bounded* A*-linear operator is the map (see [34, p. 391])* $f : H \to A$ *such that*

$$f(x) = x_\tau \qquad \forall x \in H.$$

Remark 7 *We could have defined* $M^{2,\infty}(\mathcal{G})$ *dealing with continuous time. In that case, we would have needed a filtration space* $\left(\Omega, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathcal{F}, P\right)$ *where* $\mathcal{F} = \sigma\left(\cup_{t\in[0,\infty)}\mathcal{F}_t\right)$*. With this specification,* M^2 *would have been the space of cadlag* $\mathcal{L}^2$ *bounded martingales. Our results would have remained the same.*

7 Appendix

Proof of Proposition 1. (i) implies (ii). Define $l = \bar{\varphi} \circ T^{-1} : C(K) \to \mathbb{R}$. Since $\bar{\varphi}$ and T^{-1} are both linear and continuous, $l \in C^*(K)$. Since $\bar{\varphi}$ and T^{-1} are positive and such that $\bar{\varphi}(e) = 1$ and $T^{-1}(1_K) = e$, l is positive and such that $l(1_K) = 1$. By [4, Theorem 14.14], this implies that there exists a (unique) finite regular probability measure m on the Borel σ-algebra of K such that

$$l(g) = \int_K g(\varphi)\, dm(\varphi) \qquad \forall g \in C(K).$$

It follows that

$$\bar{\varphi}(a) = l(T(a)) = \int_K T(a)(\varphi)\, dm(\varphi) = \int_K \langle a, \varphi\rangle\, dm(\varphi) \qquad \forall a \in A. \tag{29}$$

By [4, Theorem 12.14] and since m is regular (thus, tight), it follows that supp$m$ exists. By contradiction, assume that supp$m \neq K$. Thus, there exists $\hat{\varphi} \in K \backslash$supp$m$. By Urysohn's Lemma (see [4, Theorem 2.46]) and since K is compact and Hausdorff, there exists a function $0 \leq \hat{g} \in C(K)$ such that $\hat{g}(\hat{\varphi}) = 1$ and $\hat{g}(\varphi) = 0$ for all $\varphi \in$supp$m$. Since T is a lattice isomorphism, there exists $\hat{a} \in A$ such that $T(\hat{a}) = \hat{g} > 0$. It follows that $\hat{a} > 0$. Since $\bar{\varphi}$ is strictly positive, we can conclude that

$$0 = \int_{\text{supp}m} \langle \hat{a}, \varphi\rangle\, dm(\varphi) = \int_K \langle \hat{a}, \varphi\rangle\, dm(\varphi) = \bar{\varphi}(\hat{a}) > 0,$$

a contradiction.

(ii) implies (iii). Since it is immediate to see that K separates the points of A, the statement trivially follows.

(iii) implies (i). It is immediate to see that (1) defines a functional $\bar{\varphi} : A \to \mathbb{R}$ which is linear, positive, and $\| \, \|_A$ continuous. Since $\bar{\varphi}$ is positive and m a probability measure, we have that $\|\bar{\varphi}\|_{A^*} = \bar{\varphi}(e) = 1$. We just need to prove that $\bar{\varphi}$ is strictly positive. Consider $a \in A$ such that $a > 0$. Since supp$m$ separates the points of A, it follows that $\langle a, \hat{\varphi} \rangle > 0$ for some $\hat{\varphi} \in \text{supp} m \subseteq K$. This implies that there exists a nonempty open set V of K such that $\langle a, \varphi \rangle > 0$ for all $\varphi \in V$ and $m(V) > 0$. We can conclude that

$$\bar{\varphi}(a) = \int_K \langle a, \varphi \rangle \, dm(\varphi) \geq \int_V \langle a, \varphi \rangle \, dm(\varphi) > 0,$$

proving the implication. ■

Proof of the Claim in Proposition 10. Since for each $n \in \mathbb{N}$, $0 \leq \langle y, y \rangle_H^{\frac{1}{2}} \leq \langle y, y \rangle_H^{\frac{1}{2}} + \frac{1}{n} e = a_n$, it follows that

