INNOCENZO GASPARINI INSTITUTE FOR ECONOMIC RESEARCH

Institutional Members: CEPR, NBER and Università Bocconi

WORKING PAPER SERIES

On the equality of Clarke-Rockafellar and Greenberg-Pierskalla differentials for monotone and quasiconcave functionals

Simone Cerreia-Vioglio, Fabio Maccheroni, and Massimo Marinacci

Working Paper n. 561

This Version: November, 2015

IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
http://www.igier.unibocconi.it

On the equality of Clarke-Rockafellar and Greenberg-Pierskalla differentials for monotone and quasiconcave functionals*

S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci
IGIER and Università Bocconi

November 2015

Abstract

We study monotone, continuous, and quasiconcave functionals defined over an M-space. We show that if g is also Clarke-Rockafellar differentiable at x and $0 \notin \partial_{CR} g(x)$, then the closure of Greenberg-Pierskalla differentials at x coincides with the closed cone generated by the Clarke-Rockafellar differentials at x. Under the same assumptions, we show that the set of normalized Greenberg-Pierskalla differentials at x coincides with the closure of the set of normalized Clarke-Rockafellar differentials at x. As a corollary, we obtain a differential characterization of quasiconcavity a la Arrow and Enthoven (1961) for Clarke-Rockafellar differentiable functions.

1 Introduction

Since the seminal studies of de Finetti [12] and Fenchel [13], quasiconvex analysis has been the subject of active research.[1] Starting by the paper of de Finetti [12], this field has been deeply influenced by economic theory. In keeping with this tradition, our purpose here is to relate two notions of differentiability that have been proven useful both in quasiconvex analysis and decision theory: Greenberg-Pierskalla differentiability and Clarke-Rockafellar differentiability.[2] Specifically, we study monotone, continuous, and quasiconcave functionals defined over an M-space. Our main results are Theorems 1 and 2. In Theorem 1, we show that if g is also Clarke-Rockafellar differentiable at x and $0 \notin \partial_{CR} g(x)$, then the set of *normalized* Greenberg-Pierskalla differentials at x coincides with the closure of the set of *normalized* Clarke-Rockafellar differentials at x. In Theorem 2, under the same assumptions, we show that the closure of Greenberg-Pierskalla differentials at x coincides with the closed cone generated by the Clarke-Rockafellar differentials at x. As a corollary, we obtain a differential characterization of quasiconcavity a la Arrow and Enthoven [2] for Clarke-Rockafellar differentiable functionals (see Corollary 3).

In what follows, we outline the economic motivation of our exercise and how it improves upon the current literature. Recall that rational preferences are complete, transitive, and monotone binary relations defined on the classic space $B_0(\Omega, \Sigma, C)$ of decision theory where Ω is a state space, Σ is an event algebra, and C

*Massimo Marinacci gratefully acknowledges the financial support of the AXA Research Fund.

[1] We refer the reader to Penot [23] for a recent survey. See also Crouzeix [8], [9], and [10], Martinez-Legaz [20] and [21], and Penot and Volle [24].

[2] One stark difference with convex analysis is the presence of different type of dualities as well as different notions of differentiability. See, for example, Komlosi [19] and Penot [22] as well as Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [6].

is a convex set of consequences. Elements $f \in B_0(\Omega, \Sigma, C)$ are simple Σ-measurable functions $f : \Omega \to C$, interpreted as the acts available to a decision maker. Under few extra suitable behavioral conditions, Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [4] shows that rational preferences admit a utility function $V : B_0(\Omega, \Sigma, C) \to \mathbb{R}$, that is,

$$f \succsim g \qquad \Longleftrightarrow \qquad V(f) \geq V(g).$$

Moreover, they show that $V = I \circ u$ where $I : B_0(\Omega, \mathcal{F}, \mathbb{R}) \to \mathbb{R}$ is a normalized, monotone, and continuous functional and $u : C \to \mathbb{R}$ is an affine function with $u(C) = \mathbb{R}$.[3] An interesting behavioral object is the *revealed unambiguous* binary relation $\succsim^*$ of Ghirardato, Maccheroni, and Marinacci [15] where $\succsim^*$ is defined as

$$f \succsim^* g \quad \Longleftrightarrow \quad \lambda f + (1-\lambda) h \succsim \lambda g + (1-\lambda) h \qquad \forall \lambda \in (0,1], \forall h \in B_0(\Omega, \Sigma, C).$$

It is immediate to see that $\succsim^*$ is a subrelation of $\succsim$. The comparison $f \succsim^* g$ is meant to capture the idea that f is robustly preferred to g. For, no matter how we hedge f and g with a third act h, the mixture of f with h dominates the one of g with h. In [4], it is shown that

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C^*$$

where C^* is a uniquely determined convex and closed set of probabilities in Δ. This latter set of probability models can be interpreted as the set of probabilities that are relevant for the decision maker (see [15]). Thus, it is important to be able to compute the set C^* in terms of the functional I. In fact, there are several works in the literature that deal with this question:

1. Ghirardato, Maccheroni, and Marinacci [15] achieves this characterization for the important class of invariant biseparable preferences, that is, those preferences for which I is positively homogeneous and translation invariant.[4] In this case, the set C^* coincides with the Clarke's differential of I computed at 0.

2. Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [5] characterizes C^* for the class of uncertainty averse preferences, that is, those preferences for which I is quasiconcave. In this case, the set C^* coincides with the closed convex hull of the union of normalized Greenberg-Pierskalla differentials of I. Given $\varphi \in B_0(\Omega, \mathcal{F}, \mathbb{R})$, the set of normalized Greenberg-Pierskalla differentials at φ is the set:

$$\partial^N_{GP} I(\varphi) = \{p \in \Delta : \forall \psi \quad \langle \psi, p \rangle \leq \langle \varphi, p \rangle \quad \Longrightarrow \quad I(\psi) \leq I(\varphi)\}.$$

3. Ghirardato and Siniscalchi [16] (see also its working paper version) characterizes C^* for the class of monotonic, Bernoullian, and continuous preferences, that is, those preferences for which I admits a continuous extension to the completion of $B_0(\Omega, \mathcal{F}, \mathbb{R})$. In this case, the set C^* coincides with the closed convex hull of the union of normalized Clarke-Rockafellar differentials of I. Given $\varphi \in B_0(\Omega, \mathcal{F}, \mathbb{R})$, the set of normalized Clarke-Rockafellar differentials at φ is the set

$$\partial^N_{CR} I(\varphi) = \{p \in \Delta : p = p' \backslash \|p'\| \text{ for some } p' \in \partial_{CR} I(\varphi)\},$$

where $\partial_{CR} I(\varphi)$ is the usual Clarke-Rockafellar differential at φ.

[3] I is normalized if and only if $I(k1_\Omega) = k$ for all $k \in \mathbb{R}$.

[4] That is, $I(\lambda\varphi + k1_\Omega) = \lambda I(\varphi) + kI(1_\Omega)$ for all $\varphi \in B_0(\Omega, \Sigma, \mathbb{R})$, for all $\lambda \geq 0$, and for all $k \in \mathbb{R}$.

