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Marco Maffezzoli, Tommaso Monacelli

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IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
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Deleverage and Financial Fragility*

Marco Maffezzoli[†]

Università Bocconi and IGER

Tommaso Monacelli[‡]

Università Bocconi, IGER, and CEPR

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Abstract

Severe economic downturns, characterized by deleverage, are typically preceded by phenomena of debt overhang. This evidence suggests that large recessions may not be the result of large shocks, but, rather, of the interaction between typical shocks and the current state of the economy. We study the transmission of deleverage shocks in a stochastic economy with heterogeneous agents and occasionally binding collateral constraints, where debt evolves endogenously. Our key finding is that the impact effect of a deleverage shock on aggregate output is a non-linear, *S*-shaped, function of the accumulated level of debt. At low levels of debt, deleverage is almost neutral, whereas its negative impact is largely magnified when debt reaches a critical threshold, i.e., when financial fragility is sufficiently high. At this threshold, the constraint on borrowing becomes endogenously binding. However, when the level of debt is already high before the shock hits, the borrowers are constrained both ex-ante and ex-post. In this case, the effect on output of a deleverage shock is the highest, but, at the margin, roughly insensitive to the level of debt. This non-linearity is much more pronounced for deleverage shocks than for productivity shocks. Our results cast doubts on the accuracy of gauging the effects of financial disturbances in linearized, certainty-equivalence environments.

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[†]Department of Economics, Università Bocconi, Via Sarfatti 25, 20136 Milano (MI), Italy. E-mail: marco.maffezzoli@unibocconi.it. Web page: <http://www.iger.unibocconi.it/maffezzoli>.

[‡]Department of Economics, Università Bocconi. E-mail: tommaso.monacelli@unibocconi.it. Web page: <http://www.iger.unibocconi.it/monacelli>.

1 Introduction

What are the implications on aggregate fluctuations of debt deleverage in the private sector? This question has received heightened attention in light of the large and persistent effects of the recent financial crisis. As a matter of principle, deleverage should be roughly neutral on economic activity. The reduction in consumption by the ultimate borrowers, who want to deleverage, should be compensated by a rise in consumption (or investment) by the ultimate savers. In other words, the implications of deleverage should be mainly redistributive. For this mechanism to be at work, the real interest rate must fall sufficiently, precisely to make the savers, in equilibrium, willing to provide less funds.¹

There are two potential forces that might render deleverage non-neutral. First, a fraction of the agents in the private sector could be borrowing constrained. Thus, the fall in consumption by the borrowers might be larger than the increase in consumption/investment by the savers, thereby leading to a fall in aggregate spending. Second, the real interest rate might not fall enough, due to the zero lower bound constraint. In this paper we focus on the first friction.

A recent literature in macroeconomics typically obtains deleverage as a result of a (negative) “financial” shock. The latter is usually modeled as a random perturbation to the agents’ ability to borrow, orthogonal to the value of the same agents’ collateral (see Jermann and Quadrini, 2012, Liu et al., 2013, Justiniano et al., 2013, among others).² In most cases, the interest is in tracing out the implications on aggregate activity of such a shock, conditional on some form of financial friction being *always* binding.

In this paper we focus on the following question: under what conditions can deleverage produce sizable recessions? Like previous contributions, we show that financial shocks can trigger deleverage-driven recessions only in the presence of financial frictions. But differently from many of the existing papers, we emphasize that a key ingredient to generate (potentially) large recessions lies in a non-linearity. This non-linearity stems from the interaction between two features: *financial fragility* and financial shocks.³ Financial fragility arises in an economy where the process of debt accumulation is endogenous and the constraint on borrowing is only occasionally binding. In such a context, the effects of financial shocks are state-dependent, i.e., they are a function of the previously accumulated level of debt. Hence a shock of typical size, that forces the agents to deleverage, can trigger radically different effects on output, asset prices and the real interest rate depending on the current level of indebtedness. In a state of financial fragility, i.e., of sufficiently high accumulated leverage, it is therefore not necessary to assume “large shocks” to engineer

¹See, e.g., Aiyagari (1994).

²In other recent contributions, such as Guerrieri and Lorenzoni (2011), Eggertsson and Krugman (2012), and Midrigan and Philippon (2013), the characterization of deleverage induced by financial shocks is in terms of exogenous variations in a previously fixed borrowing limit.

³Interchangeably the literature has referred to a concept of financial fragility also as “credit overhang” or “debt overhang”.

large recessions.

We first lay out a two-period deterministic economy with two agents (borrowers and savers), who are heterogeneous in their patience rates. This structure allows to study the equilibrium in the credit market in a simple manner. We analyze the effects of “deleverage shocks,” i.e., disturbances that affect the constrained agents’ ability to borrow independently of the value of their collateral. The simple model builds the intuition for three main insights. First, a negative deleverage shock is non-neutral (on agents’ consumption) only if the borrowing constraint is binding. Second, in response to such a shock, the real interest rate must fall, with the effect being magnified by a binding borrowing constraint. Third, nonlinearities play an important role. Shocks of typical size can produce non-neutral effects only if the agents are sufficiently close to the constraint. More importantly, depending on the initial conditions - e.g., a high or low level of debt - the same shock can or cannot trigger a binding borrowing constraint, producing sharply different effects on consumption and the real interest rate.

We then study whether these three results carry over to an infinite horizon dynamic stochastic economy (with both deleverage and productivity shocks) where the borrowing constraint is occasionally binding. Unlike the previous simplified setting, the important feature of the infinite horizon economy is twofold: private debt is an endogenous state, and the price of capital is an endogenous variable, whose equilibrium movements affect the agents’ ability to borrow.

The fact that the economy is stochastic has two major implications. First, shocks of normal size exert an effect depending on the current state of the economy (i.e., the agents’ decision rules are non-linear). Hence, whether or not the agents are highly or lowly leveraged makes a crucial difference to the ability of even small perturbations in financial and/or product markets to exert large real effects in the economy. Second, the anticipation of borrowing constraints becoming binding in the future affects decisions taken in the current period. This implies that the agents may decide to take precautionary measures and borrow up to the constraint only occasionally. Relative to a certainty equivalence environment, this precautionary motive might limit the likelihood of states of the world in which the agents are financially fragile and therefore the economy be prone to large contractions in economic activity.

We find that the response of output to a deleverage shock is (in absolute value) an increasing, non-linear, function of the underlying degree of financial fragility. We capture this non-linearity by describing the impact effect on output of a shock of given size as a function of the debt-to-output ratio (the endogenous state). It turns out that the non-linear effect of financial fragility is *S-shaped*, i.e., it features three regions, labeled, for simplicity, “low,” “intermediate,” and “high” debt respectively.

The intuition for the S-shaped non-linearity is as follows. If the state of the economy is such that agents feature a “low” debt-to-income ratio, a deleverage shock of typical size

produces a mild impact effect on output, triggers a small fall in asset prices and even a small rise in the real interest rate. In the low-debt region, the impact effect of a deleverage shock on output is roughly state-independent, i.e., it does not depend on the current level of debt. This is because, before the shock hits, the borrowers are unconstrained, and remain as such also after the shock has materialized. If, however, the underlying level of debt falls in the “intermediate” region, the impact effect on output of a shock of the same size is an increasing function of the level of debt. This happens because, in that region, the borrowers are not constrained *ex-ante*, but they become so *ex-post*, precisely as a result of the shock. Hence, the fall in private debt and asset prices reinforce each other, leading to a large contraction in output. To quantify the relative effect of being in the low vs. intermediate debt region, when the current debt-to-output ratio is, e.g., 0.2 (belonging to the “low” debt region), a one standard deviation exogenous drop in the loan-to-value ratio (the baseline size of our deleverage shock) triggers a fall in output (relative to its unconditional mean) of about 0.2 percent; however, when the current debt-to-output ratio is 0.3 (belonging to the “intermediate” region), a deleverage shock of the same size produces a contraction in output of about 2 percent, i.e., almost ten times larger. Finally, if the current level of debt falls in the “high” region, the borrowers are already constrained *ex-ante*, and remain such also *ex-post*. In this region, the impact effect on output of a deleverage shock is the largest, but the *marginal* effect of a higher level of debt is roughly zero. The combined shape of the three regions makes the impact response of output to a deleverage shock a S-shaped function of the current level of debt.

Our results are relevant on two distinct grounds. First, recent empirical contributions studying historical episodes of credit and asset price booms and busts, such as Jorda et al. 2013, have emphasized that large buildups in private borrowing before a recession predict the severity of the subsequent downturn. Our theoretical framework precisely rationalizes these facts, emphasizing the financial fragility side of the credit boom phase. Second, our work shows that gauging the effects of financial shocks in a certainty equivalence, linearized environment can be misleading about their quantitative relevance. In such a setup financial frictions are assumed to be always binding and the impact of a financial shock is necessarily the result of an average across states of the world which are potentially very different depending on the endogenously accumulated level of debt.

1.1 Relation to the literature

Our paper speaks, more generally, to the recent soaring literature on credit market imperfections in macroeconomics. Surveys on the subject include Gertler and Kiyotaki (2010), Christiano and Ikeda (2011), Quadrini (2011), and Brunnermeier et al. (2013). These papers illustrate the role of alternative microeconomic foundations of credit market im-

perfections.⁴ A recurrent feature of this vast literature, however, is that the equilibrium is typically analyzed conditional on financial frictions being always in place.⁵

In our environment such frictions arise endogenously. Our meaning of endogenous is threefold. First, in the presence of incomplete contracts, the agents' ability to borrow is limited by the (time-varying) value of their collateral. Second, asset prices are a key determinant of the fluctuations in collateral values. Third, and as hinted above, borrowing constraints become binding only occasionally. While the first two features (incomplete contracts and time-varying collateral values) are shared with other papers in the literature (as, e.g., Kiyotaki and Moore, 1997, Cooley et al., 2004, Iacoviello, 2008, Del Negro et al., 2011, Jermann and Quadrini, 2012, Liu et al., 2013, Justiniano et al., 2013, Christiano et al., 2014), the third feature is more uncommon. Most often, in fact, the existing literature in macroeconomics builds on certainty equivalence. This assumption rules out any role for precautionary saving motives and makes the agents willing to be always at the constraint.⁶

Noticeable recent exceptions to the “certainty equivalence approach” include Perri and Quadrini (2011), Brunnermeier and Sannikov (2014), and a set of papers in the so-called “sudden stop” literature, such as Mendoza (2010), Bianchi and Mendoza (2010), Bianchi (2011), and Benigno et al. (2013). Our work is mostly related to Mendoza (2010) and Bianchi (2011), who study under what conditions a competitive economy characterized by financial market frictions leads to over-borrowing. As in these papers, we allow collateral constraints to be endogenously binding, and the borrowing limit to depend upon the fluctuations in asset prices. But our work also differs in two main respects. First, and foremost, we study an economy with heterogeneous agents, whereas those papers study a representative-agent small open economy. This implies that the equilibrium real interest rate is endogenous in our framework, as opposed to being exogenously determined in the world capital markets. This feature is key because the coexistence of borrowers and savers in the economy highlights the conditions under which deleverage shocks, who induce some agents to save more but others to consume more, can lead to an aggregate contraction in

⁴For example, asymmetric information, limited enforcement as a manifestation of the incompleteness of contracts, and agency frictions, due to costly state verification.

⁵For instance, in models with limited enforcement, as those, e.g., originating from the seminal contribution of Kiyotaki and Moore (1997), the resulting constraint on borrowing is usually always binding. Alternatively, in models in the Bernanke and Gertler (1989) costly-state-verification tradition, the assumption that borrowers have finite lives insures that they never accumulate enough wealth so that their credit constraint becomes eventually non-binding. In other words, in all these contributions, the occurrence of financial frictions is not treated as an endogenous event.

⁶Another important work is by Guerrieri and Iacoviello (2013). They focus on the asymmetric effect of positive vs. negative housing preference shocks depending on whether or not the underlying borrowing constraint is binding. This non-linearity is accounted for via a piece-wise linear solution method which provides a local approximation as a function of a prespecified number of regimes (e.g., binding or not binding constraint). The difference with respect to our approach is that the piece-wise linear method in their work abstracts from the role of future uncertainty in shaping current decision rules, thereby ruling out precautionary saving motives in determining the accumulation of debt. This feature applies also to recent work by Jensen et al. (2015).

consumption and output. Second, in addition to traditional neutral productivity shocks, we study (and center the analysis on) the effects of deleverage shocks.

Our work is also related to Guerrieri and Lorenzoni (2011) - *GL* henceforth - who study the effect of a “credit crunch” on the consumption and saving behavior of a Bewley-type economy with a continuum of heterogeneous agents. Our work differs from *GL* in three main respects. First, we place financial frictions on the production side (i.e, affecting labor demand) and focus primarily on the role of nonlinearities, whereas *GL* place those frictions on the households’ side, emphasizing their effect on labor supply. Second, we fully characterize the stochastic equilibrium in response to deleverage (and productivity) shocks, whereas *GL* focus their analysis on the dynamic response of the system to a one-time permanent reduction in the (exogenous) borrowing limit. Third, we treat the price of the collateral asset (physical capital in our case) as endogenous. This element is crucial in shaping the effects of deleverage on the agents’ ability to borrow and on aggregate activity in general.