$$\langle y, y \rangle_H \leq a_n^2 \qquad \forall n \in \mathbb{N}.$$

Since $y_n = a_n^{-1} \cdot y$ for all $n \in \mathbb{N}$, this implies that $\left(a_n^{-1}\right)^2 \langle y, y \rangle_H \leq e$, that is,

$$\langle y_n, y_n \rangle_H \leq e \qquad \forall n \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$. We have that $\langle y_n, y \rangle_H = a_n^{-1} \langle y, y \rangle_H$. First, observe that $a_n \geq a_{n+1}$. This implies that $a_{n+1}^{-1} \geq a_n^{-1}$ and $\langle y_n, y \rangle_H = a_n^{-1} \langle y, y \rangle_H \uparrow$. Note that

$$\langle y, y \rangle_H \leq \langle y, y \rangle_H + \frac{1}{n} \langle y, y \rangle_H^{\frac{1}{2}} = \left(\langle y, y \rangle_H^{\frac{1}{2}} + \frac{1}{n} e \right) \langle y, y \rangle_H^{\frac{1}{2}} = a_n \langle y, y \rangle_H^{\frac{1}{2}}.$$

It follows that

$$\langle y_n, y \rangle_H = a_n^{-1} \langle y, y \rangle_H \leq \langle y, y \rangle_H^{\frac{1}{2}}.$$

Also note that

$$\langle y, y \rangle_H^{\frac{1}{2}} \leq \langle y, y \rangle_H^{\frac{1}{2}} + \frac{1}{n} e = a_n,$$

which yields

$$0 \leq a_n^{-1} \langle y, y \rangle_H^{\frac{1}{2}} \leq e.$$

Moreover, we can conclude that

$$0 \leq \langle y, y \rangle_H^{\frac{1}{2}} - a_n^{-1} \langle y, y \rangle_H = a_n^{-1} \left(a_n \langle y, y \rangle_H^{\frac{1}{2}} - \langle y, y \rangle_H \right) \leq \frac{1}{n} a_n^{-1} \langle y, y \rangle_H^{\frac{1}{2}}.$$

Since $\| \, \|_A$ is a lattice norm, it follows that

$$\left\| \langle y, y \rangle_H^{\frac{1}{2}} - \langle y_n, y \rangle_H \right\|_A \leq \frac{1}{n} \left\| a_n^{-1} \langle y, y \rangle_H^{\frac{1}{2}} \right\|_A \leq \frac{1}{n} \|e\|_A.$$

Since n was arbitrarily chosen, the statement follows. ■

Proof of Lemma 3. We first show, by steps, that N_* is a vector-valued norm.

Step 1. $N_*(f) = 0$ *if and only if* $f = 0$.

Proof of the Step. It is immediate to see that if $f = 0$, then $N_*(f) = 0$. Viceversa, note that if $N_*(f) = 0$, then

$$\begin{aligned}\sup_{x\in H}\left(\sup_{n\in\mathbb{N}}\frac{|f(x)|}{N(x)+\frac{1}{n}e}\right)\leq 0 \implies & \sup_{n\in\mathbb{N}}\frac{|f(x)|}{N(x)+\frac{1}{n}e}\leq 0 \qquad \forall x\in H,\\ \implies & \frac{|f(x)|}{N(x)+\frac{1}{n}e}\leq 0 \qquad \forall n\in\mathbb{N},\forall x\in H,\\ \implies & |f(x)|\leq 0 \qquad \forall x\in H,\\ \implies & f(x)=0 \qquad \forall x\in H,\end{aligned}$$

proving that $f = 0$. □

Step 2. $N_*(a\cdot f) = |a|\,N_*(f)$ *for all* $a\in A$ *and for all* $f\in H^\sim$.