In this paper, as mentioned, we focus on the quasiconcave case. In light of the results contained in points 1–3, we ask ourselves when the different sets of differentials coincide, that is, when $\partial^N_{GP} I(\varphi) = \partial^N_C I(\varphi)$ and when $\partial^N_{GP} I(\varphi) = \partial^N_{CR} I(\varphi)$. The set $\partial^N_C I(\varphi)$ denotes the set of normalized Clarke-Rockafellar differentials of I at φ. Inter alia, under quasiconcavity a positive answer to this question provides a unifying framework for points 1–3. To the best of our knowledge, this problem was never directly addressed in the literature. Nevertheless, the equality $\partial^N_{GP} I(\varphi) = \partial^N_C I(\varphi)$ can be proven in the Banach space case by grouping some of the results that are scattered in the literature (see also Remark 2). In particular, Penot [22, Proposition 16] yields that the set of Greenberg-Pierskalla differentials at φ is contained in the cone generated by the Clarke differentials at φ. As for the opposite inclusion, it can be derived as a consequence of Daniilidis, Hadjisavvas, and Martinez-Legaz [11, Proposition 7 and Corollary 12]. By further using some of our results (see Lemma 2), one can also derive that the equality holds for the normalized sets, that is, $\partial^N_{GP} I(\varphi) = \partial^N_C I(\varphi)$. In an earlier paper (see Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini [7, Theorem 2]), not aware of [22] and [11], we proved directly this latter equality for the non-Banach space $B_0(S, \Sigma, \mathbb{R})$.

Here, we address the problem in its full generality. To the best of our knowledge, the only relevant result is Daniilidis, Hadjisavvas, and Martinez-Legaz [11, Proposition 7 and Corollary 12] which only yields the inclusion $\partial_{GP} I(\varphi) \supseteq \partial_{CR} I(\varphi) \setminus \{0\}$ and in the particular case the domain is a Banach space.

2 Mathematical preliminaries

2.1 Basic notions

We refer to any functional analysis textbook (e.g., Aliprantis and Border [1]) for the definitions of normed space $(X, \| \|)$ and ordered vector space $(X, \geq)$. As usual, if X is a normed space, we denote by X^* its norm dual. If X is ordered, then we denote by $X_+ = \{x \in X : x \geq 0\}$ its positive cone. If it is both normed and ordered, we denote by $X^*_+ = \{\xi \in X^* : \xi(X_+) = [0, \infty)\}$. An ordered vector space $(X, \geq)$ is an Archimedean Riesz space with unit if and only if

- *Archimedean property*: $0 \leq nx \leq y$ for all $n \in \mathbb{N}$ implies $x = 0$;
- *Riesz property*: X is a lattice with respect to $\geq$;
- *Existence of a unit*: there exists $e \in X_+ \setminus \{0\}$ such that for each $x \in X$ eventually $-ne \leq x \leq ne$.

The element e is called a unit and it naturally induces a norm

$$\|x\|_e = \inf \{\lambda \in [0, \infty) : -\lambda e \leq x \leq \lambda e\} \qquad \forall x \in X.$$

The completion of X with respect to $\|\cdot\|_e$ is an AM-space with unit (see Aliprantis and Border [1, ch. 9]). In this paper, we will consider an Archimedean Riesz space with unit, X, endowed with the norm $\|\cdot\|_e$. For brevity, we will call X an M-space and denote $\|\cdot\|_e$ by $\|\cdot\|$.

We denote by $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ the dual pairing $(\xi, x) \mapsto \langle \xi, x \rangle = \xi(x)$. We endow X^* and any of its subsets with the weak* topology. We set

$$\Delta = \left\{\xi \in X^*_+ : \langle \xi, e \rangle = 1\right\}.$$

If we denote by $\|\cdot\|_*$ the dual norm, then $\|\xi\|_* = \langle \xi, e \rangle$ for all $\xi \in X^*_+$. It is then immediate to see that Δ is convex and, by the Banach-Alaoglu theorem, also compact.

Example 1 Consider a measurable space (S, Σ) where S is a nonempty set and Σ a σ-algebra. The space $B_0(S, \Sigma)$ of real valued and Σ-measurable simple functions is an M-space.[5] The order is the usual pointwise order between functions and the unit is 1_S. The norm $\|\cdot\|_e$ coincides with the supnorm. The supnorm completion of $B_0(S, \Sigma)$, that is $B(S, \Sigma)$ (the space of real valued, bounded, and Σ-measurable functions), is also an M-space. Both spaces play a central role in decision theory. In both cases, the norm dual can be identified with the set $ba(S, \Sigma)$, that is, the set of all bounded, finitely additive, and signed set functions on Σ. The set Δ coincides with the set of finitely additive probabilities.

Example 2 Consider a probability space (S, Σ, P) where S is a nonempty set, Σ a σ-algebra, and P is a probability measure on Σ. The space $L^\infty(S, \Sigma, P)$ is an M-space. The order is the usual P-a.s. pointwise order and the unit is (the equivalence class including) 1_S. The norm $\|\cdot\|_e$ coincides with the essential supnorm. This space plays an important role in mathematical finance (see, e.g., Follmer and Schied [14]). In this case, the norm dual can be identified with the set $ba(S, \Sigma, P)$, that is, the subset of elements of $ba(S, \Sigma)$ that are absolutely continuous wrt P. The set Δ coincides with the set of finitely additive probabilities that are absolutely continuous wrt P.

Example 3 Consider an Hausdorff compact space (S, τ). The space $C(S)$ of real valued continuous functions on S is an M-space. The order is the usual pointwise order and the unit is 1_S. The norm $\|\cdot\|_e$ coincides with the supnorm. In this case, the norm dual can be identified with the set $ca(S, \mathcal{B})$, that is, the set of all bounded, regular, and signed measures on the Borel σ-algebra generated by τ. The set Δ coincides with the set of regular Borel probability measures.

Given an interval $I \subseteq \mathbb{R}$, we define

$$X(I) = \{x \in X : \exists \lambda, \mu \in I \text{ such that } \lambda e \leq x \leq \mu e\}.$$

It is easy to check that $X(I)$ is convex and that it is open if and only if I is open. Since the set of positive elements X_+ is the set $\{x \in X : x \geq 0\} = X([0, \infty))$, it is then also immediate to see that X_+ has nonempty interior. We next provide a couple of ancillary lemmas.[6]

Lemma 1 *Let C_1 and C_2 be two subsets of Δ. If C_1 and C_2 are such that, given $x \in X$,*

$$\langle \xi, x \rangle \geq 0 \quad \forall \xi \in C_1 \qquad \Longrightarrow \qquad \langle \xi, x \rangle \geq 0 \quad \forall \xi \in C_2, \tag{1}$$

then $\overline{\mathrm{co}}(C_1) \supseteq \overline{\mathrm{co}}(C_2)$.

Given a set $C \subseteq X_+^*$, we define the *normalized* (version of) C by

$$C^N = \left\{\xi' \in \Delta : \xi' = \xi \backslash \|\xi\|_* \text{ for some } \xi \in C\right\}.$$

We define also $\mathrm{cone}\, C$ to be the set

$$\left\{\xi' \in X_+^* : \xi' = \lambda \xi \text{ for some } \xi \in C \text{ and } \lambda > 0\right\}.$$

Lemma 2 *Let $C \subseteq X_+^*$ be such that $0 \notin C$. The following statements are true:*

1. *If C is convex, $\mathrm{cl}\left(C^N\right)$ is convex and compact;*
2. *If C is convex and compact, C^N is convex and compact;*
3. $\mathrm{cone}\, C = \mathrm{cone}\, C^N$.

[5] In the Introduction, we also denoted this space $B_0(S, \Sigma, \mathbb{R})$.
[6] The proofs are in the Appendix.