2 Intuition: a two-period model

In order to facilitate the intuition on a number of results, we begin by constructing a stylized, two-period, deterministic economy that allows a simple graphical representation of the equilibrium. Consider an endowment economy that lasts for two periods, and is populated by two types of agents which differ in terms of patience rate. Since in equilibrium the impatient agents will act as borrowers, and the patient agents as savers, we will henceforth label each group as *borrowers* and *savers*.⁷

Each representative agent starts out with zero savings. We assume that the impatient agent initially also holds a durable asset. Both agents maximize the intertemporal utility:

$$u(c_j) \equiv \log c_j + \beta_j \log c'_j, \quad j \in \{b, s\},$$

with $\beta_s > \beta_b$, and where a superscript denote the second period.

⁷The model can be thought of as a simplified version of classic equilibrium models with incomplete markets, such as Bewley (1980), Huggett (1993), and Aiyagari (1994). Although with a smaller degree of heterogeneity, our setup has the advantage of allowing to deal with aggregate uncertainty in a much simpler way. A key difference is that who borrows and lends is predetermined by preferences in our model whereas it is determined by idiosyncratic uncertainty in the Bewley-Aiyagari-Huggett setup. Recent examples of models along the lines of the current paper are Monacelli and Perotti (2008) and Eggertsson and Krugman (2012). Curdia and Woodford (2010) allow agents to differ in their impatience to consume, but (differently from our framework) limit the ability to borrow by assuming that agents can have access to financial markets (in the form of purchase of state contingent securities) only randomly.

Borrowers The representative borrower is subject to following intratemporal budget constraints:

$$c_b + qk = y_b + d_b, \quad (1)$$

$$c'_b + (1 + r)d_b = q'k + y'_b, \quad (2)$$

where d_b is borrowing, y_b is endowment/income, $1 + r$ is the gross real interest rate, and k is a durable asset which is in fixed supply. Notice that, in (2), $q'k$ denotes the resale value of the asset.

The borrower is also subject to the following collateral constraint:

$$d_b \leq \chi q'k. \quad (3)$$

Equation (3) states that current borrowing cannot exceed a fraction χ of the resell value of the durable asset. A constraint of this kind can be generally justified by the presence of limited commitment (see more below on this point). We will think of shifts in parameter χ as “deleverage shocks”⁸, i.e., exogenous variations in the agent’s ability to borrow that are independent of the future value of the collateral at the time of repayment.

The borrower’s intertemporal budget constraint can be obtained by combining (1) and (2):

$$c_b + \frac{c'_b}{1 + r} = y_b + \frac{y'_b}{1 + r} + \left(\frac{q'}{1 + r} - q \right) k. \quad (4)$$

The left hand side is the present value of consumption, while the right hand side is the present value of income, which includes the capital gain from holding the asset from period 1 to period 2.

Combining (1) with (3) yields:

$$c_b \leq y_b + (\chi q' - q) k. \quad (5)$$

The presence of the collateral constraint implies that current consumption cannot exceed the sum of current income and of a suitably defined expected return on the asset - the second term on right hand side of (5). Notice that such return is affected by the loan-to-value parameter χ . The intersection of (4) and (5) defines the budget set of the borrower.

The borrower’s problem implies the first order condition:

$$\frac{1}{c_b} = \frac{\beta_b (1 + r)}{c'_b} + \psi, \quad (6)$$

⁸The two period model is deterministic, therefore the expression “shock” is an abuse of terminology. We maintain the definition of “shock” to be consistent with the terminology in the fully stochastic model developed later.

where $\psi \geq 0$ is the multiplier on constraint (3). The above equation states that when the borrowing constraint is binding (i.e., $\psi > 0$), the marginal utility of current consumption (the left-hand side) exceeds the marginal utility of saving (the component $\beta_b(1+r)/c'_b$ in the right-hand side).

Savers The representative saver is subject to the period-by-period budget constraints:

$$c_s = y_s + d_s, \quad (7)$$

$$c'_s + (1+r)d_s = y'_s, \quad (8)$$

where d_s is savers' borrowing.

The saver's intertemporal budget constraint reads:

$$c_s + \frac{c'_s}{1+r} = y_s + \frac{y'_s}{1+r}. \quad (9)$$

The saver's first order (necessary) condition is the standard Euler equation:

$$\frac{1}{c_s} = \frac{\beta_s(1+r)}{c'_s}. \quad (10)$$

Equilibrium Equilibrium in the credit market requires:

$$d_b = -d_s. \quad (11)$$

When constraint (3) is not binding ($\psi = 0$), combining (4), (6), and the credit market equilibrium condition (11) yields the following equilibrium negative relationship between the real interest rate and the amount of debt:

$$1+r = \frac{q'k + y'_b}{d(1+\beta_b) + \beta_b(y_b - qk)}. \quad (12)$$

Equation 12 describes the *demand* schedule for debt in the economy. In Figure 1 we label it as *DB* curve. Notice that in the case of binding borrowing constraint, the relationship becomes vertical:

$$d_b = d = \chi q'k. \quad (13)$$

Combining (9) and (10), and using (11), we obtain an upward sloping schedule describing the equilibrium *supply* of debt (that we label *DS*):

$$1+r = -\frac{y'_s}{d(1+\beta_s) - \beta_s y_s}. \quad (14)$$

The initial unconstrained equilibrium in the credit market is represented at point A in the top panel of Figure 1. The bottom panel represents the equilibrium allocation of consumption for the borrower in the two periods. Notice that the borrower's budget set has a kink at the point $y_b + (\chi q' - q)k$, which corresponds to the right hand side of (5). Point A in the bottom panel represents the consumption choice of the borrower when the borrowing constraint is *not* binding.

Equilibrium real interest rate When the constraint is not binding, combining (12) and (14) yields the following expression for the equilibrium real interest rate:

$$1 + r = \frac{(1 + \beta_s)(q'k + y'_b) + (1 + \beta_b)y'_s}{y_s\beta_s(1 + \beta_b) + \beta_b(1 + \beta_s)(y_b - qk)}. \quad (15)$$

The key implication of 15 is that credit shocks (variations in χ) do not exert any effect on the equilibrium real interest rate when the constraint is not binding.

Conversely, when the borrowing constraint is binding, the expression for the equilibrium real interest rate can be obtained by combining (13) and (14), which yield:

$$1 + r = \frac{y'_s}{y_s\beta_s - (1 + \beta_s)\chi q'k}. \quad (16)$$

Expression (16) shows that there exists a positive relationship between variations in the credit constraint parameter χ and the real interest rate. Hence, a fall in χ , possibly ensuing from a credit contraction, determines a fall in the equilibrium real interest rate.

Effect of a deleverage shock Suppose the impatient agent (the borrower) is hit by a negative deleverage shock, in the form of a fall in χ , from χ_{high} to χ_{low} , with $\chi_{low} < \chi_{high}$. In principle we could have two scenarios, depending on whether the initial unconstrained equilibrium is far or close to the kink point. The first scenario is illustrated in Figure 2. If the initial equilibrium is far from the kink point (or, equivalently, the shock is sufficiently small), the same kink point shifts to the left, but the final equilibrium remains unaffected. The reason is that the shock is not sufficient to make the borrowing constraint bind, so that the equilibrium continues to be the one described in point A , with the level of debt and the real interest rate being unaffected.

The second scenario is depicted in Figure 3. In this case the initial unconstrained equilibrium is sufficiently close to the kink point (or, equivalently, the shock is sufficiently large), so that after the shock the constraint becomes binding. In the top panel, the kink point of the DB schedule shifts to the left, and the new equilibrium is at point B , with a lower real interest rate and lower debt. The reason for the fall in the real interest rate is simple. The negative deleverage shock induces a fall in the demand for debt, which generates - *ceteris paribus* - an excess supply of debt. The real interest rate must therefore

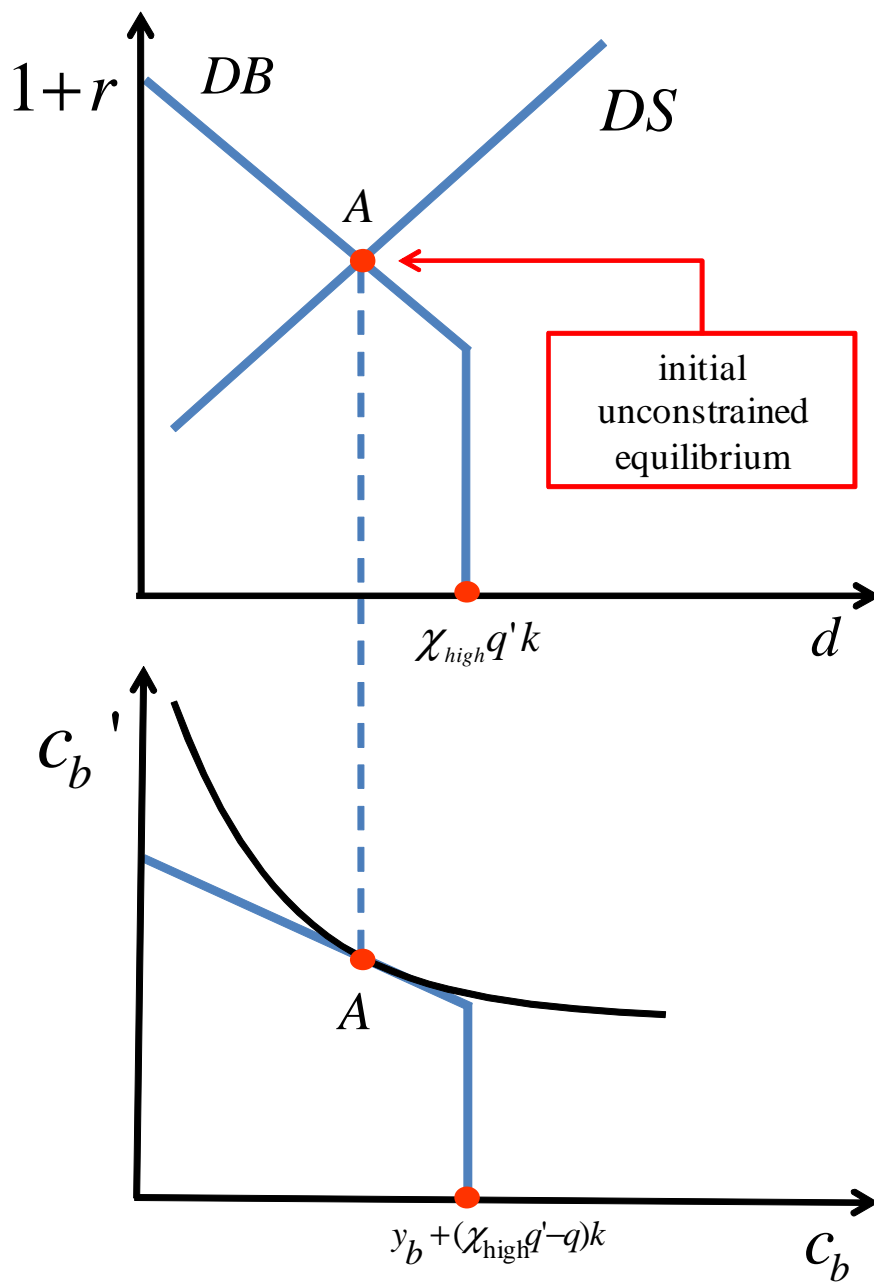


Figure 1: Credit market equilibrium (top) and borrower's consumption allocation (bottom): starting from unconstrained equilibrium.

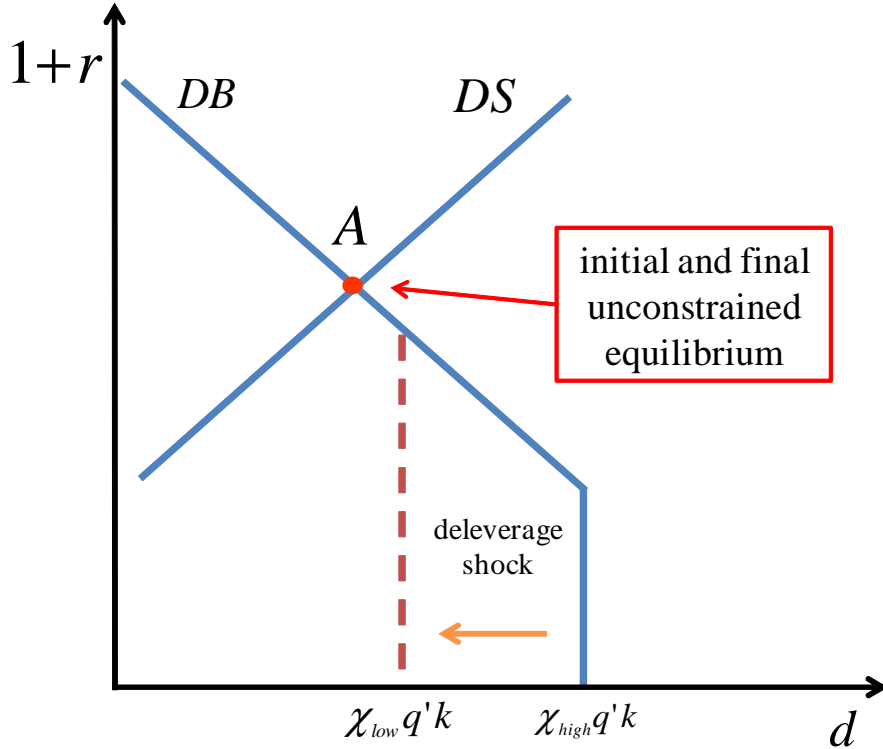


Figure 2: Effect of a deleverage shock ($\chi_{low} < \chi_{high}$) starting from an unconstrained equilibrium far from the constraint.

drop in order to re-establish the equilibrium in the credit market. Put differently, the fall in the real interest rate is the price mechanism that induces the patient agents to accept to save less (to meet the lower demand for debt by the borrowers) and, correspondingly, increase their consumption.