Proof of the Step. Consider $a\in A$ and $f\in H^\sim$. It follows that

$$\begin{aligned}N_*(a\cdot f) &= \sup_{x\in H}\left(\sup_{n\in\mathbb{N}}\frac{|af(x)|}{N(x)+\frac{1}{n}e}\right) = \sup_{x\in H}\left(\left(\sup_{n\in\mathbb{N}}\frac{|a|\,|f(x)|}{N(x)+\frac{1}{n}e}\right)\right)\\ &= \sup_{x\in H}\left(|a|\left(\sup_{n\in\mathbb{N}}\frac{|f(x)|}{N(x)+\frac{1}{n}e}\right)\right) = |a|\sup_{x\in H}\left(\left(\sup_{n\in\mathbb{N}}\frac{|f(x)|}{N(x)+\frac{1}{n}e}\right)\right)\\ &= |a|\,N_*(f),\end{aligned}$$

proving the statement.[18] □

Step 3. $N_*(f+g)\leq N_*(f)+N_*(g)$ *for all* $f,g\in H^\sim$.

Proof of the Step. Consider $f,g\in H^\sim$. It follows that

$$\begin{aligned}N_*(f+g) &= \sup_{x\in H}\left(\sup_{n\in\mathbb{N}}\frac{|(f+g)(x)|}{N(x)+\frac{1}{n}e}\right) = \sup_{x\in H}\left(\left(\sup_{n\in\mathbb{N}}\frac{|f(x)+g(x)|}{N(x)+\frac{1}{n}e}\right)\right)\\ &\leq \sup_{x\in H}\left(\left(\sup_{n\in\mathbb{N}}\frac{|f(x)|+|g(x)|}{N(x)+\frac{1}{n}e}\right)\right)\\ &\leq \sup_{x\in H}\left(\left(\sup_{n\in\mathbb{N}}\frac{|f(x)|}{N(x)+\frac{1}{n}e}\right)+\left(\sup_{n\in\mathbb{N}}\frac{|g(x)|}{N(x)+\frac{1}{n}e}\right)\right)\\ &= \sup_{x\in H}\left(\sup_{n\in\mathbb{N}}\frac{|f(x)|}{N(x)+\frac{1}{n}e}\right)+\sup_{x\in H}\left(\sup_{n\in\mathbb{N}}\frac{|g(x)|}{N(x)+\frac{1}{n}e}\right) = N_*(f)+N_*(g),\end{aligned}$$

proving the statement. □

Steps 1-3 prove that N_* is a well defined vector-valued norm. Finally, observe that

$$\begin{aligned}d_{H^\sim}(f,g)=0 &\iff d(0,N_*(f-g))=0 \iff N_*(f-g)=0\\ &\iff f-g=0 \iff f=g.\end{aligned}$$

[18] In Step 2, we implicitly used the fact that if a nonempty subset B of A is bounded from above and $c\in A_+$, then $\sup cB = c\sup B$ (see [11, Footnote 26]).

It is immediate to see that $d_{H^\sim}(f,g) = d_{H^\sim}(g,f)$ for all $f,g \in H^\sim$ as well as $d_{H^\sim}(f+h,g+h) = d_{H^\sim}(f,g)$ for all $f,g,h \in H^\sim$. Finally, by definition of d and [6, Lemma 1.4] and since $N_*(f+g) \leq N_*(f) + N_*(g)$ for all $f,g \in H^\sim$, we can conclude that

$$
\begin{aligned}
d_{H^\sim}(f,g) &= d(0,N_*(f-g)) = \bar{\varphi}(N_*(f-g) \wedge e) = \bar{\varphi}(N_*((f-h)+(h-g)) \wedge e) \\
&\leq \bar{\varphi}((N_*(f-h)+N_*(h-g)) \wedge e) \leq \bar{\varphi}(N_*(f-h) \wedge e + N_*(h-g) \wedge e) \\
&= d_{H^\sim}(f,h) + d_{H^\sim}(h,g) \qquad \forall f,g,h \in H^\sim,
\end{aligned}
$$

proving the statement. ■

References

[1] Y. A. Abramovich, C. D. Aliprantis, and W. R. Zame, A Representation Theorem for Riesz Spaces and its Applications to Economics, *Economic Theory*, 5, 527–535, 1995.

[2] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Springer, New York, 2006.

[3] F. Albiac and N. J. Kalton, A Characterization of Real $\mathcal{C}(K)$-spaces, *The American Mathematical Monthly*, 114, 737–743, 2007.

[4] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, 3rd ed., Springer Verlag, Berlin, 2006.

[5] C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, 2nd ed., American Mathematical Society, Providence, 2003.