2.2 Quasiconcave duality

Given a function $g : X(I) \to [-\infty, \infty]$, we say that g is:

1. *lower semicontinuous* if and only if $\{x \in X(I) : g(x) \leq \alpha\}$ is closed (in the relative topology) for all $\alpha \in \mathbb{R}$;

2. *upper semicontinuous* if and only if $\{x \in X(I) : g(x) \geq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$;

3. *continuous* if and only if g is lower and upper semicontinuous;

4. *monotone* if and only if $x \geq y$ implies $g(x) \geq g(y)$;

5. *quasiconcave* if and only if $\{x \in X(I) : g(x) \geq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

Given a function $G : \mathbb{R} \times \Delta \to [-\infty, \infty]$, we say that G is:[7]

1. *lower semicontinuous* if and only if $\{(t, \xi) \in \mathbb{R} \times \Delta : G(t, \xi) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$;

2. *quasiconvex* if and only if $\{(t, \xi) \in \mathbb{R} \times \Delta : G(t, \xi) \leq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

An important function from $\mathbb{R} \times \Delta$ to $[-\infty, \infty]$ is the function

$$(t, \xi) \mapsto G_\xi(t) = \sup\{g(y) : y \in X(I) \text{ and } \langle \xi, y \rangle \leq t\}.$$

If $I = \mathbb{R}$ and g is monotone, then it is easy to see that $X(I) = X$ and

$$G_\xi(t) = \sup\{g(y) : \langle \xi, y \rangle = t\} \qquad \forall (t, \xi) \in \mathbb{R} \times \Delta. \tag{2}$$

Proposition 1 *Let I be an open interval. If $g : X(I) \to [-\infty, \infty]$ is monotone and quasiconcave, then*

1. *$(t, \xi) \mapsto G_\xi(t)$ is quasiconvex over $\mathbb{R} \times \Delta$;*

2. *If g is also lower semicontinuous, $t \mapsto G_\xi(t)$ is monotone for each $\xi \in \Delta$;*

3. *If g is also lower semicontinuous, $(t, \xi) \mapsto G_\xi(t)$ is lower semicontinuous;*

4. *If g is also lower semicontinuous and real valued, then*

$$g(x) = \min_{\xi \in \Delta} G_\xi(\langle \xi, x \rangle) \qquad \forall x \in X(I).$$

Proof. See [5] and [6]. In particular, point 1 follows from [5, Lemma 31]. Points 2 and 3 follow from [5, Lemma 32]. Point 4 follows from [5, Theorem 36]. ■

A notion of superdifferential which comes from quasiconvex analysis is the one due to Greenberg and Pierskalla [17]. A functional $\xi \in X_+^*$ is a Greenberg-Pierskalla (super)differential at $x \in X(I)$ if and only if for each $y \in X(I)$

$$\langle \xi, y \rangle \leq \langle \xi, x \rangle \quad \Longrightarrow \quad g(y) \leq g(x).$$

We denote the collection of Greenberg-Pierskalla differentials at x by $\partial_{GP} g(x)$. An interesting subset of $\partial_{GP} g(x)$ is the set of normalized Greenberg-Pierskalla differentials

$$\partial_{GP}^N g(x) = \Delta \cap \partial_{GP} g(x).$$

This subset is particularly important in decision theory (see, e.g., [25, p. 1169], [18], [5, p. 1288] and [7]).

This notion of differential is an ordinal notion well suited for quasiconcave functions, in fact:

[7] We endow $\mathbb{R} \times \Delta$ with the product topology.

1. $\partial_{GP} g(x)$ is a cone, that is, if $\xi \in \partial_{GP} g(x)$, then $\lambda\xi \in \partial_{GP} g(x)$ for all $\lambda > 0$;

2. If $f : [-\infty, \infty] \to [-\infty, \infty]$ is strictly increasing, then $\partial_{GP} g(x) = \partial_{GP}(f \circ g)(x)$;

3. If g is real valued, monotone, and lower semicontinuous, then g is quasiconcave if and only if $\partial^N_{GP} g(x) \neq \emptyset$ for all $x \in X(I)$.

Proposition 2 *Let I be an open interval. If $g : X(I) \to \mathbb{R}$ is monotone, quasiconcave, and lower semicontinuous, then for each $x \in X(I)$*

$$\tilde{\xi} \in \partial^N_{GP} g(x) \quad \iff \quad \tilde{\xi} \in \operatorname{argmin} G_\xi(\langle \xi, x\rangle).$$

In particular, $\partial^N_{GP} g(x)$ is nonempty, convex, and compact.

Proof. Let $x \in X(I)$. Observe that for each $\xi \in \Delta$

$$G_\xi(\langle \xi, x\rangle) = \sup\{g(y) : y \in X(I) \text{ and } \langle \xi, y\rangle \leq \langle \xi, x\rangle\} \geq g(x).$$

If $\tilde{\xi} \in \partial^N_{GP} g(x)$, then $g(y) \leq g(x)$ for all $y \in X(I)$ such that $\left\langle \tilde{\xi}, y\right\rangle \leq \left\langle \tilde{\xi}, x\right\rangle$. This implies that $G_{\tilde{\xi}}\left(\left\langle \tilde{\xi}, x\right\rangle\right) \leq g(x)$, that is, $G_{\tilde{\xi}}\left(\left\langle \tilde{\xi}, x\right\rangle\right) = g(x)$. By Proposition 1, we can conclude that $\tilde{\xi} \in \operatorname{argmin} G_\xi(\langle \xi, x\rangle)$. Viceversa, if $\tilde{\xi} \in \operatorname{argmin} G_\xi(\langle \xi, x\rangle)$, then

$$\sup\left\{g(y) : y \in X(I) \text{ and } \left\langle \tilde{\xi}, y\right\rangle \leq \left\langle \tilde{\xi}, x\right\rangle\right\} = G_{\tilde{\xi}}\left(\left\langle \tilde{\xi}, x\right\rangle\right) = g(x),$$

proving that if $y \in X(I)$ and $\left\langle \tilde{\xi}, y\right\rangle \leq \left\langle \tilde{\xi}, x\right\rangle$, then $g(y) \leq g(x)$, that is, $\tilde{\xi} \in \partial^N_{GP} g(x)$.

By Proposition 1 and since Δ is convex and compact and $(t, \xi) \mapsto G_\xi(t)$ is quasiconvex and lower semicontinuous, we have that $\xi \mapsto G_\xi(\langle \xi, x\rangle)$ is quasiconvex and lower semicontinuous and

$$\partial^N_{GP} g(x) = \operatorname{argmin} G_\xi(\langle \xi, x\rangle) \neq \emptyset$$

is convex and compact. ■

Lemma 3 *Let I be an open interval and $g : X(I) \to \mathbb{R}$ a monotone and lower semicontinuous function. If g is such that*

$$t > 0 \text{ and } x, x + te \in X(I) \implies g(x + te) > g(x), \tag{3}$$

then $\xi \in \partial^N_{GP} g(x)$ if and only if

$$y \in X(I) \text{ and } \langle \xi, y\rangle < \langle \xi, x\rangle \quad \implies g(y) < g(x). \tag{4}$$

Proof. Assume ξ satisfies (4). Next, consider $y \in X(I)$ such that $\langle \xi, y\rangle \leq \langle \xi, x\rangle$. If $\langle \xi, y\rangle < \langle \xi, x\rangle$, then $g(y) \leq g(x)$. If $\langle \xi, y\rangle = \langle \xi, x\rangle$, then for $\bar{n} \in \mathbb{N}$ large enough $y - \frac{1}{\bar{n}}e \in X(I)$. It follows that $\{y_n\}_{n \geq \bar{n}}$ defined as $y_n = y - \frac{1}{n}e \in X(I)$ is such that $\langle \xi, y_n\rangle < \langle \xi, x\rangle$ for all $n \geq \bar{n}$. By (4), it follows that $g(y_n) \leq g(x)$ for all $n \geq \bar{n}$. By passing to the limit and since g is lower semicontinuous, we have that $g(y) \leq g(x)$. We can conclude that $\xi \in \partial^N_{GP} g(x)$. Assume now $\xi \in \partial^N_{GP} g(x)$. Consider $y \in X(I)$ such that $\langle \xi, y\rangle < \langle \xi, x\rangle$. If we choose $t > 0$ small enough, we have that $y + te \in X(I)$ and $\langle \xi, y + te\rangle \leq \langle \xi, x\rangle$. Since $\xi \in \partial^N_{GP} g(x)$ and g satisfies (3), this implies that $g(y) < g(y + te) \leq g(x)$, proving that ξ satisfies (4). ■