The bottom panel of Figure 3 illustrates the implications for borrowers' consumption. There are two effects. First, the kink point of the budget set shifts to the left. Second, since the equilibrium real interest rate has fallen, the budget set becomes flatter in its downward sloping section. Formally, this corresponds to a fall in the slope in equation (4). The final equilibrium will be at point B , with the agent's current consumption dropping by an amount proportional to the variation in χ . Clearly, in this scenario, credit shocks do exert a real effect on agents' consumption.

2.1 Low vs. high initial debt

Recent (and different) empirical studies have highlighted the close connection between the initial level of debt (either at the household or at the country level) and subsequent economic outcomes. Jorda et al. 2013 emphasize, looking at a large sample of historical episodes and across different countries, that large buildups in private borrowing (a "credit overhang") before a recession predict the severity of the subsequent downturn. Mian et al. (2013) show that the drop in US households' spending after the decline in house prices

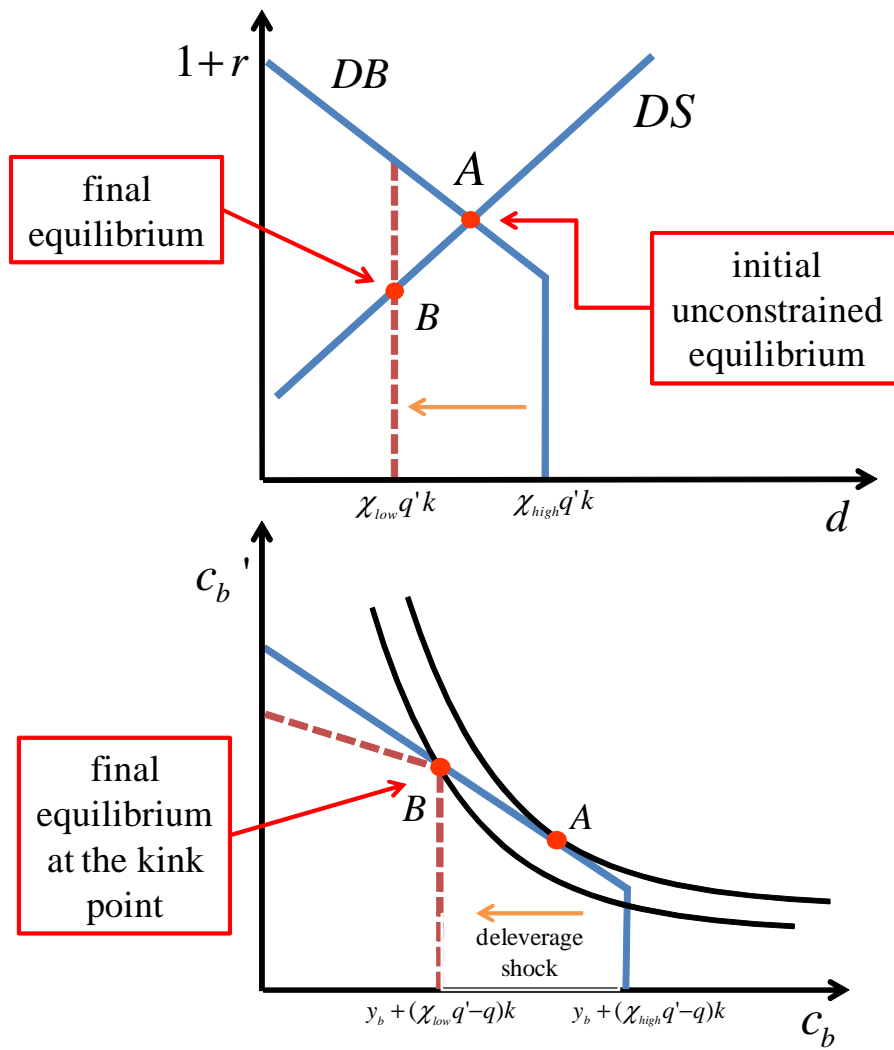


Figure 3: Effect of a deleverage shock ($\chi_{low} < \chi_{high}$) starting from an unconstrained equilibrium close to the constraint.

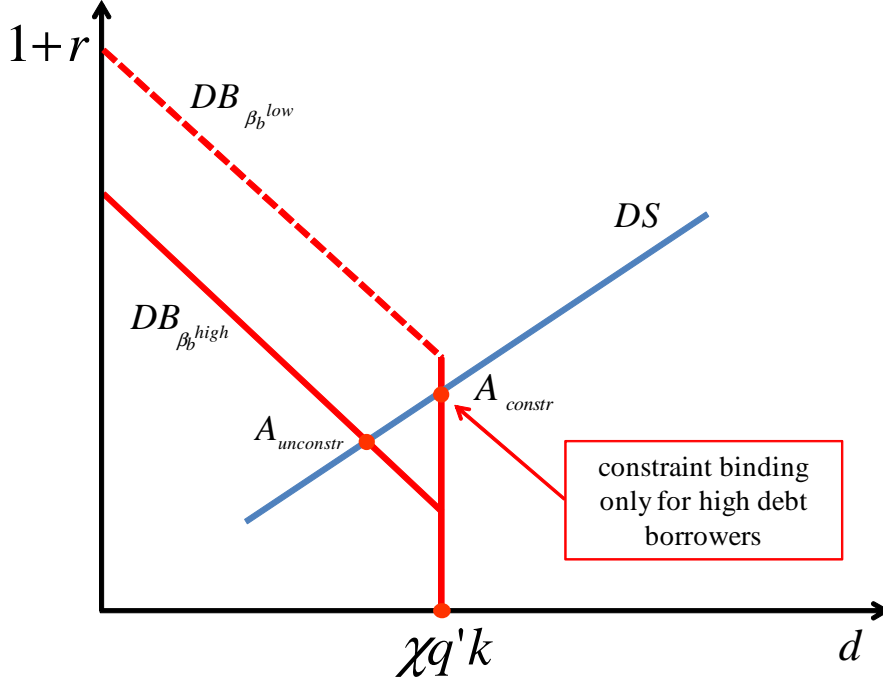


Figure 4: Different equilibria depending on preference parameter $\beta_b^{low} < \beta_b^{high}$. The borrowing constraint is initially binding only for the β_b^{low} types.

was significantly larger in those *counties* in which households featured higher debt levels at the peak of the housing boom. Mian et al. (2013) specifically emphasize the interaction between the fall in house prices and the existing level of debt as the key factor generating a cross-*county* variation in the response of consumption. More recently, International Monetary Fund (2012) has shown a similar cross-sectional pattern across *OECD* countries. During the last recession, aggregate consumption (both in durables and non-durables) has fallen more in those countries where the drop in housing prices interacted with high levels of initial households' debt (at the peak of the housing boom).

Our model provides a simple illustration of the role of the *initial* level of debt as a key factor in generating a differential response of consumption to financial disturbances. Suppose the economy is populated by two types of borrowers, with low and high patience rate, denoted by β_b^{low} and β_b^{high} respectively, with $\beta_b^{low} < \beta_b^{high} < \beta_s$. This difference in patience rates can capture heterogeneity at the individual, or at the county or country level.

Figure 4 shows two types of *DB* schedule, depending on the assumed value of β_b . The *DB* schedule for borrowers of the *low* type is shifted upward relative to the *DB* schedule for borrowers of the *high* type. As a result we can have two equilibria, depending on whether or not the borrowing constraint is binding. The *DB* schedule for β_b^{high} borrowers crosses the *DS* schedule at point $A_{unconstr}$, where the borrowing constraint is not binding, while the *DB* schedule for β_b^{low} borrowers crosses the *DS* schedule at A_{constr} , where the equilibrium level of debt is higher and the borrowing constraint is binding. Intuitively,

agents of the β_b^{low} type are relatively more impatient, therefore accumulate more debt and find themselves up against the constraint.

Next, consider a negative deleverage shock in the form of a fall in parameter χ , as depicted in Figure 5. Depending on the initial level of debt the same shock produces starkly different effects on consumption. For agents starting from the unconstrained equilibrium (point $A_{unconstr}$) the shock is neutral, whereas for those that are constrained in the initial equilibrium (point A_{constr}) the shock produces a contraction in consumption.

Summary of the results in the simplified model. To summarize, the prototype economy teaches us that the presence of financial imperfections is only a necessary condition for deleverage shocks to exert real effects on the economy. For those effects to take place one also needs that borrowing frictions become binding endogenously. There are two ways in which borrowing frictions can become binding: first, by having sufficiently large shocks; second, by having that typical shocks generate non-linear effects depending on the state of the economy (i.e., the underlying level of debt). We turn to these points in the following sections.

3 A stochastic economy with occasionally binding constraints

The analysis of the simple model misses at least four main features. First, while the effects of a deleverage shock depend on the current level of debt, the latter is assumed to be “high” or “low” exogenously. We wish therefore to model debt as an *endogenous state* variable, so that the effect of shocks of any given size will possibly depend on the agents’ degree of “financial fragility.”

Second, the price of the durable asset, q , is assumed to be given. When a deleverage shock hits, the agent is willing to reduce her holdings of the durable asset, trying to smooth the effects on consumption. This, however, exerts a downward pressure on the price of the asset, further impairing the agent’s ability to borrow, and in turn depressing the price of the asset even further. This vicious spiral is the “debt-deflation” channel that has been widely emphasized in the recent literature.⁹

Third, although the real interest rate is endogenous, the model is too simplistic to allow to quantify the magnitude of the required fall in the real interest rate in response to situations of financial distress. One goal of our analysis, in fact, is to develop a theory of the natural real interest rate in the presence of financial imperfections, and to quantify its equilibrium movements in response to financial (and other types of) shocks. This is a key feature, for it is precisely equilibrium movements in the real interest rate that generate

⁹See Mendoza (2010) for more details.

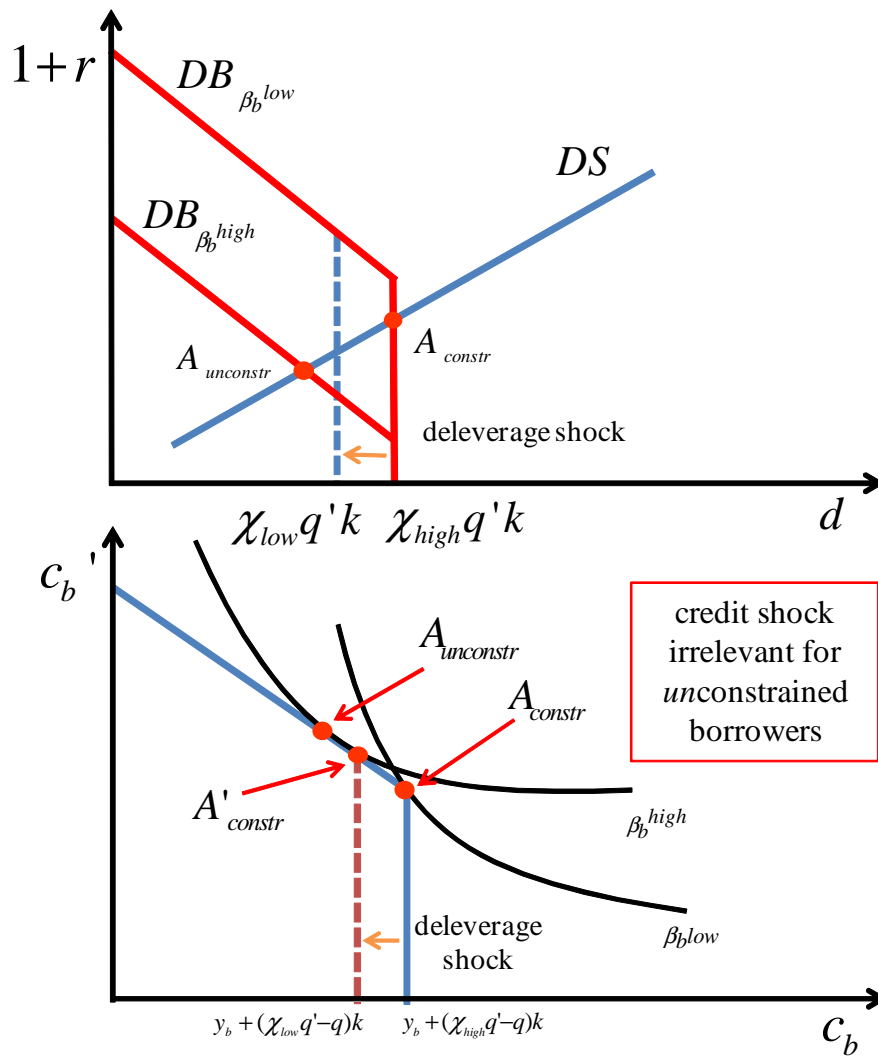


Figure 5: Effect of a deleverage shock starting from different initial equilibria: high vs. low debt.

the logical possibility of an aggregate neutrality of deleverage. In other words, when the borrowers decide to deleverage, the required fall in the real interest rate will induce the savers to consume more, potentially neutralizing the effects on aggregate spending.