[6] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, 2nd ed., Springer, Dordrecht, 2006.

[7] C. D. Aliprantis and O. Burkinshaw, *Principles of Real Analysis*, 3rd ed., Academic Press, San Diego, 2008.

[8] R. Arens, Representations of *-Algebras, *Duke Mathematical Journal*, 14, 269–282, 1947.

[9] S. K. Berberian, *Introduction to Hilbert Spaces*, Oxford University Press, New York, 1961.

[10] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Verlag, New York, 2010.

[11] S. Cerreia-Vioglio, M. Kupper, F. Maccheroni, M. Marinacci, and N. Vogelpoth, Conditional L_p-spaces and the Duality of Modules over f-algebras, IGIER Working Paper 535, 2014.

[12] J. Diestel and J. J. Uhl, *Vector Measures*, American Mathematical Society, Providence, 1977.

[13] R. M. Dudley, *Real Analysis and Probability*, Cambridge University Press, Cambridge, 2002.

[14] D. Filipovic, M. Kupper, and N. Vogelpoth, Approaches to Conditional Risk, *SIAM Journal of Financial Mathematics*, 3, 402–432, 2012.

[15] M. Frank, Self-duality and C^*-reflexivity of Hilbert C^*-modules, *Zeitschrift für Analysis und ihre Anwendungen*, 9, 165–176, 1990.

[16] M. Frank, Hilbert C^*-Modules over Monotone Complete C^*-algebras, mimeo, 2010.

[17] T. Guo, Relations Between Some Basic Results Derived from Two Kinds of Topologies for a Random Locally Convex Module, *Journal of Functional Analysis*, 258, 3024–3047, 2010.

[18] T. Guo, On Some Basic Theorems of Continuous Module Homomorphisms between Random Normed Modules, *Journal of Function Spaces and Applications*, 1–13, 2013.

[19] L. P. Hansen and S. F. Richard, The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models, *Econometrica*, 55, 587–613, 1987.

[20] R. Haydon, M. Levy, and Y. Raynaud, *Randomly Normed Spaces*, Hermann, Paris, 1991.

[21] R. B. Holmes, *Geometric Functional Analysis and its Applications*, Springer, New York, 1975.

[22] C. B. Huijsmans and B. de Pagter, Ideal Theory in f-Algebras, *Transactions of the American Mathematical Society*, 269, 225–245, 1982.

[23] C. B. Huijsmans and B. de Pagter, Averaging Operators and Positive Contractive Projections, *Journal of Mathematical Analysis and Applications*, 113, 163–184, 1986.

[24] S. Kaijser, A Note on Dual Banach Spaces, *Mathematica Scandinavica*, 41, 325–330, 1977.

[25] I. Kaplansky, Modules Over Operator Algebras, *American Journal of Mathematics*, 75, 839–858, 1953.

[26] J. L. Kelley and R. L. Vaught, The Positive Cone in Banach Algebras, *Transactions of the American Mathematical Society*, 74, 44–55, 1953.

[27] M. Loeve, *Probability Theory II*, 4th ed., Springer Verlag, New York, 1978.

[28] P. Malliavin, *Integration and Probability*, Springer Verlag, New York, 1995.

[29] V. M. Manuilov and E. V. Troitsky, *Hilbert C^*-Modules*, American Mathematical Society, Providence, 2005.

[30] S-T. C. Moy, Characterizations of Conditional Expectation as a Transformation on Function Spaces, *Pacific Journal of Mathematics*, 4, 47–64, 1954.

[31] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, 1998.

[32] K. Ng, On a Theorem of Dixmier, *Mathematica Scandinavica*, 29, 279–280, 1971.

[33] W. L. Paschke, Inner Product Modules over B^*-algebras, *Transactions of the American Mathematical Society*, 182, 443–468, 1973.

[34] S. I. Resnick, *A Probability Path*, Birkhauser, Boston, 1999.

[35] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.

[36] Z. Semadeni, *Banach Spaces of Continuous Functions*, Polish Scientific Publishers, Warsaw, 1971.

[37] K. R. Stromberg, *Probability for Analysts*, Chapman and Hall, New York, 1994.