2.3 Nonsmooth differentials

Consider a continuous functional $g : X(I) \to \mathbb{R}$ and $x \in \operatorname{int} X(I)$. Define the

(i) *Clarke-Rockafellar upper (directional) derivative* $g^{\uparrow}(x;\cdot) : X \to [-\infty,\infty]$ at x by:

$$g^{\uparrow}(x;y) = \limsup_{\substack{x' \to x \\ t \downarrow 0}} \inf_{y' \to y} \frac{g(x'+ty') - g(x')}{t} \qquad \forall y \in X,$$

(ii) *Clarke upper (directional) derivative* $g^{\circ}(x;\cdot) : X \to [-\infty,\infty]$ at x by:

$$g^{\circ}(x;y) = \limsup_{\substack{x' \to x \\ t \downarrow 0}} \frac{g(x'+ty) - g(x')}{t} \qquad \forall y \in X,$$

(iii) *Clarke lower (directional) derivative* $g_{\circ}(x;\cdot) : X \to [-\infty,\infty]$ at x by:

$$g_{\circ}(x;y) = \liminf_{\substack{x' \to x \\ t \downarrow 0}} \frac{g(x'+ty) - g(x')}{t} \qquad \forall y \in X,$$

(iv) *Clarke-Rockafellar lower (directional) derivative* $g^{\downarrow}(x;\cdot) : X \to [-\infty,\infty]$ at x by:

$$g^{\downarrow}(x;y) = \liminf_{\substack{x' \to x \\ t \downarrow 0}} \sup_{y' \to y} \frac{g(x'+ty') - g(x')}{t} \qquad \forall y \in X.$$

The (possibly empty) *Clarke-Rockafellar differential* $\partial_{CR} g(x)$ at x is defined as

$$\partial_{CR} g(x) = \left\{ \xi \in X^* : \forall y \in X \quad \langle \xi, y \rangle \leq g^{\uparrow}(x;y) \right\}.$$

We will say that g is *Clarke-Rockafellar differentiable at* x if and only if $\partial_{CR} g(x) \neq \emptyset$. Similarly, g is *Clarke-Rockafellar differentiable* if and only if $\partial_{CR} g(x) \neq \emptyset$ for all $x \in X(I)$. If g is locally Lipschitz at x, then $g^{\circ}(x;\cdot)$ is a finite, sublinear, continuous functional and g is *Clarke differentiable at* x, that is, $\partial_C g(x) \neq \emptyset$ where

$$\partial_C g(x) = \{ \xi \in X^* : \forall y \in X \quad \langle \xi, y \rangle \leq g^{\circ}(x;y) \}.$$

Remark 1 Rockafellar [26] defines $g^{\uparrow}(x;y)$, $g^{\downarrow}(x;y)$, and $\partial_{CR} g(x)$, for a function g defined over the entire space X, that is, $g : X \to [-\infty,\infty]$. The definitions of $\limsup\inf$ and $\liminf\sup$ and their characterizations at [26, p. 260] show that these are local notions. Thus, if g is defined on $X(I)$ with I open, any extension of g will have the same directional derivatives and differential at $x \in X(I)$. In the proof of Lemmas 4 and 5, we will implicitly use the extension $\hat{g} : X \to [-\infty,\infty]$ such that

$$\hat{g}(x) = \sup \{ g(y) : X(I) \ni y \leq x \} \qquad \forall x \in X.$$

We adopt the usual convention $sup\emptyset = -\infty$. If g is real valued, monotone, and continuous, its extension $\hat{g}$ is monotone and lower semicontinuous. If g is also quasiconcave, then $\hat{g}$ is quasiconcave (see the proof of [5, Theorem 36]).

The next lemma collects few useful facts about Clarke-Rockafellar differentials contained in Rockafellar [26].[8]

[8] The proof is in the Appendix.

Lemma 4 *Let I be an open interval. If $g : X(I) \to \mathbb{R}$ is monotone, continuous, and Clarke-Rockafellar differentiable at $x \in X(I)$, then $\partial_{CR} g(x)$ is a convex and closed subset of X_+^*,*

$$\partial_{CR} g(x) = \left\{\xi \in X^* : \forall y \in X \quad \langle \xi, y\rangle \geq g^{\downarrow}(x; y)\right\} = \{\xi \in X^* : \forall y \in X \quad \langle \xi, y\rangle \leq g^{\circ}(x; y)\}, \tag{5}$$

and

$$-\infty < g^{\uparrow}(x; y) = \sup_{\xi \in \partial_{CR} g(x)} \langle \xi, y\rangle \leq g^{\circ}(x; y) \qquad \forall y \in X.$$

In particular, $y \mapsto g^{\circ}(x; y)$ is a monotone, lower semicontinuous at 0, and hypolinear functional such that $g^{\circ}(x; 0) = 0$.

3 Results

Before stating the main results, we need an ancillary lemma.

Lemma 5 *Let I be an open interval. If $g : X(I) \to \mathbb{R}$ is monotone, continuous, and Clarke-Rockafellar differentiable at x, and such that $0 \notin \partial_{CR} g(x)$, then*

$$\sup_{\xi \in \partial_{CR} g(x)} \langle \xi, -e\rangle = g^{\circ}(x; -e) < 0.$$

In particular, g satisfies (3).

Proof. By Lemma 4 and since g is monotone, continuous, and Clarke-Rockafellar differentiable at x, we have that $-\infty < g^{\uparrow}(x; -e) \leq g^{\circ}(x; -e) \leq g^{\circ}(x; 0) \leq 0$. By an inspection of the proof of [26, Proposition 4], it follows that g is directionally Lipschitzian at x wrt each $y \in -\operatorname{int} X_+$, thus, in particular, wrt $-e$. By [26, Theorem 3], we have that $g^{\uparrow}(x; -e) = g^{\circ}(x; -e) \leq 0$. By Lemma 4, it follows that

$$0 \geq g^{\circ}(x; -e) = g^{\uparrow}(x; -e) = \sup_{\xi \in \partial_{CR} g(x)} \langle \xi, -e\rangle.$$

By contradiction, assume that $g^{\circ}(x; -e) = 0$, that is,

$$\sup_{\xi \in \partial_{CR} g(x)} \langle \xi, -e\rangle = 0.$$

Consider $\beta = -1$. It follows that the set

$$C = \{\xi \in \partial_{CR} g(x) : \langle \xi, -e\rangle \geq \beta\} \neq \emptyset.$$

Since $\partial_{CR} g(x)$ is closed and $\partial_{CR} g(x) \subseteq X_+^*$, it is immediate to see that C is closed and contained in

$$\left\{\xi \in X_+^* : \langle \xi, e\rangle \leq 1\right\}.$$

Since this latter set is convex, closed, and $\| \|_*$ bounded, we can conclude that it is compact and so is C. It follows that $\partial_{CR} g(x)$ is nonasymptotic relative to $-e$. By [26, Theorem 6], it follows that

$$g^{\circ}(x; -e) = \max_{\xi \in \partial_{CR} g(x)} \langle \xi, -e\rangle. \tag{6}$$

Thus, there exists $\bar{\xi} \in \partial_{CR} g(x)$ such that $\langle \bar{\xi}, e\rangle = 0$. Since $\bar{\xi} \in X_+^*$ and e is an order unit, it follows that $\bar{\xi} = 0$, a contradiction with $0 \notin \partial_{CR} g(x)$.