Fourth, the two-period model is deterministic. In a stochastic economy, however, a precautionary saving motive - whereby agents anticipate that the borrowing constraint might become binding in the future- affects the agents' current consumption-saving decisions, even in states of the world in which the constraint is not binding.

In the following we describe a relatively standard infinite horizon dynamic economy with imperfect credit markets, in which both the price of the asset and the real interest rate are determined in equilibrium. There are two types of agents. The impatient borrowers consume, hire labor, and borrow in order to purchase a physical asset (capital or land), which is in fixed supply and is required for production (their source of income). Hence they can be likened to entrepreneurs. The savers are standard intertemporal maximizers with a relatively lower impatience rate.

Essentially ours is a Kiyotaki and Moore (1997, KM) type economy. The difference with respect to *KM* is threefold. First, the borrowing constraint is only occasionally binding, whereas it is always binding in *KM*. Second, the real interest rate is endogenous and time-varying, whereas it is constant in *KM*, due to the assumption of linear utility. Third, the economy is stochastic, whereas it is deterministic in *KM*.¹⁰

3.1 Borrowers/Entrepreneurs

A typical *borrower* maximizes the following intertemporal utility function:

$$\mathcal{U}_{b,t} \equiv \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta_b^t U(c_{b,t}) \right],$$

subject to an intertemporal flow of funds constraint:

$$c_{b,t} + q_t (k_{b,t+1} - k_{b,t}) + W_t n_{b,t} + \frac{d_{b,t+1}}{R_t} = d_{b,t} + y_{b,t}, \quad (17)$$

where $c_{b,t}$ is consumption, $k_{b,t}$ represents individual holdings of “land” at the beginning of period t , i.e. holdings of a non-reproducible asset available in aggregate fixed supply, W_t is the real wage paid to households, $n_{b,t}$ is the amount of labor demanded by the entrepreneur, $d_{b,t}$ is the (negative) holdings of risk-less discount bonds that pay one unit of consumption at the beginning of period t , R_t is the gross real interest rate on those bonds, and $y_{b,t}$ is output of a homogeneous final good.

¹⁰See Liu et al. (2013) for the estimation of a model similar to ours. In that work, however, the collateral constraint is assumed to be always binding in the neighborhood of the steady state, and the analysis is limited to a log-linearized (certainty equivalence) solution. A similar property applies also to Iacoviello (2005).

Output is produced via an aggregate production function that combines the services of land and labor:

$$y_{b,t} = F(z_t, k_{b,t}, n_{b,t}), \quad (18)$$

where z_t is a productivity factor, which evolves exogenously. In particular, we assume that the log of z_t follows a stationary $AR(1)$ stochastic process:

$$\ln z_{t+1} = (1 - \rho_Z) \ln \bar{z} + \rho_Z \ln z_t + \epsilon_{z,t+1}, \quad (19)$$

where $\rho_z \in [0, 1)$ and $\epsilon_{z,t} \sim N(0, \sigma_Z^2)$.

The entrepreneur is also subject to the following borrowing constraint:

$$-\frac{d_{b,t+1}}{R_t} \leq \chi_t \mathbb{E}_t(q_{t+1} k_{b,t+1}), \quad (20)$$

where χ_t evolves according to a stationary $AR(1)$ stochastic process:

$$\chi_{t+1} = (1 - \rho_\chi) \bar{\chi} + \rho_\chi \chi_t + \epsilon_{\chi,t+1}, \quad (21)$$

where $\rho_\chi \in [0, 1)$ and $\epsilon_{\chi,t} \sim N(0, \sigma_{\chi,t}^2)$.

Equation (20) states that new borrowing at time t cannot exceed a (stochastic) fraction χ_t of the expected future value of the capital holdings. As in our prototype two-period economy, this constraint can be justified on the basis of limited commitment (as in Kiyotaki and Moore, 1997). If the debtor defaults on his/her obligations, it is assumed that, due to costly contract commitment, the lender can seize only up to a fraction χ_t of the future value of the collateral assets. Notice that the borrowing limit depends on the *expected* future market value of the collateral asset (capital), i.e., at the value of capital at the time when repayment is expected. This differs, e.g., from the specification in other papers in the literature, such as Mendoza (2010) and Bianchi and Mendoza (2010), where the borrowing limit is assumed to depend on current market prices.

Henceforth we will refer to the process $\{\chi_t\}$ as *maximum* leverage, and to innovations to this process as (de)leverage shocks. We will interpret stochastic movements in χ_t as shocks to the ability to borrow which are orthogonal to the expected resale value of the asset. In short, random movements in χ_t capture shocks to the maximum leverage. In Geanakoplos (2010), equilibrium “leverage cycles” are the key factor leading to possibly pronounced fluctuations. Our analysis, in the spirit of *KM*, does not endogenize maximum leverage, but makes it exogenously time-varying, emphasizing the aggregate implications of (de)leverage shocks.

To better understand the role of an occasionally binding collateral constraint, let’s define leverage \mathcal{L}_t as:

$$\frac{-d_{b,t+1}/R_t}{\mathbb{E}_t(q_{t+1} k_{b,t+1})} \equiv \mathcal{L}_t \leq \chi_t.$$

Thus, in our framework, we have:

$$\mathcal{L}_t \leq \chi_t. \quad (22)$$

On the other hand, in models a la Kiyotaki and Moore (1997), \mathcal{L}_t is always constant, i.e.,:

$$\mathcal{L}_t = \chi = \bar{\chi}, \quad \forall t, \quad (23)$$

whereas in models a la Jermann and Quadrini (2012), and Liu et al. (2013), \mathcal{L}_t is exogenously time varying, but condition (22) is assumed to be always holding with equality:

$$\mathcal{L}_t = \chi_t \quad \forall t.$$

3.2 Savers/Households

A second category of agents, called savers, act as standard permanent income agents. They consume and provide labor supply, and will be the ones providing the supply of savings in the equilibrium (from here the subscript s). A typical saver maximizes the following intertemporal utility function:

$$U_{s,t} = \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta_s^t U(c_{s,t}, n_{s,t}) \right], \quad (24)$$

subject to the flow of funds constraint:

$$c_{s,t} + \frac{d_{s,t+1}}{R_t} = d_{s,t} + W_t n_{s,t}. \quad (25)$$

where $n_{s,t}$ denotes labor supply by the savers.

3.3 Equilibrium

In equilibrium, the market for physical capital, private debt and labor must clear. This requires the following three conditions to hold:

$$k_{b,t+1} = \bar{K}, \quad (26)$$

$$d_{b,t+1} + d_{s,t+1} = 0, \quad (27)$$

$$n_{b,t} = n_{s,t}. \quad (28)$$

The full set of equilibrium conditions is reported in Appendix A.1.

3.4 Calibration and solution procedure

We employ the following functional forms for the utility functions of the two agents (we drop time indexes for simplicity):

$$U(c_j, n_j) = \frac{c_j^{1-\mu} - 1}{1-\mu} - \frac{\varphi_j}{1+v} n_j^{1+v} \quad j = b, s$$

where $\varphi_b = 0$ and $\varphi_s > 0$. Notice that the savers feature an elastic labor supply, whereas the borrowers do not supply labor. Following Bianchi and Mendoza (2010), we set $\beta_s = 0.96$, $\beta_b = 0.95$, and $\mu = 2$. Furthermore, we set the inverse Frisch elasticity $v = 1$, and set the preference parameter $\varphi_s = 28.4$. The latter value implies a (savers') labor supply equal to 0.33 on average in the long-run.

The production function is of Cobb-Douglas form, with decreasing returns to scale:

$$F(z, k_b, n_b) = z k_b^{\alpha_k} n_b^{\alpha_n}, \quad (29)$$

where $\alpha_k > 0$, $\alpha_n > 0$, and $\alpha_k + \alpha_n < 1$. Following again Bianchi and Mendoza (2010), we set $\alpha_k = 0.05$ and $\alpha_n = 0.64$. Notice that α_k represents the share of fixed assets in *GDP*, and not the standard share of capital income in *GDP*.

Shock processes Following Cooley and Prescott (1995), we set $\rho_z = 0.95$ and $\sigma_z = 0.007$, while \bar{z} is selected in order to guarantee that $\mathbb{E}(z_t) = 1$. Hence, productivity shocks will be highly persistent but transitory, and neutral in the long run. The calibration of the deleverage shock process is more problematic. Available empirical evidence (e.g., Jermann and Quadrini, 2012, Liu et al., 2013) suggests that credit shocks are very persistent, and more volatile than productivity shocks. The unconditional expectation of χ is set to $\bar{\chi} = 0.3$, while $\rho_\chi = 0.95$ and $\sigma_\chi = 0.02$. Hence, our leverage shocks are as persistent as technology shocks, but significantly more volatile, by an order of magnitude similar to the estimates reported in the Bayesian exercises of Jermann and Quadrini (2012) and Liu et al. (2013).

The two autoregressive processes are approximated by discrete Markov chains using Tauchen's method. We use respectively 5 and 9 nodes to approximate the processes for z_t and χ_t . This allows us to refer to low, average and high values for z and χ respectively. Note that the two chains can be aggregated into a single one, that jointly represents the two underlying stochastic processes.

Solution procedure Thanks to the limited number of endogenous state variables involved, we employ a global non-linear solution method, taking the role of uncertainty fully into account. This has a few relevant implications. First, it allows risk aversion and prudence to have an explicit role in shaping the policy functions, introducing precaution-

ary saving motives that are absent under certainty equivalence. Second, precautionary savings imply that the borrowing constraint is not necessarily binding at the “risky” steady state,¹¹ so that the constraint becomes truly “occasionally binding”. The latter feature differentiates our work from other contributions that allow for “occasionally non-binding” constraints under a certainty equivalence framework, such as Guerrieri and Iacoviello (2013) and Jensen et al. (2015), where the constraint is always binding in the steady state.

Our approach can be summarized as follows. We build a grid of 1,000 values for debt $d_b = -d_s$, uniformly distributed over the $[0, 0.2]$ interval, and choose an initial guess for the values of R on the grid. Then, given our current guess for the values of R , we solve both the borrower and the saver’s problems using fixed point iteration on the Euler equations. Linear interpolation is used to approximate the future policy functions and prices. To solve the borrower’s problem, we follow a two-stage approach: given the current guess of the pricing function q , we solve for the policy function of consumption by iterating on the Euler equation, solve for a new guess for the pricing function q by iterating on the corresponding first order condition, and iterate this two-stage process until convergence. Finally, we imitate the Walrasian auctioneer and compare d'_b and d'_s : if they do not (approximately) sum to zero, we adjust the current guess for R accordingly, repeating the process until convergence. Appendix A.2 describes the solution procedure in greater detail.

4 Narrative of recessions: the role of financial stress

In this section we study the stochastic properties of the model. We wish to investigate the following questions. What are the typical features of an episode of deleverage? What is the size of the ensuing contraction in output (if any) during these episodes? To what extent does the magnitude of the recession depend on whether (i) the borrowing constraint is binding; (ii) the predetermined level of debt is high (or low)? What is the behavior of the real interest rate and the price of capital during deleverage-driven recessions?

We proceed as follows. We first simulate the joint discrete Markov chain for 3,000,000 periods conditional on both productivity and leverage shocks, and recover all episodes in which output happens to fall below its unconditional mean. Within these, we then further distinguish between those episodes characterized by a state labeled as *financial stress* (i.e., such that the borrowing constraint is binding, $\psi_t > 0$) and those in which financial stress is absent ($\psi_t = 0$).

In Figure 6 we plot the ergodic distribution of output, employment, the price of capital

¹¹The risky steady-state is the point where agents choose to stay at a given date if they expect future risk and if the realization of shocks is 0 at this date. See Coeurdacier et al. (2011) for more details.