By contradiction, assume that (3) is violated. By working hypothesis and since g is monotone, there exist $z \in X(I)$ and $\bar{t} > 0$ such that $z + \bar{t}e \in X(I)$ and $g(z + \bar{t}e) = g(z)$. If we call $x = z + \bar{t}e$, then $z = x - \bar{t}e$. By definition of $X(I)$ and since g is monotone, this implies that $x - te \in X(I)$ and $g(x) \geq g(x - te) \geq g(x - \bar{t}e) = g(x)$, that is, $g(x) = g(x - te)$ for all $t \in (0, \bar{t})$. We can conclude that $g^{\circ}(x; -e) \geq 0$, a contradiction. ■

Proposition 3 *Let I be an open interval. If $g : X(I) \to \mathbb{R}$ is monotone, continuous, quasiconcave, and Clarke-Rockafellar differentiable at $x \in X(I)$, then*

$$\partial^N_{CR} g(x) \subseteq \partial^N_{GP} g(x). \tag{7}$$

Moreover, if $0 \notin \partial_{CR} g(x)$, then

$$\partial^N_{GP} g(x) \subseteq \mathrm{cl}\left(\partial^N_{CR} g(x)\right). \tag{8}$$

Proof. Consider $\bar{\xi} \in \partial^N_{CR} g(x)$. By definition, there exists $\xi \in X^*$ such that $0 \neq \xi \in \partial_{CR} g(x)$ and $\bar{\xi} = \xi / \|\xi\|_* \in \Delta$. Consider now $y \in X(I)$. We prove two facts:

1. $\langle \bar{\xi}, y \rangle < \langle \bar{\xi}, x \rangle \implies g(y) \leq g(x)$. Define $\varepsilon = \langle \xi, x - y \rangle$. By assumption, we have that $\varepsilon > 0$. Since $\xi \in \partial_{CR} g(x)$, $g^{\uparrow}(x; x - y) \geq \langle \xi, x - y \rangle = \varepsilon > 0$. By definition of $g^{\uparrow}(x; x - y)$ (see also Rockafellar [26, p. 260]), we have that there exist $\{t_n\}_{n \in \mathbb{N}} \subseteq (0, \infty)$, $\{x_n\}_{n \in \mathbb{N}} \subseteq X(I)$ and $\{z_n\}_{n \in \mathbb{N}} \subseteq X$ such that for n large enough
$$\frac{g(x_n + t_n z_n) - g(x_n)}{t_n} \geq g^{\uparrow}(x; x - y) - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2} > 0$$
where $0 < t_n \to 0$, $x_n \to x$, and $z_n \to x - y$. It follows that for n large enough
$$\frac{g(x_n + t_n z_n) - g(x_n)}{t_n} \geq \frac{\varepsilon}{2}.$$
Define $\{y_n\}_{n \in \mathbb{N}}$ to be such that $y_n = x_n - z_n$ for all $n \in \mathbb{N}$. Note that $y_n \to y$. It follows that for n large enough
$$\frac{g(x_n + t_n(x_n - y_n)) - g(x_n)}{t_n} > 0.$$
We can finally conclude that for n large enough
$$g(x_n + t_n(x_n - y_n)) > g(x_n). \tag{9}$$
Define $\alpha_n = (1 + t_n)^{-1} \in (0, 1)$ for all $n \in \mathbb{N}$. Note that $x_n = \alpha_n(x_n + t_n(x_n - y_n)) + (1 - \alpha_n) y_n$ for all $n \in \mathbb{N}$. Since g is quasiconcave and by (9), we have that for n large enough
$$g(x_n) \geq \min\{g(x_n + t_n(x_n - y_n)), g(y_n)\} = g(y_n).$$
Since g is continuous, it follows that $g(y) = \lim_n g(y_n) \leq \lim_n g(x_n) = g(x)$.

2. $\langle \bar{\xi}, y \rangle = \langle \bar{\xi}, x \rangle \implies g(y) \leq g(x)$. Since $\xi \neq 0$, there exists $z \in X(I)$ such that $\langle \bar{\xi}, z \rangle < 0$. Define
$$y_n = y + \frac{1}{n} z \qquad \forall n \in \mathbb{N}.$$
Since $y \in X(I)$ and the latter set is open, note that $\{y_n\}_{n \in \mathbb{N}}$ eventually belongs to $X(I)$. It is also immediate to see that $\langle \bar{\xi}, y_n \rangle < \langle \bar{\xi}, y \rangle$ for all $n \in \mathbb{N}$. By point 1 and since g is continuous, we have that $g(y) = \lim_n g(y_n) \leq g(x)$.

By points 1 and 2, we proved that, if $y \in X(I)$ is such that $\langle \bar{\xi}, y \rangle \leq \langle \bar{\xi}, x \rangle$, then $g(y) \leq g(x)$. Thus, $\xi \backslash \|\xi\|_* = \bar{\xi} \in \partial^N_{GP} g(x)$, proving (7).

Suppose $0 \notin \partial_{CR} g(x)$. Let $\bar{\xi} \in \partial^N_{GP} g(x)$. If $y \in \ker \bar{\xi} = \{z \in X : \langle \xi, z \rangle = 0\}$, then we have that $\langle \bar{\xi}, x + ty \rangle \leq \langle \bar{\xi}, x \rangle$ for all $t \geq 0$. Since $\bar{\xi} \in \partial^N_{GP} g(x)$, this implies that $g(x + ty) \leq g(x)$ for all $t \geq 0$ such that $x + ty \in X(I)$. It follows that $g_\circ(x; y) \leq 0$. Since y was a generic element of $\ker \bar{\xi}$, it follows that $g_\circ(x; y) \leq 0$ for all $y \in \ker \bar{\xi}$. By Lemma 4, we have that $g_\circ(x; 0) = 0$. Define $\tilde{g} : X \to [-\infty, \infty)$ to be such that $\tilde{g}(y) = g_{\circ}(x;y)$ for all $y \in Y$. Define also $\bar{g} : X \to [-\infty, \infty)$ to be such that $\bar{g}(y) = \inf_{\xi \in \partial_{CR} g(x)} \langle \xi, y \rangle$ for all $y \in Y$. By Lemma 4 and [26, Theorem 3], we have that $\tilde{g}$ is monotone, hyperlinear, and such that

$$\tilde{g}(y) \leq \bar{g}(y) \qquad \forall y \in X \text{ and } \bar{g}(y) = \inf_{Y \in \mathcal{N}(y)} \sup_{y' \in Y} \tilde{g}(y').[9]$$

Note also that

$$0 = \tilde{g}(0) \leq \sup_{\langle \bar{\xi}, y \rangle = 0} \tilde{g}(y) \leq 0,$$

that is, by (2), $\tilde{G}_{\bar{\xi}}(0) = \sup_{\langle \bar{\xi}, y \rangle = 0} \tilde{g}(y) = 0 = \tilde{g}(0)$. In other words, if $\langle \bar{\xi}, y \rangle \leq 0 = \langle \bar{\xi}, 0 \rangle$, then $\tilde{g}(y) \leq \tilde{g}(0)$. This implies that

$$\tilde{g}(y) > 0 \implies \langle \bar{\xi}, y \rangle > 0. \tag{10}$$

Consider $y \in X$ such that $\bar{g}(y) > 0$. It follows that for each $n \in \mathbb{N}$ there exists $y'_n \in B_{\frac{1}{n}}(y)$ such that $\tilde{g}(y'_n) > 0$. It is immediate to see that $y'_n \to y$. By (10), we have that

$$\tilde{g}(y'_n) > 0 \quad \forall n \in \mathbb{N} \implies \langle \bar{\xi}, y'_n \rangle > 0 \quad \forall n \in \mathbb{N} \implies \langle \bar{\xi}, y \rangle \geq 0.$$

We proved that

$$\bar{g}(y) > 0 \implies \langle \bar{\xi}, y \rangle \geq 0. \tag{11}$$

By Lemma 5 and since $0 \notin \partial_{CR} g(x) \neq \emptyset$, we have that $\infty > \tilde{g}(e) = -g^{\circ}(x; -e) = \inf_{\xi \in \partial_{CR} g(x)} \langle \xi, e \rangle = \bar{g}(e) > 0$. If $\bar{g}(y) = 0$, then

$$\infty > \bar{g}\left(y + \frac{1}{n} e\right) \geq \bar{g}(y) + \bar{g}\left(\frac{1}{n} e\right) = \bar{g}(y) + \frac{1}{n} \bar{g}(e) > 0.$$

By (11), this implies that if $\bar{g}(y) = 0$, then for each $n \in \mathbb{N}$

$$\bar{g}\left(y + \frac{1}{n} e\right) > 0 \implies \left\langle \bar{\xi}, y + \frac{1}{n} e \right\rangle \geq 0,$$

that is,

$$\bar{g}(y) = 0 \implies \langle \bar{\xi}, y \rangle \geq 0.$$

We can thus conclude that

$$\bar{g}(y) \geq 0 \implies \langle \bar{\xi}, y \rangle \geq 0. \tag{12}$$

This implies that

$$\bar{g}(y) \geq 0 \iff \langle \xi, y \rangle \geq 0 \quad \forall \xi \in \partial g_{CR}(x) \iff \langle \xi, y \rangle \geq 0 \quad \forall \xi \in \partial^N_{CR} g(x).$$

By (12), it follows that

$$\langle \xi, y \rangle \geq 0 \qquad \forall \xi \in \partial^N_{CR} g(x) \implies \langle \bar{\xi}, y \rangle \geq 0.$$

By Lemma 1, we can conclude that $\bar{\xi} \in \overline{\text{co}}\left(\partial^N_{CR} g(x)\right)$. By point 1 of Lemma 2 and Lemma 4, we have that $\text{cl}\left(\partial^N_{CR} g(x)\right)$ is a convex and closed set. This yields that $\overline{\text{co}}\left(\partial^N_{CR} g(x)\right) = \text{cl}\left(\partial^N_{CR} g(x)\right)$, proving the statement. ■

Theorem 1 *Let I be an open interval. If $g : X(I) \to \mathbb{R}$ is monotone, continuous, quasiconcave, and Clarke-Rockafellar differentiable at $x \in X(I)$, with $0 \notin \partial_{CR} g(x)$, then*

$$\partial^N_{GP} g(x) = \text{cl}\left(\partial^N_{CR} g(x)\right).$$

[9] $\mathcal{N}(y)$ is the collection of open neighbourhoods of y.

Proof. By Proposition 2, $\partial^N_{GP} g(x)$ is closed. By (7), we have that $\partial^N_{CR} g(x) \subseteq \partial^N_{GP} g(x)$. Since the latter set is closed, it follows that $\mathrm{cl}\left(\partial^N_{CR} g(x)\right) \subseteq \partial^N_{GP} g(x)$. By (8), the opposite inclusion follows, proving the statement. ■

Theorem 2 *Let I be an open interval. If $g : X(I) \to \mathbb{R}$ is monotone, continuous, quasiconcave, and Clarke-Rockafellar differentiable at $x \in X(I)$, with $0 \notin \partial_{CR} g(x)$, then*

$$\mathrm{cl}\left(\partial_{GP} g(x)\right) = \mathrm{cl}\left(\mathrm{cone}\left(\partial_{CR} g(x)\right)\right).$$

Proof. First note that, g satisfies (3). This implies that $0 \notin \partial_{GP} g(x)$. By point 3 of Lemma 2 and Theorem 1, we have that

$$\partial_{GP} g(x) = \mathrm{cone}\left(\partial_{GP} g(x)\right) = \mathrm{cone}\left(\partial^N_{GP} g(x)\right) = \mathrm{cone}\left(\mathrm{cl}\left(\partial^N_{CR} g(x)\right)\right). \tag{13}$$

It follows that

$$\mathrm{cl}\left(\partial_{GP} g(x)\right) = \mathrm{cl}\left(\mathrm{cone}\left(\mathrm{cl}\left(\partial_{CR} g(x)\right)\right)\right).$$

At the same time, $\mathrm{cl}\left(\mathrm{cone}\left(\mathrm{cl}\left(\partial_{CR} g(x)\right)\right)\right) = \mathrm{cl}\left(\mathrm{cone}\left(\partial_{CR} g(x)\right)\right)$. For, it is immediate to see that

$$\mathrm{cl}\left(\mathrm{cone}\left(\mathrm{cl}\left(\partial_{CR} g(x)\right)\right)\right) \supseteq \mathrm{cl}\left(\mathrm{cone}\left(\partial_{CR} g(x)\right)\right).$$

Viceversa, observe that

$$\mathrm{cl}\left(\mathrm{cone}\left(\partial_{CR} g(x)\right)\right) \supseteq \mathrm{cl}\left(\partial_{CR} g(x)\right),$$

proving the statement. ■

Corollary 1 *Let I be an open interval. If $g : X(I) \to \mathbb{R}$ is monotone, continuous, quasiconcave, and locally Lipschitz at $x \in X(I)$, with $0 \notin \partial_C g(x)$, then*

$$\partial_{GP} g(x) = \mathrm{cone}\left(\partial_C g(x)\right) \text{ and } \partial^N_{GP} g(x) = \partial^N_C g(x).$$

Proof. By [26, Corollary 1 p.268 and Corollary 2 p.275] and since g is locally Lipschitz at x, we have that $\partial_C g(x)$ is nonempty, compact, and coincides with $\partial_{CR} g(x)$. By point 2 of Lemma 2, we have that $\partial^N_C g(x)$ is also compact. By Theorem 1, we can conclude that $\partial^N_{GP} g(x) = \partial^N_C g(x)$. Since g satisfies (3), we have that $0 \notin \partial_{GP} g(x)$. By point 3 of Lemma 2, we can conclude that $\partial_{GP} g(x) = \mathrm{cone}\left(\partial_{GP} g(x)\right) = \mathrm{cone}\left(\partial^N_{GP} g(x)\right) = \mathrm{cone}\left(\partial^N_C g(x)\right) = \mathrm{cone}\left(\partial_C g(x)\right)$. ■

Remark 2 The corollary above can be proved also using the following results. By Penot [22, Proposition 16], we have that $\partial_{GP} g(x) \subseteq \mathrm{cone}\left(\partial_C g(x)\right)$. Viceversa, in the Banach space case, by Daniilidis, Hadjisavvas, and Martinez-Legaz [11, Proposition 7 and Corollary 12], we have that $\partial_C g(x) \subseteq \partial_{GP} g(x)$), that is, $\mathrm{cone}\left(\partial_C g(x)\right) \subseteq \mathrm{cone}\left(\partial_{GP} g(x)\right) = \partial_{GP} g(x)$.

If g is strictly differentiable (in the full limit sense), then $\partial_{CR} g(x)$ is a singleton that we denote by $\nabla g(x)$ (see Rockafellar [27, Proposition 4 and p. 340]).

Corollary 2 *Let I be an open interval. If $g : X(I) \to \mathbb{R}$ is monotone, continuous, quasiconcave, and strictly differentiable, with $\nabla g(x) \neq 0$, then*

$$\partial_{GP} g(x) = \left\{\lambda \nabla g(x) : \lambda > 0\right\}.$$

Proof. By the proof of Theorem 2, we can conclude that $\partial_{GP} g(x) = \text{cone}\left(\text{cl}\left(\partial^N_{CR} g(x)\right)\right)$. Since g is strictly differentiable, $\partial_{CR} g(x)$ is a singleton. Thus, $\text{cl}\left(\partial^N_{CR} g(x)\right) = \partial^N_{CR} g(x) = \{\nabla g(x) \setminus \|\nabla g(x)\|_*\}$. By point 3 of Lemma 2, the statement follows. ■

The last corollary presents a differential characterization of quasiconcavity which generalizes the one established by Arrow and Enthoven [2] in the case of differentiable g (see also Komlosi [19, Theorem 10.4] for a characterization in terms of Dini derivatives).