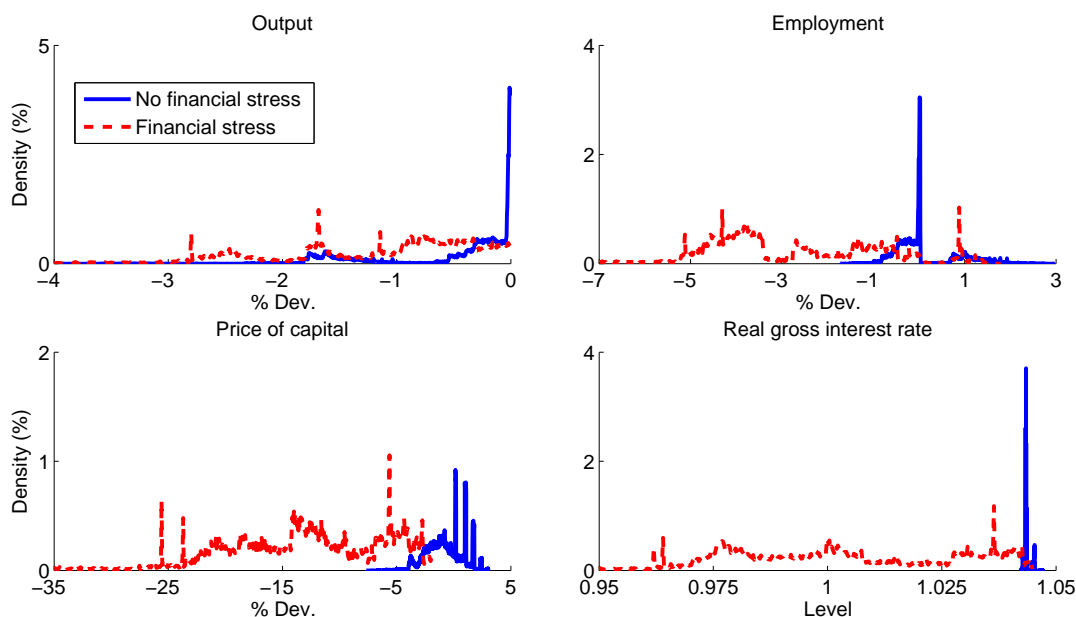


Figure 6: Ergodic distributions conditional on output being below its unconditional mean: all shocks. Note: financial stress denotes a state with a binding collateral constraint.

and the real interest rate *conditional* on output being below its unconditional mean.¹² All variables, except for the real interest rate, are expressed in percent deviations from their respective unconditional means. Two main results are worth noticing. First, in periods of financial stress, the frequency distribution of all variables shifts to the left (dashed lines), relative to periods where such stress is absent (solid lines). In other words, recessions characterized by a large fall in output, employment and asset prices are significantly more likely when the borrowing constraint is binding. It is also worth noticing that, during recessions characterized by financial stress, the price of capital can fall dramatically (up to a range between -20 and -30 percent). Second, the (gross) real interest rate can fall below 1, sometimes even significantly, but this happens only conditional on financial stress. Recall that, in general, we are not imposing any constraint on the possibility for the (net) real interest rate to fall below zero. If we assumed that the savers in the economy could resort to a real storage technology, the zero lower bound would become a binding constraint in those events now characterized by financial stress and a real interest rate falling below 1. In those cases, the ensuing fall in output would be much larger.

In Figure 7 we select the recession episodes in a different way. We plot the frequency distribution of the same selected variables as in Figure 6, but conditional on productivity being equal to its average value. We wish therefore to isolate only those episodes in which the maximum leverage χ_t falls below its unconditional mean. There are two main results.

¹²Please note that the irregular shape of these distributions is mainly due to the coarse discretization of the stochastic processes governing the dynamics of the exogenous state variables.

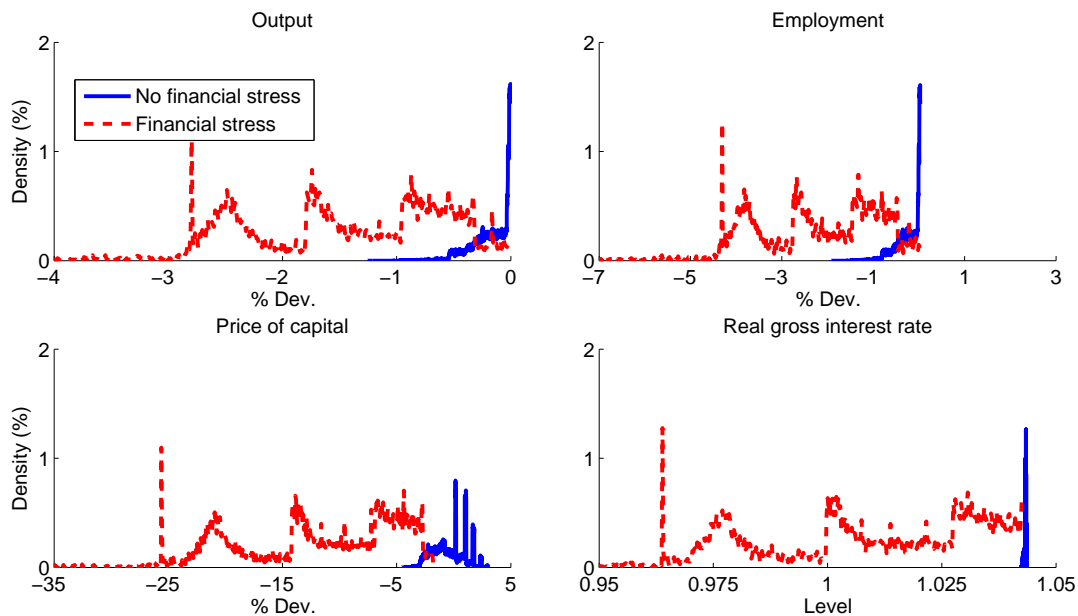


Figure 7: Ergodic distributions conditional on output being below its unconditional mean: deleverage shocks only. Note: financial stress denotes a state with a binding collateral constraint.

First, the frequency and the intensity of deleverage driven recessions (with output and employment falling below their respective means), and especially under financial stress, is larger than in the more general case, where both types of shocks are present. Second, events in which the real interest rate falls below 1 are in this case significantly more frequent.

What happens to debt during recessions? Figure 8 plots the ergodic distribution of (new) debt and output, conditional on output being below its unconditional mean. One main result is worth noticing. When the economy is not subject to financial stress, i.e., when the collateral constraint is not binding, recessions are characterized, most frequently, by an increase in private debt. The intuition is simple. In those states of the world, entrepreneurs resort to more debt, as opposed to a contraction in labor demand, to smooth consumption. However, when the collateral constraint is binding (i.e., during periods of financial stress), the borrowers need to deleverage in response to either a negative productivity or financial shock. In other words, deleverage is a feature mostly characterizing economies subject to financial stress.

5 Dynamics: a deleverage experiment

We now turn to a characterization of the dynamic evolution of the economy in response to exogenous deleverage episodes. Figure 9 reports the dynamic response of selected variables to an unexpected and *persistent reduction* in the maximum leverage χ_t (a “deleverage

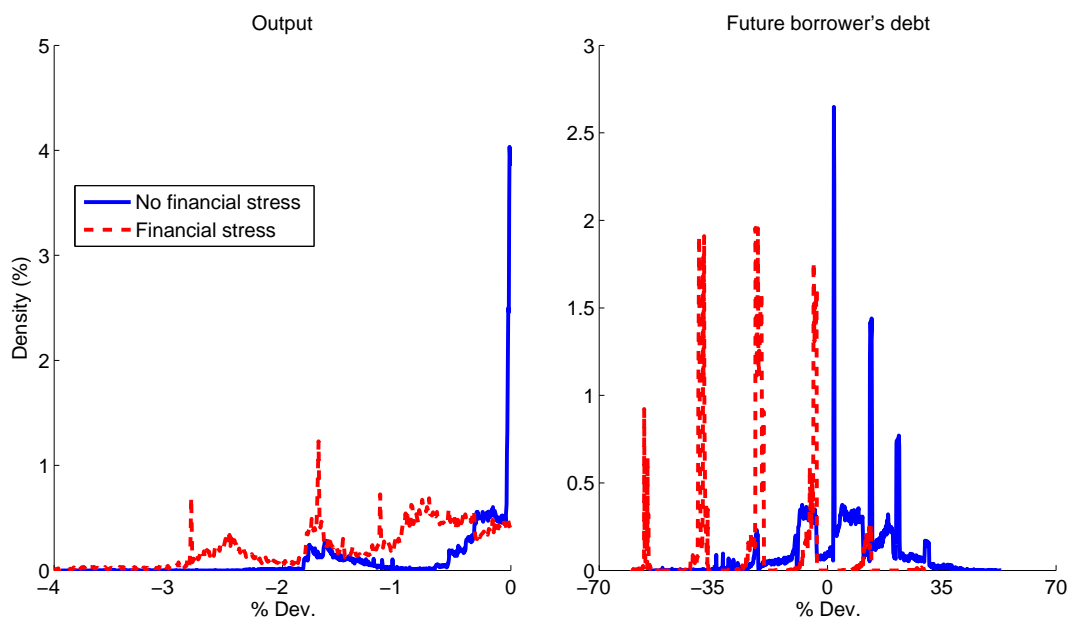


Figure 8: Ergodic distributions of output and (new) debt conditional on output being below its unconditional mean: all shocks. Note: financial stress denotes a state with a binding collateral constraint.

shock”).

Our shock experiment is constructed as follows. We normalize the initial value of each state variable to its unconditional mean. Then we assume that the maximum leverage χ_t unexpectedly falls in such a way to induce a contraction in the debt-to-output ratio, $d_{b,t}/y_t$, of about 10 percentage points, with respect to the pre-shock ratio. It turns out that the necessary reduction in χ_t corresponds roughly to a 2 standard deviation negative shock. This case is labeled “large shock” and is identified with a dashed line. It is compared to a benchmark case (solid line), where the assumed change χ_t is the largest feasible on the grid such that the borrowing constraint remains non binding. This case is labeled “small” shock. In both cases, the shock is persistent, i.e., χ_t assumes the new value from period 1 onward, until the end of the simulation horizon. Notice that the initial conditions are such that, in both cases, the entrepreneur starts out as unconstrained. In the figure, all variables are expressed in percent deviations from a benchmark “no-shock scenario,” except for the real interest rate and the multiplier on the borrowing constraint that are expressed in levels. The “no-shock scenario” corresponds to the dynamic adjustment path that the economy would follow in the absence of any shock to the exogenous state variables, given the varying initial value of debt: for any given initial debt level below (above) its unconditional mean, the system would converge to the unconditional mean from below (above) in the absence of exogenous shocks.

The effects of the shock are very different in the two scenarios. In the “small shock”

scenario, the borrowing constraint remains non binding. This can be gauged by the bottom-left panel, that plots the multiplier on the borrowing constraint (20). As a result, the effects of deleverage are neutral on all variables, consistent with our simple, two-period model.

When the shock is sufficiently large (dashed line), however, the borrowing constraint becomes binding. In this scenario, the entrepreneurs must sharply reduce their debt holdings. The initial (required) fall in debt induces the borrowers to contract their expenditure through two channels: by reducing their demand of production inputs (both labor and capital) and their consumption. Since the supply of capital is fixed, the equilibrium price of capital falls, further tightening the borrower's collateral constraint, in a classic debt-deflation dynamic.¹³ In turn, this further reduces the entrepreneur's demand of labor and consumption. The net effect, relative to the unconstrained case, is a much sharper contraction in borrowers' consumption (of roughly -25 percent), which is not compensated by the rise in savers' consumption. The implication is a fall in aggregate consumption and output of about -7.6 percent.

Notice also that while in the unconstrained scenario the real interest rate remains virtually constant, in the constrained scenario the real interest rate falls sharply below 1. The intuition is simple. Due to the more sizable contraction in debt, also the households have to reduce their asset holdings (i.e., their savings). In order to induce them to save unexpectedly so much less, the real interest rate must fall to clear the credit market.

To summarize, a deleverage shock normalized to yield a reduction in the debt-to-output ratio of 10 percent is far from being neutral, and is able to generate a contraction in aggregate output of about 5 percent. Importantly, this happens despite the absence of any form of nominal rigidity and without imposing a zero bound constraint on the interest rate. The main channel at work is a borrowing friction with the possibility of that friction becoming endogenously binding.

6 The role of nonlinearities

In this section we focus our attention on the main theme of our paper: the role of nonlinearities and their interaction with the degree of financial fragility in shaping the effects of financial shocks.

6.1 Decision rules

Figure 10 plots the decision rule for output as a function of $d_{b,t}$, the outstanding level of debt at the beginning of time t . For each panel, there are two lines, corresponding

¹³See Mendoza (2010) for an analysis of Fisherian-style debt deflation dynamic in a general equilibrium model of a small open economy.