Corollary 3 *Let I be an open interval and $g : X(I) \to \mathbb{R}$ a monotone, continuous, and Clarke-Rockafellar differentiable function with $0 \notin \cup_{x\in X(I)} \partial_{CR} g(x)$. The following statements are equivalent:*

(i) g is quasiconcave;

(ii) For each $x \in X(I)$ and for each $\xi \in \partial_{CR} g(x)$

$$g(y) > g(x) \quad \Longrightarrow \quad \langle \xi, y\rangle > \langle \xi, x\rangle ; \tag{14}$$

(iii) For each $x \in X(I)$ and for each $\xi \in \partial_{CR} g(x)$

$$g(y) \geq g(x) \quad \Longrightarrow \quad \langle \xi, y\rangle \geq \langle \xi, x\rangle ; \tag{15}$$

(iv) $\partial^N_{GP} g(x) \neq \emptyset$ for all $x \in X(I)$.

Proof. (i) implies (ii). Let $x \in X(I)$ and $\xi \in \partial_{CR} g(x)$. Since $0 \notin \partial_{CR} g(x)$, we have that $\xi \setminus \|\xi\|_* \in \partial^N_{CR} g(x)$. By Proposition 3, this implies that $\xi \setminus \|\xi\|_* \in \partial^N_{GP} g(x) \subseteq \partial_{GP} g(x)$. Since $\partial_{GP} g(x)$ is a cone, it follows that $\xi \in \partial_{GP} g(x)$, yielding (14).

(ii) implies (iii). Let $x \in X(I)$ and $\xi \in \partial_{CR} g(x)$. Since $\xi \neq 0$ satisfies (14), it follows that $\xi \setminus \|\xi\|_* \in \partial^N_{GP} g(x)$. By Lemma 5 and Lemma 3, we have that $\xi \setminus \|\xi\|_*$ satisfies (4), that is, it satisfies (15) and so does ξ.

(iii) implies (iv). Let $x \in X(I)$ and $\xi \in \partial_{CR} g(x)$. Since ξ satisfies (15), it follows that $\xi \setminus \|\xi\|_*$ satisfies (4). By Lemma 5 and Lemma 3, we have that $\xi \setminus \|\xi\|_* \in \partial^N_{GP} g(x)$.

(iv) implies (i). Let $x \in X(I)$. Observe that for each $\xi \in \Delta$

$$G_\xi(\langle \xi, x\rangle) = \sup\{g(y) : y \in X(I) \text{ and } \langle \xi, y\rangle \leq \langle \xi, x\rangle\} \geq g(x).$$

In particular, we have that

$$g(x) \leq \inf_{\xi\in\Delta} G_\xi(\langle \xi, x\rangle).$$

Let $\xi_x \in \partial^N_{GP} g(x) \neq \emptyset$. It follows that $G_{\xi_x}(\langle \xi_x, x\rangle) = \sup\{g(y) : y \in X(I) \text{ and } \langle \xi_x, y\rangle \leq \langle \xi_x, x\rangle\} \leq g(x)$, that is,

$$g(x) \leq \inf_{\xi\in\Delta} G_\xi(\langle \xi, x\rangle) \leq G_{\xi_x}(\langle \xi_x, x\rangle) \leq g(x).$$

Since x was arbitrarily chosen, we have that $g(x) = \inf_{\xi\in\Delta} G_\xi(\langle \xi, x\rangle)$ for all $x \in X(I)$. Since, for each $\xi \in \Delta$, $x \mapsto G_\xi(\langle \xi, x\rangle)$ is a monotonic transformation of an affine function, $x \mapsto G_\xi(\langle \xi, x\rangle))$ is quasiconcave for all $\xi \in \Delta$. This implies that g is quasiconcave. ■

4 Appendix

Proof of Lemma 1. Let $i \in \{1,2\}$. Consider $x \in X$. It is immediate to see that

$$\langle \xi, x \rangle \geq 0 \quad \forall \xi \in C_i \qquad \Longleftrightarrow \qquad \langle \xi, x \rangle \geq 0 \quad \forall \xi \in \overline{\mathrm{co}}\,(C_i)\,.$$

So in what follows, wlog, we will think of C_1 and C_2 as convex and closed. By contradiction, assume that $C_2 \not\subseteq C_1$. It follows that there exists $\bar{\xi} \in C_2$ which does not belong to C_1. By [28, Theorem 3.4] and since C_1 is convex and closed, it follows that there exists $x \in X$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ such that

$$\left\langle \bar{\xi}, x \right\rangle < \varepsilon_1 < \varepsilon_2 < \langle \xi, x \rangle \qquad \forall \xi \in C_1.$$

Since $\bar{\xi} \in \Delta$ and $C_1 \subseteq \Delta$, if we choose $\hat{x} = x - \varepsilon_2 e$, it follows that there exists $\varepsilon > 0$ such that

$$\left\langle \bar{\xi}, \hat{x} \right\rangle < -\varepsilon < 0 < \langle \xi, \hat{x} \rangle \qquad \forall \xi \in C_1,$$

but $\hat{x}$ violates (1), a contradiction. ■

Proof of Lemma 2. 1. Consider $\xi_1', \xi_2' \in C^N$. By construction, there exist $\xi_1, \xi_2 \in C$ such that $\xi_i' = \xi_i \backslash \left\| \xi_i \right\|_*$ for $i \in \{1,2\}$. Define $\lambda_i = \left\| \xi_i \right\|_* > 0$ for $i \in \{1,2\}$. Since C is convex, it follows that

$$\xi = \frac{\lambda_2}{\lambda_1 + \lambda_2}\xi_1 + \frac{\lambda_1}{\lambda_1 + \lambda_2}\xi_2 \in C.$$

At the same time, $\left\| \xi \right\|_* = \langle \xi, e \rangle = 2\lambda_1\lambda_2 / \left(\lambda_1 + \lambda_2\right) > 0$. We thus have that

$$C \ni \frac{\xi}{\left\| \xi \right\|_*} = \frac{\lambda_2 \xi_1}{2\lambda_1\lambda_2} + \frac{\lambda_1 \xi_2}{2\lambda_1\lambda_2} = \frac{1}{2}\xi_1' + \frac{1}{2}\xi_2',$$

proving that C^N is a midpoint convex set. It is routine to check that $\mathrm{cl}\left(C^N\right)$ is also midpoint convex. By [29, p. 701], we can conclude that $\mathrm{cl}\left(C^N\right)$ is convex and closed. Since Δ is compact, so is $\mathrm{cl}\left(C^N\right) \subseteq \Delta$.

2. In light of point 1, it is enough to prove that C^N is closed. Consider a net $\left\{\xi_\alpha'\right\}_{\alpha \in A} \subseteq C^N$ such that $\xi_\alpha' \to \xi'$. By construction, there exists a net $\{\xi_\alpha\}_{\alpha \in A} \subseteq C$ such that $\xi_\alpha' = \xi_\alpha \backslash \left\| \xi_\alpha \right\|_*$ for all $\alpha \in A$. Since C is compact, there exists a subnet $\left\{\xi_{\alpha_\beta}\right\}_{\beta \in B} \subseteq \{\xi_\alpha\}_{\alpha \in A}$ such that $\xi_{\alpha_\beta} \to \bar{\xi} \in C$. In particular, since $C \subseteq X_+^*$, we have that $\left\| \xi_{\alpha_\beta} \right\|_* = \left\langle \xi_{\alpha_\beta}, e \right\rangle \to \left\langle \bar{\xi}, e \right\rangle = \left\| \bar{\xi} \right\|_*$. Since $0 \notin C$ and $\bar{\xi} \in C$, we have that $\left\| \bar{\xi} \right\|_* \neq 0$. It follows that $\xi_{\alpha_\beta}' = \xi_{\alpha_\beta} \backslash \left\| \xi_{\alpha_\beta} \right\|_* \to \bar{\xi} \backslash \left\| \bar{\xi} \right\|_*$. By the uniqueness of the limit and since $\xi_\alpha' \to \xi'$, we can conclude that

$$\xi_{\alpha_\beta}' \to \xi' = \frac{\bar{\xi}}{\left\| \bar{\xi} \right\|_*} \in C^N,$$

proving the statement.