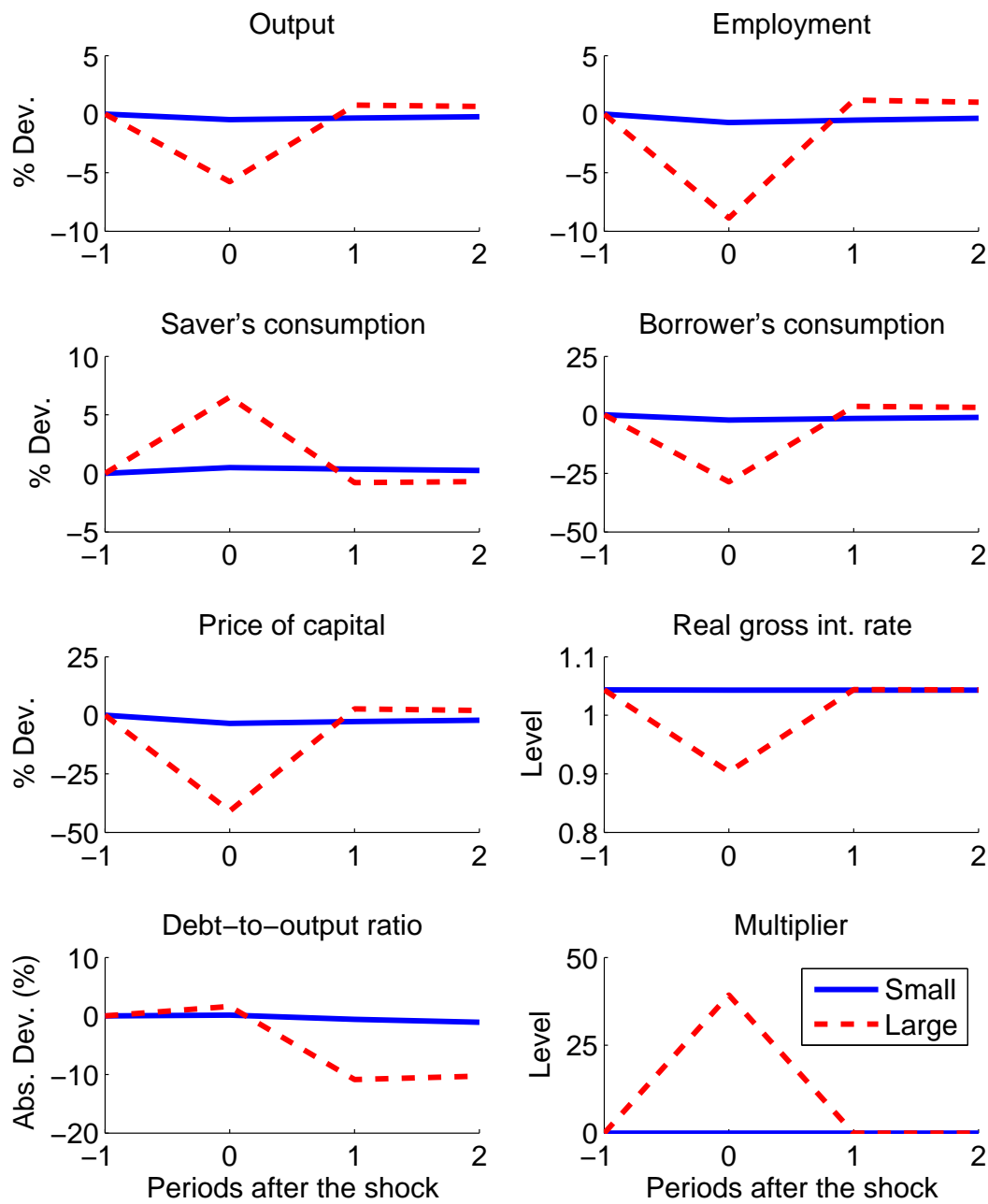


Figure 9: Large vs. small deleverage shock. Note: a large shock is a persistent contraction in χ_t such that the debt to output ratio, $d_{b,t}/y_t$, falls by 10% after the shock. All variables, except the real interest rate and the multiplier on the borrowing constraint, expressed in percent deviations from the no-shock scenario.

to two different states. In the solid line case, the maximum leverage χ_t is (one standard deviation) below its mean, whereas productivity is at its average value; conversely, in the dashed line case, productivity is (one standard deviation) below its mean, whereas the maximum leverage χ_t is at its average value. Output, on the vertical axis, is measured in percentage deviations from its unconditional mean.

Consider, first, the state of the economy in which χ_t is low (and productivity is average). Notice that, conditional on a low value of χ_t , the policy function is downward sloping and features a kink around a threshold value of debt $d_{b,t}^*$ (which is roughly equal to 0.12 under our baseline calibration). This has a twofold implication. First, the impact effect of a deleverage shock is state-dependent, i.e. it is larger (in absolute value) the larger is the outstanding level of debt. Second, the degree of state-dependency is significantly more pronounced to the right as opposed to the left of $d_{b,t}^*$. To the left of $d_{b,t}^*$ (i.e., when debt is sufficiently low), the current level of debt has a limited impact on the sign of output, which can vary between mildly negative and mildly positive. Conversely, for values of current debt higher than $d_{b,t}^*$, output is invariably below its unconditional mean and, most importantly, the size of this deviation varies significantly with the outstanding level of debt. In particular, in this region, the size of the output deviation (holding constant the deviation of χ_t from its mean) can vary between -0.4 and -5 per cent.

This result contrasts with the case in which the state of the economy is characterized by a low value of productivity (low z_t , with χ_t being equal to its average value). Conditional on a low z_t , the sign of the output deviation is always negative - a standard result from real business cycle theory; but, most importantly, it remains insensitive to the outstanding level of debt in a larger range of values of debt. Thus, in our context, state-dependency in the response of output is significantly more pronounced when the economy is hit by (de)leverage shocks than when it is hit by productivity shocks.

Figure 11 plots decision rules for selected variables other than output. The two main results highlighted above are confirmed. First, conditional on a low realization of the maximum leverage parameter χ_t , the policy function features a kink, with this being particularly pronounced for the price of capital and the real interest rate. Notice that the sensitivity of both the real interest rate and the price of capital to a deleverage shock depends dramatically on the outstanding level of debt. In particular, the real interest rate can assume a value below 1 only when two conditions are simultaneously present: a low value of χ_t and a sufficiently high level of outstanding debt. Notice also that the decision rule of savers' consumption is, qualitatively, the mirror image of the borrowers' consumption decision rule. The difference, however, lies in the magnitudes: conditional on $d_{b,t} > d_{b,t}^*$, borrowers' consumption is much more responsive than savers' consumption to a deleverage shock. Second, the kink in the policy function corresponds to a lower level of current debt in the case of deleverage shocks relative to the case of productivity shocks. In other words, for any given degree of financial fragility, deleverage shocks make the

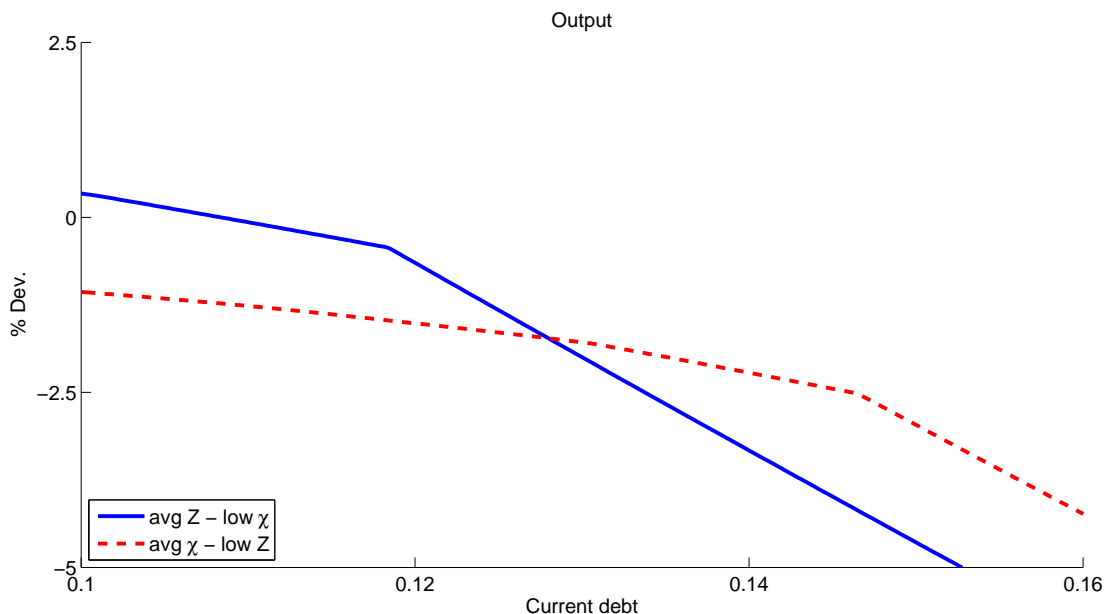


Figure 10: Decision rule of aggregate output (in percent deviations from its unconditional mean) as a function of current debt. Solid line: z_t average and χ_t one standard deviation below its unconditional mean. Dashed line: z_t one standard deviation below its unconditional mean and χ_t average.

borrowing constraint be more likely to become binding. This feature, which is intuitive, is common to the policy function of all variables.

6.2 The S-shaped effect of financial fragility

In this section we show that even small differences in the current state of the economy, and in the level of debt in particular, can trigger sizeable differences in the response to a deleverage shock of the same size and sign. Our “deleverage shock” experiment is constructed as follows. We normalize the initial value of each exogenous state variable to its unconditional mean, and assume that the maximum leverage χ_t unexpectedly falls, respectively by one or two standard deviations below its mean. We then measure the *distance* between the pre-shock and the after-shock decision rule of output, and plot this distance as a function of the outstanding level of debt. We label this as the *impact function* of output, and report it in Figure 12. The solid line depicts the impact function of output for the baseline case of one standard deviation shock, whereas the dashed line depicts the case of a two standard deviation shock.

To clarify, in the case of a linearized solution, the impact function of any given variable would be a flat line. It is clear however that the impact response of output is highly non-linear in the pre-shock level of debt, and features a *S*-shape. The intuition for the *S*-shaped non-linearity is as follows. To start with, notice that, for each shock size, we can identify

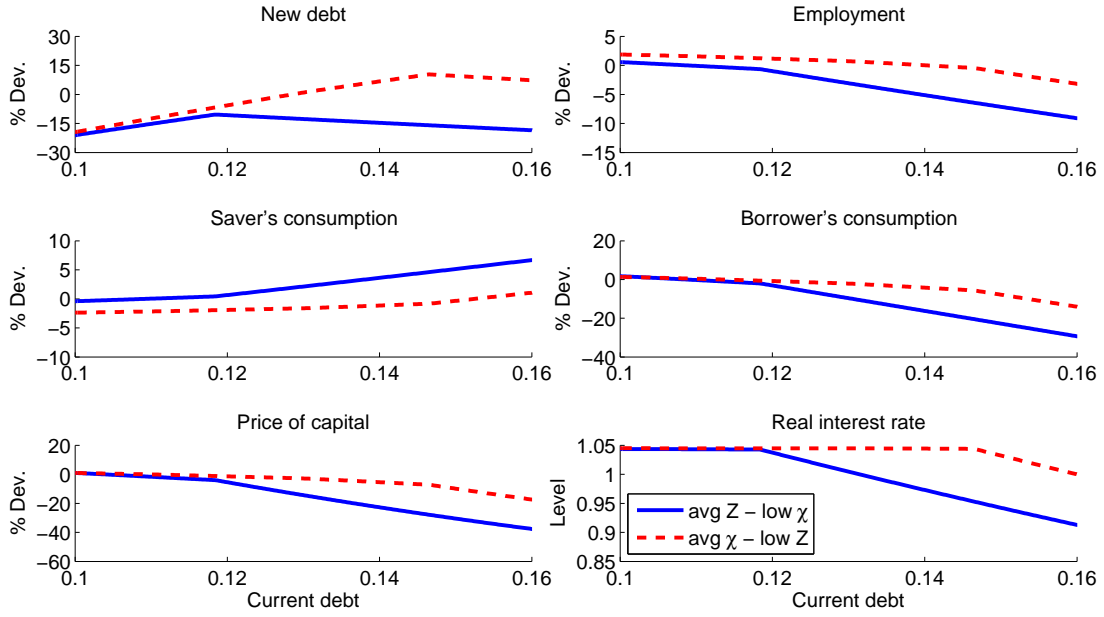


Figure 11: Decision rules for selected variables as a function of current debt. All expressed in % deviations from respective unconditional means, with the exception of R_t expressed in units. Solid line: z_t average and χ_t one standard deviation below its mean. Dashed line: z_t one standard deviation below its unconditional mean and χ_t average.

three debt regions, labeled “low,” “intermediate” and “high” debt respectively. If the state the economy is such that agents feature a “low” debt-to-income ratio, a deleverage shock of typical (one standard deviation) size produces a mild impact effect on output, triggers a small fall in asset prices and even a small rise in the real interest rate (not shown). In this region, the impact effect of the shock on output is roughly state-independent, i.e., it is unaffected by the current level of debt. This is because, before the shock hits, the borrowers are unconstrained, and remain such also after the shock has materialized. If, however, the underlying level of debt falls in the “intermediate” region, the impact effect on output of a shock of the same size is an increasing function of the level of debt. This happens because, in that region, the borrowers are not constrained *ex-ante*, but they become so *ex-post*, precisely as a result of the shock. Hence, the fall in private debt and asset prices reinforce each other, leading to a large contraction in output. To quantify the relative effect of being in the low vs. intermediate debt region, when the current debt-to-output ratio is, e.g., 0.2 (belonging to the “low” debt region), a (one standard deviation) drop in the maximum leverage χ_t triggers a fall in output (relative to its unconditional mean) of about 0.2 percent ; however, when the current debt-to-output ratio is 0.3 (now belonging to the “intermediate” region), a deleverage shock of the same size produces a contraction in output of up to 4 percent, i.e., almost twenty times larger. Not surprisingly, this effect is largely magnified in the case we consider a drop in χ_t equal to two standard

deviations below its mean. Finally, if the current level of debt falls in the “high” region, the borrowers are already constrained ex-ante, and remain such also ex-post. In this region, the impact effect on output of a deleverage shock is the largest, but the *marginal* effect of a higher level of debt tends to flatten out. Thus, the combined shape of the three regions makes the impact response of output to a deleverage shock a *S*-shaped function of the current level of debt.