3. Let $\xi'' \in \mathrm{cone}\, C$. By construction, it follows that there exist $\xi \in C$ and $\lambda > 0$ such that $\xi'' = \lambda \xi$. Since $0 \notin C$, we have that $\left\| \xi \right\|_* > 0$. It follows that $C^N \ni \xi' = \xi \backslash \left\| \xi \right\|_*$ and $\lambda \left\| \xi \right\|_* > 0$. We can conclude that $\xi'' = \left(\lambda \left\| \xi \right\|_*\right) \xi'$, proving that $\xi'' \in \mathrm{cone}\, C^N$. Viceversa, let $\xi'' \in \mathrm{cone}\, C^N$. By construction, it follows that there exist $\xi' \in C^N$ and $\lambda > 0$ such that $\xi'' = \lambda \xi'$. At the same time, there exists $\xi \in C$ such that $\xi' = \xi \backslash \left\| \xi \right\|_*$. It follows that $\lambda \backslash \left\| \xi \right\|_* > 0$. We can conclude that $\xi'' = \left(\frac{\lambda}{\left\| \xi \right\|_*}\right) \xi$, proving that $\xi'' \in \mathrm{cone}\, C$. ■

Proof of Lemma 4. By Rockafellar [26, Theorem 4], $\partial_{CR} g\,(x)$ is convex and closed. By Rockafellar [26, Corollary 3] and since g is monotone and finite at x, $\partial_{CR} g\,(x) \subseteq X_+^*$. By Rockafellar [26, Proposition 4] and since g is monotone and finite at x, g is directionally Lipschitzian at x. By [26, Theorem 6], equation (5) follows. By [26, Theorem 4] and (5) and since g is Clarke-Rockafellar differentiable at x, it follows that

$$-\infty < g^{\uparrow}(x; y) = \sup_{\xi \in \partial_{CR} g(x)} \langle \xi, y \rangle \leq g^{\circ}(x; y) \qquad \forall y \in Y.$$

Since g is monotone, it is immediate to verify that $g^\circ(x;y) \leq 0$ for all $y \in -X_+$. In particular, $g^\circ(x;0) \leq 0$. By [26, Theorem 3], we can conclude that $y \mapsto g^\circ(x;y)$ is hypolinear.[10] By [3, Lemma 1.7], it follows that $y \mapsto g^\circ(x;y)$ is monotone. By [3, Theorem 5.1], it follows that $y \mapsto g^\circ(x;y)$ is also lower semicontinuous at 0. ■

References

[1] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*, 3rd ed., Springer Verlag, Berlin, 2006.

[2] K. J. Arrow and A. C. Enthoven, Quasi-concave programming, *Econometrica*, 29, 779–800, 1961.

[3] B. Anger and J. Lembcke, Hahn-Banach type theorems for hypolinear functionals on preordered topological vector spaces, *Pacific Journal of Mathematics*, 54, 13–33, 1974.

[4] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi, Rational preferences under ambiguity, *Economic Theory*, 48, 341–375, 2011.

[5] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, *Journal of Economic Theory*, 146, 1275–1330, 2011.

[6] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Complete monotone quasiconcave duality, *Mathematics of Operations Research*, 36, 321–339, 2011.

[7] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and A. Rustichini, Variational preferences and their structure, *Journal of Mathematical Economics*, 57, 12–19, 2015.

[8] J.-P. Crouzeix, Polaires quasi-convexes et dualité, *Compte Rendue, Acad. Sci. Paris Series A*, 279, 955–958, 1974.

[9] J.-P. Crouzeix, Conditions for convexity of quasiconvex functions, *Mathematics of Operations Research*, 5, 120–125, 1980.

[10] J.-P. Crouzeix, Duality between direct and indirect utility functions. Differentiability properties, *Journal of Mathematical Economics*, 12, 149–165, 1983.

[11] A. Daniilidis, N. Hadjisavvas, and J.-E. Martinez-Legaz, An appropriate subdifferential for quasiconvex functions, *SIAM Journal on Optimization*, 12, 407–420, 2002.

[12] B. de Finetti, Sulle stratificazioni convesse, *Annali di Matematica Pura e Applicata*, 30, 173–183, 1949.

[13] W. Fenchel, Convex cones, sets, and functions, *mimeo*, Princeton University, Princeton, 1953.

[14] H. Follmer and A. Schied, *Stochastic finance: An introduction in discrete time*, 2nd ed., De Gruyter, Berlin, 2004.

[15] P. Ghirardato, F. Maccheroni, and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 133–173, 2004.

[10] That is, 1) $g^\circ(x;y) > -\infty$ for all $y \in X$; 2) $g^\circ(x;y+z) \leq g^\circ(x;y) + g^\circ(x;z)$ for all $y,z \in X$; 3) $g^\circ(x;\lambda y) = \lambda g^\circ(x;y)$ for all $\lambda \geq 0$ and for all $y \in X$. By [26, Theorem 3], we have that 3) holds for all $y \in X$ and all $\lambda > 0$. Since $0 \geq g^\circ(x;0) > -\infty$, this implies that $g^\circ(x;0) = 0$, yielding that 3) holds also for $\lambda = 0$.

[16] P. Ghirardato and M. Siniscalchi, Ambiguity in the small and in the large, *Econometrica*, 80, 2827–2847, 2012.

[17] H. P. Greenberg and W. P. Pierskalla, Quasiconjugate functions and surrogate duality, *Cahiers du Centre d'Etude de Recherche Operationelle*, 15, 437–448, 1973.

[18] E. Hanany and P. Klibanoff, Updating ambiguity averse preferences, *The B. E. Journal of Theoretical Economics (Advances)*, 9, Article 37, 2009.

[19] S. Komlosi, Generalized convexity and generalized derivatives, in *Handbook of Generalized Convexity and Generalized Monotonicity*, eds. N. Hadjisavvas, S. Komlosi, and S. Schaible, Springer, Boston, 2005.

[20] J.-E. Martinez-Legaz, A generalized concept of conjugation, in *Optimization Theory and Algorithmsm*, eds. A. Auslander, J. B. Hiriart-Urruty, and W. Oetti, Marcel Dekker, New York, 1983.

[21] J.-E. Martinez-Legaz, Duality between direct and indirect utility functions under minimal hypothesis, *Journal of Mathematical Economics*, 20, 199–209, 1991.

[22] J.-P. Penot, Are generalized derivatives useful for generalized convex functions?, in *Generalized Convexity, Generalized Monotonicity: Recent Results*, eds. J.-P. Crouzeix, J.-E. Martinez-Legaz, and M. Volle, Kluwer Academic Publishers, Dordrecht, 1998.

[23] J.-P. Penot, What is quasiconvex analysis?, *Optimization*, 47 35–110, 2000.

[24] J.-P. Penot and M. Volle, On quasi-convex duality, *Mathematics of Operations Research,* 15, 597–625, 1990.

[25] L. Rigotti, C. Shannon, and T. Strzalecki, Subjective beliefs and ex-ante trade, *Econometrica*, 76, 1167–1190, 2008.

[26] R. T. Rockafellar, Generalized directional derivatives and subgradients of nonconvex functions, *Canadian Journal of Mathematics*, 32, 257–280, 1980.

[27] R. T. Rockafellar, Directionally Lipschitzian functions and subdifferential calculus, *Proceedings of the London Mathematical Society*, 39, 331–355, 1979.

[28] W. Rudin, *Functional analysis*, 2nd ed., McGraw-Hill, Boston, 1991.

[29] E. Schechter, *Handbook of analysis and its foundations*, Academics Press, San Diego, 1996.