To better gauge the intuition behind the S-shape of the impact function of output, Figure 13 plots the decision rule for output as a function of current debt, and conditional on two alternative values of the state χ_t . In the first case, represented with a solid line, χ_t is equal to its average value, whereas in the second case, represented with a dashed line, χ_t is one standard deviation below its unconditional mean. In both cases, productivity z_t is kept unchanged at its unconditional mean. Hence what the impact function measures is, holding constant the size of the change in χ_t (from average to low), the vertical distance between the two decision rules. The key feature to notice is that the drop in χ_t does not translate into a parallel shift of the decision rule. Hence the vertical distance between the decision rules varies with the current level of debt: it is narrow and roughly constant for low levels of debt (the “low” region); it is large, but once again roughly constant for high levels of debt (the “high” debt region); and it is increasing in the level of debt for intermediate values of the latter (the “intermediate” debt region). Overall, the crucial element that determines the *S*-shape of the impact function is that, when χ_t falls, e.g., from average to low, the *kink* in the decision rule also *shifts* to the left. Put differently, in a state of the world where χ_t is lower than its average value, the borrowing constraint becomes binding at lower levels of debt, i.e., financial fragility starts to exert an effect at lower levels of accumulated debt.

Figure 14 displays the impact function for other selected variables. As done earlier, in each panel, we distinguish two cases: a one standard deviation (solid) and a two standard deviation (dashed) drop of χ_t below its average value. As long as the initial debt-to-output ratio remains below a given critical value (which varies depending on the size of the deleverage shock), the borrowing constraint is not binding. This can be seen from the right bottom panel, which plots the impact response of the Lagrange multiplier ψ_t as a function of the pre-shock debt to output ratio. As a result, we observe a negligible adjustment in debt, a muted fall in the price of capital and a rough constancy of the real interest rate. In addition, the response of employment and of consumption by each agent is roughly zero. However, when the deleverage shock hits an economy with a sufficiently high outstanding level of debt, the borrowing constraint becomes binding. Holding constant the size of the shock, the impact response of the price of capital and of the real interest rate become sharply increasing (in absolute value) in the outstanding level of debt. The fall in the price of capital makes the necessary reduction in debt much larger, and this is reflected in a stronger reduction in entrepreneur’s consumption and a sharper fall in the

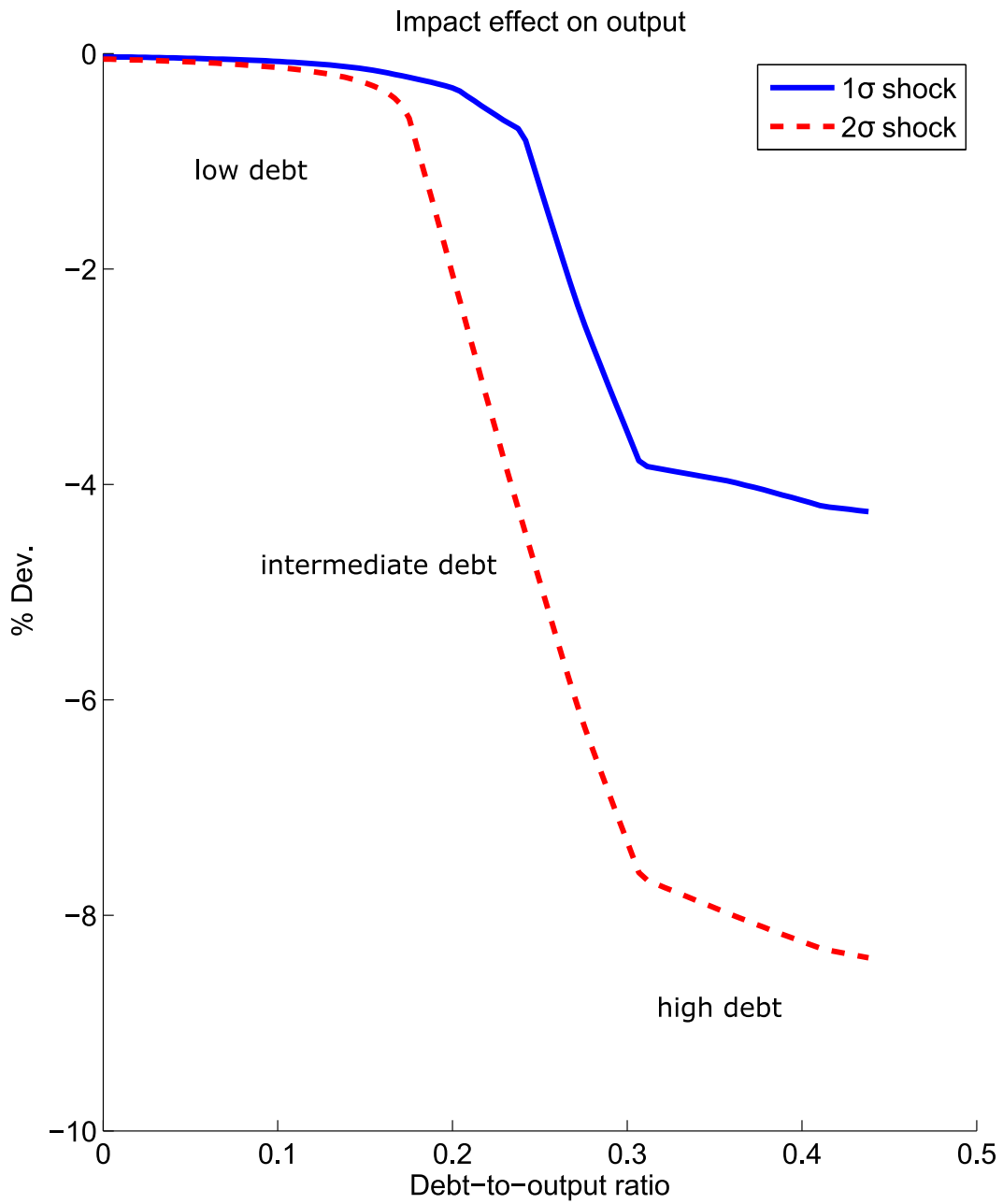


Figure 12: Impact function of output: impact effect on output of a deleverage shock as a function of the outstanding, pre-shock, level of debt.

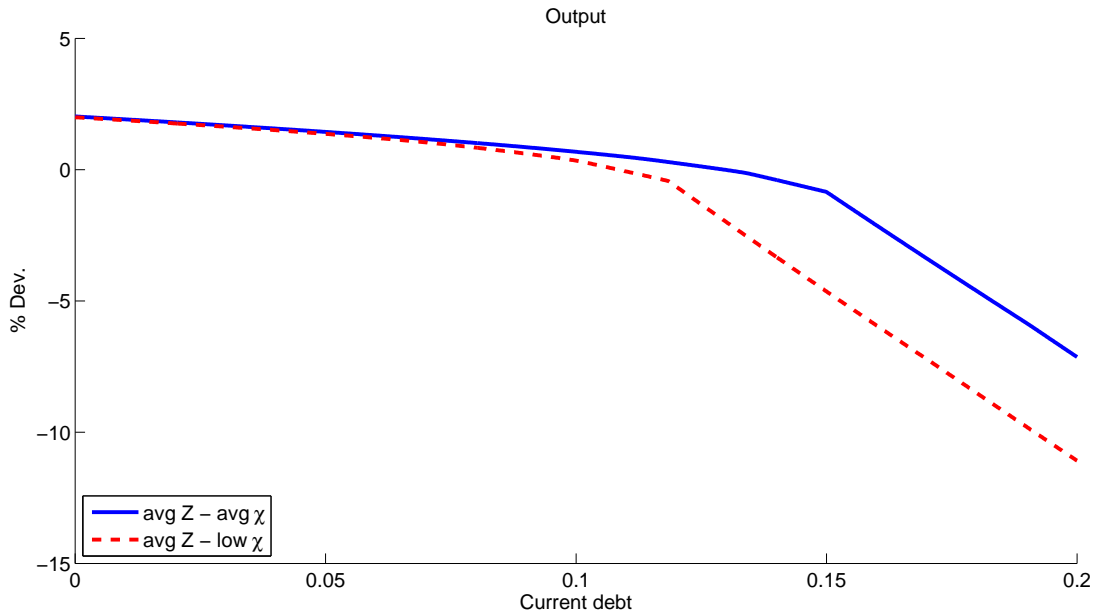


Figure 13: Decision rule of output: conditional on average χ_t (solid) vs. low χ_t (dashed). In both cases productivity z_t is kept unchanged at its average value.

real interest rate. Notice that, for the given range of values of debt, all impact functions (with the possible exception of the real interest rate) feature a *S*-shape similar to the one described in detail earlier.

7 Conclusions

Severe economic downturns, characterized by deleverage, are typically preceded by phenomena of credit overhang (see Jorda et al., 2013, and Mian and Sufi, 2010). This evidence suggests that large recessions are not the result of large shocks, but rather of the interaction between “typical” shocks and the current state of the economy. From a theoretical standpoint, this requires making credit overhang endogenous, so that the aggregate implications of deleverage shocks are state dependent. This paper is a step in this direction.

We have studied a dynamic general equilibrium model augmented with financial frictions, in the form of a constraint on borrowing rooted in limited commitment. Relative to many recent contributions on the subject, we have treated this constraint as only occasionally binding. We have shown that, in this context, the effects of financial disturbances are highly state dependent. In particular, those effects are a highly non-linear function of the degree of financial fragility, the latter being proxied by the endogenously evolving level of debt. Our results show that gauging the effects of financial shocks in a certainty equivalence, linearized environment can be misleading about their quantitative relevance. In such a setup financial frictions are assumed to be always binding and the impact of a

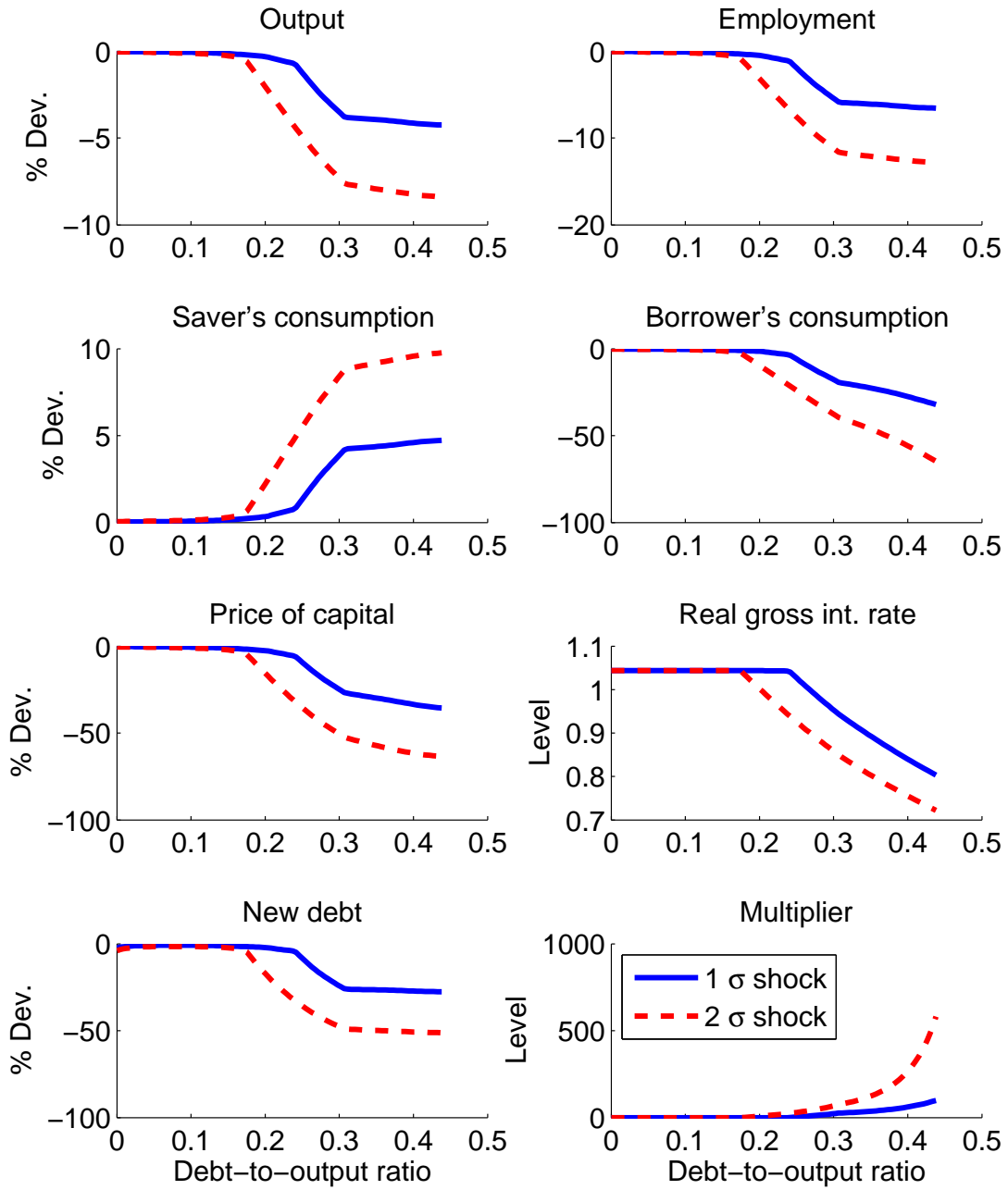


Figure 14: Impact effect on selected variables of a deleverage shock as a function of the outstanding, pre-shock, level of debt.

financial shock is necessarily the result of an average across states of the world which are potentially very different depending precisely on the degree of financial fragility.

Our analysis is intentionally based on a relatively standard setup with financial frictions, widely employed in the more recent literature in macroeconomics. There are several further directions this analysis can take. We mention at least three. First, it would be important to allow for endogenous investment and capital accumulation to assess more broadly the quantitative significance of nonlinearities. Second, a distinction should be made between deleverage in the household vs business sector. It is easy to modify our setup to allow for heterogeneous households, with borrowers consuming a durable asset and using that asset as a collateral in borrowing. The key difference of such a setup, relatively to ours, would be that financial shocks would directly affect the agents who supply, rather than demand, labor. If, in light of a negative balance sheet shock, households become borrowing constrained, they increase their labor supply to smooth their consumption. As a result, and *ceteris paribus*, output can rise in response to a financial shock. Obtaining a contraction in output would therefore represent an additional challenge. Third, it would be relevant, and timely, to extend our setup to allow for nominal rigidities. Such a setup would lend itself to the analysis of the role of the zero lower bound constraint and, simultaneously, to the relevance of nonlinearities in the transmission of monetary policy impulses.¹⁴ In addition, the presence of nominal rigidities would reintroduce a role for labor demand effects even in a context in which the agents subject to borrowing constraints are also the ones who supply labor. All these endeavors are part of our ongoing research.

¹⁴See for instance Guerrieri and Iacoviello (2013) on this point.

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A Appendix

A.1 Equilibrium conditions

Let $\psi_{b,t}$ denote the Lagrange multiplier on the borrowing constraint (20), and $U_{x,t}$ as the marginal utility of variable x . The optimality conditions for the borrowers read:

$$U_{c_{b,t}}q_t - \psi_{b,t}\chi_t\mathbb{E}_tq_{t+1} = \beta_b\mathbb{E}_t\left[U_{c_{b,t+1}}(F_{k,t+1} + q_{t+1})\right], \quad (30)$$

$$U_{c_{b,t}} = \beta_b R_t \mathbb{E}_t U_{c_{b,t+1}} + \psi_{b,t}, \quad (31)$$

$$F_{n_{b,t}} = W_t, \quad (32)$$

$$c_{b,t} + q_t(k_{b,t+1} - k_{b,t}) + W_t n_{b,t} + \frac{d_{b,t+1}}{R_t} = d_{b,t} + y_{b,t}, \quad (33)$$

$$\psi_{b,t} \left[\frac{d_{b,t+1}}{R_t} - \chi_t \mathbb{E}_t (q_{t+1} k_{b,t+1}) \right] \geq 0. \quad (34)$$

The optimality conditions for the households are more standard:

$$U_{c_{s,t}} = \beta_s R_t \mathbb{E}_t U_{c_{s,t+1}}, \quad (35)$$

$$-U_{n_{s,t}} = W_t U_{c_{s,t}}. \quad (36)$$

The aggregate equilibrium conditions remain the following:

$$k_{b,t+1} = \bar{K}, \quad (37)$$

$$d_{b,t+1} + d_{s,t+1} = 0, \quad (38)$$

$$n_{b,t} = n_{s,t}. \quad (39)$$

Conditional on stochastic processes for the exogenous variables $\{\chi_t\}$ and $\{z_t\}$, a rational expectations equilibrium is a vector process for the endogenous variables:

$$\{c_{b,t}, c_{s,t}, n_{b,t}, n_{s,t}, d_{b,t}, d_{s,t}, k_{b,t}, W_t, q_t, R_t\}$$

solving the dynamic system (30)-(39).

A.2 Solution procedure

We employ a global solution method based on fixed-point iteration over the Euler equations. Rendhal (2013) shows that time iteration on the Euler equation converges to the solution obtained with value function iteration also in the presence of occasionally binding constraints. Fixed point iteration is faster, but is not guaranteed to converge in general, and often some damping is necessary; in case of convergence, it converges by construction to the same solution obtained via time iteration.

Define a grid over bond holdings on the real line, say $\mathbf{d}_s = \{d_{s,i}\}_{i=1}^h$, where i identifies the node on the grid and h denotes the total number of nodes. Hence, $\mathbf{d}_b = -\mathbf{d}_s$. Discretize the exogenous stochastic processes for Z and χ using Tauchen's method. The resulting independent discrete Markov chains can be combined in a single process, characterized by a transition matrix $\pi = \pi_Z \otimes \pi_\chi$ such that $\pi(\varepsilon_j, \varepsilon_i) \geq 0$ denotes the probability that $\varepsilon_{t+1} = \varepsilon_j$ if $\varepsilon_t = \varepsilon_i$, where, for the sake of notational convenience, $\varepsilon \equiv \{Z, \chi\}$. Choose initial guesses for R and q at each grid point, i.e. vectors $\mathbf{R}_{\varepsilon,0} = \{R_{\varepsilon,0,i}\}_{i=1}^h$ and $\mathbf{q}_{\varepsilon,0} = \{q_{\varepsilon,0,i}\}_{i=1}^h$, one for each exogenous state. Choose initial guesses for c'_s and c_b at each grid point, i.e. vectors $\mathbf{c}_{s,\varepsilon,0} = \{c_{s,\varepsilon,0,i}\}_{i=1}^h$ and $\mathbf{c}_{b,\varepsilon,0} = \{c_{b,\varepsilon,0,i}\}_{i=1}^h$.

Main loop The main loop mimics a Walrasian auctioneer, and iterates on the interest rate \mathbf{R} until convergence to an equilibrium:

Algorithm 1. *Given the initial guess $\mathbf{R}_{\varepsilon,0}$, for $j \geq 0$:*

1. *Solve the savers problem for $\mathbf{c}_{s,\varepsilon,j}$.*
2. *Solve the borrowers problem for $\mathbf{c}_{b,\varepsilon,j}$ and $\mathbf{q}_{\varepsilon,j}$.*
3. *Compute $\mathbf{d}'_{s,\varepsilon,j}$, $\mathbf{d}'_{b,\varepsilon,j}$, and update the guess for \mathbf{R} via:*

$$\mathbf{R}_{\varepsilon,j+1} = \mathbf{R}_{\varepsilon,j} - \vartheta \left(\mathbf{d}'_{s,\varepsilon,j} + \mathbf{d}'_{b,\varepsilon,j} \right). \quad (40)$$

where $\vartheta > 0$ is a dumping factor.

4. *Iterate on (1) – (3) until $\mathbf{d}'_{s,\varepsilon,j} + \mathbf{d}'_{b,\varepsilon,j} \approx \mathbf{0}$.*

Households The saver problem in step (2) of the main loop is solved via fixed point iteration on the Euler equation:

Algorithm 2. *Given the current guess $\mathbf{R}_{\varepsilon,j}$, choose an initial guess for $\mathbf{c}_{s,\varepsilon,0}$; then, for $z \geq 0$:*

1. *Compute:*

$$\mathbf{d}'_{s,\varepsilon,z} = \mathbf{R}_{\varepsilon,j} \left(\mathbf{d}_s + s_N \mathbf{y}_{b,\varepsilon,z} - \mathbf{c}_{s,\varepsilon,z} \right), \quad (41)$$

where:

$$\mathbf{y}_{b,\varepsilon,z} = \mathbf{Z}_\varepsilon \mathbf{n}_{\varepsilon,z}^{s_N} \bar{K}^{s_K}, \quad (42)$$

$$\mathbf{n}_{\varepsilon,z} = \left(\frac{\mathbf{c}_{s,\varepsilon,z}^{-\mu} s_N \mathbf{Z}_\varepsilon \bar{K}^{s_K}}{\varphi} \right)^{\frac{1}{v+1-s_N}}. \quad (43)$$

2. *Given $\mathbf{d}'_{s,\varepsilon,z}$, compute $\mathbf{c}'_{s,\varepsilon,z}$ via interpolation on \mathbf{d}_s and $\mathbf{c}_{s,\varepsilon,z}$.*

3. Compute:

$$\bar{\mathbf{c}}_{s,\varepsilon,z} = \left\{ \beta_s \mathbf{R}_{\varepsilon,j} \mathbb{E} \left[\left(\mathbf{c}'_{s,\varepsilon,z} \right)^{1-\mu} \right] \right\}^{-\frac{1}{\mu}}, \quad (44)$$

and:

$$\hat{\mathbf{c}}_{s,\varepsilon,z} = \min(\bar{\mathbf{c}}_{s,\varepsilon,z}, \mathbf{d}_s + s_N \mathbf{y}_{b,\varepsilon,z}). \quad (45)$$

4. Update the current guess:

$$\mathbf{c}_{s,\varepsilon,z+1} = \varrho \mathbf{c}_{s,\varepsilon,z} + (1 - \varrho) \hat{\mathbf{c}}_{s,\varepsilon,z}, \quad (46)$$

where $\varrho \in (0, 1)$ is a dumping factor.

5. Iterate on (1) – (4) until convergence.

Entrepreneurs The borrower problem in step (3) of the main loop is solved similarly:

Algorithm 3. Given the current guess $\mathbf{R}_{\varepsilon,j}$, choose initial guess for $\mathbf{c}_{b,\varepsilon,0}$, $\psi_{b,\varepsilon,0}$, and $\mathbf{q}_{\varepsilon,0}$; then, for $z \geq 0$:

1. Given $\mathbf{q}_{\varepsilon,z}$, solve for $\mathbf{c}_{b,\varepsilon,z}$ and $\psi_{b,\varepsilon,z}$; hence, for $m \geq 0$:

(a) Compute:

$$\mathbf{d}'_{b,\varepsilon,m} = \mathbf{R}_{\varepsilon,j} [\mathbf{d}_E + (1 - s_N) \mathbf{y}_{b,\varepsilon} - \mathbf{c}_{b,\varepsilon,m}]. \quad (47)$$

(b) Given $\mathbf{d}'_{b,\varepsilon,m}$, compute $\mathbf{c}'_{b,\varepsilon,m}$ via interpolation on \mathbf{d}_b and $\mathbf{c}_{b,\varepsilon,m}$.

(c) Compute:

$$\bar{\mathbf{c}}_{b,\varepsilon,m} = \left\{ \beta_b \mathbf{R}_{\varepsilon,j} \mathbb{E} \left[\left(\mathbf{c}'_{b,\varepsilon,m} \right)^{1-\mu} \right] \right\}^{-\frac{1}{\mu}}, \quad (48)$$

and:

$$\hat{\mathbf{c}}_{b,\varepsilon,m} = \min[\bar{\mathbf{c}}_{b,\varepsilon,m}, \mathbf{d}_b + (1 - s_N) \mathbf{y}_{b,\varepsilon} - \chi_\varepsilon \mathbf{q}_{\varepsilon,z} \bar{K}]. \quad (49)$$

(d) Given $\hat{\mathbf{c}}_{b,\varepsilon,m}$, compute:

$$\psi_{b,\varepsilon,m+1} = \hat{\mathbf{c}}_{b,\varepsilon,m}^{-\mu} - \beta_b \mathbf{R}_{\varepsilon,j} \mathbb{E} \left[\left(\mathbf{c}'_{b,\varepsilon,m} \right)^{1-\mu} \right]. \quad (50)$$

(e) Update the current guess for $\mathbf{c}_{b,\varepsilon}$:

$$\mathbf{c}_{b,\varepsilon,m+1} = \varrho \mathbf{c}_{b,\varepsilon,m} + (1 - \varrho) \hat{\mathbf{c}}_{b,\varepsilon,m}. \quad (51)$$

(f) Iterate on (a) – (e) until convergence of $\mathbf{c}_{b,\varepsilon,m}$.

2. Given the previously obtained $\mathbf{d}'_{b,\varepsilon,z}$, compute $\mathbf{q}'_{\varepsilon,z}$ via interpolation on \mathbf{d}_b and $\mathbf{q}_{\varepsilon,z}$.

3. Update the current guess for \mathbf{q}_ε :

$$\mathbf{q}_{\varepsilon,z+1} = \frac{\beta_b \mathbb{E} \left[\left(\mathbf{c}'_{b,\varepsilon,z} \right)^{-\mu} \left(s_K \frac{\mathbf{y}'_{b,\varepsilon}}{K} + \mathbf{q}'_{\varepsilon,z} \right) \right] + \psi_{b,\varepsilon,z} \chi_\varepsilon \mathbb{E} \left(\mathbf{q}'_{\varepsilon,z} \right)}{\mathbf{c}_{b,\varepsilon,z}^{-\mu}}. \quad (52)$$

4. Iterate on (1) – (3) until convergence of $\mathbf{q}_{\varepsilon,z}$.