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A Market Foundation for Conditional Asset Pricing

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Abstract

Hansen and Richard (1987) prove a classic representation theorem for *prices of payoffs* in a conditional asset market. In this note we study the *portfolio formation* and *portfolio pricing rules* that ensure that the prices of payoffs generated by portfolios actually satisfy the assumptions of their representation theorem. In this way, we obtain a fundamental theorem of finance for conditional asset pricing.

1 Introduction and preliminaries

1.1 Purpose

In their seminal paper Hansen and Richard (1987) investigated the role of conditioning information in asset pricing. In a two-period setup, they suppose that agents trade at time t assets that will pay some uncertain payoffs at a subsequent time T. A pricing function maps such uncertain payoffs at time T into their prices at time t. Between payoffs and prices there is a key difference in measurability. Payoffs are measurable with respect to the information available at time T (when the payoff relevant uncertainty will be resolved). Prices, in contrast, are measurable with respect to the earlier (and so coarser) information available at time t and upon which assets are traded and portfolio decisions are made. This measurability gap is the distinguishing feature, both conceptually and mathematically, of conditional asset pricing.

Hansen and Richard (1987, HR hereafter) carry out their analysis directly in terms of assets' payoffs (that is, contingent claims).[1] Our purpose is to explicitly model the underlying financial market, with its portfolio operations, and to derive a fundamental theorem of finance under conditional information that complements the representation theorem that HR establish in their payoff setup. Both in HR and here, the conditional setting significantly complicates matters and requires the use of Hilbert modules in place of Hilbert spaces.[2]

[1] Throughout we use the terms "payoffs" and "contingent claims" interchangeably.

[2] On Hilbert modules, see e.g. Cerreia-Vioglio et al. (2015).

1.2 Setup

Uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Two sigma algebras, say $\mathcal{G}_t \subseteq \mathcal{G}_T \subseteq \mathcal{F}$, represent the public information available at times t and T. The set of all contingent claims (marketed and not) consist of all conditionally square integrable random variables

$$H = \left\{ y \in L_0 \left(\mathcal{G}_T \right) : \mathrm{E}[y^2 \mid \mathcal{G}_t] < \infty \right\}$$

where $L_0 \left(\mathcal{G}_T \right)$ denotes the set of all $\mathcal{G}_T$-measurable random variables.[3]

Heuristically, H has all the properties of a standard L_2-space where $\mathcal{G}_t$-measurable random variables play the role of scalars, and standard expectations are replaced by conditional ones. Formally, H is a conditional L_2-space, an instance of an Hilbert module, with (random) norm $\|y\|_2 = \sqrt{\mathrm{E}[|y|^2 \mid \mathcal{G}_t]}$ and (scalar) metric $d_2 \left(y, y' \right) = \mathrm{E} \left[\|y - y'\|_2 \wedge 1 \right]$ for all $y, y' \in H$.[4] In particular, a sequence $\{y_k\}$ in H converges to y_∞ if and only if $\|y_k - y_\infty\|_2$ converges in probability to 0.

1.3 A representation theorem

Consider the set $M \subseteq H$ of contingent claims that are actually marketed at time t and a *payoff pricing function* $\pi : M \to L_0 \left(\mathcal{G}_t \right)$. The pair (M, π) is the *payoff market*, a convenient reduced form that does not explicitly model the trading operations at t that deliver marketed payoffs at T. HR consider payoff markets and make a few assumptions on them.

A.1 (Structure) *M is d_2-complete and $y, y' \in M$ implies $wy + w'y' \in M$ for all $w, w' \in L_0 \left(\mathcal{G}_t \right)$.*[5]

A.2 (Linearity) *$\pi \left(wy + w'y' \right) = w\pi \left(y \right) + w'\pi \left(y' \right)$ for all $w, w' \in L_0 \left(\mathcal{G}_t \right)$ and all $y, y' \in M$.*

A.3 (Continuity) *π is continuous at 0.*

A.4 (Nontriviality) *There exists $y_0 \in M$ such that $\pi \left(y_0 \right) \neq 0$.*

The nature of these assumptions is mathematical because of the reduced nature of payoff markets. Their financial underpinning will be provided by Lemma 1.

The starting point of the HR analysis is the following important result (HR, Theorem 2.1) that relies on a Riesz representation theorem for Hilbert modules that they are able to establish.

[3] Inequalities among random variables are tacitly assumed to hold $\mathbb{P}$-almost everywhere; for instance, $y < y'$ means that $\mathbb{P} \left(\{ \omega \in \Omega : y \left(\omega \right) < y' \left(\omega \right) \} \right) = 1$. To ease notation we omit the subscript $\mathbb{P}$ in the conditional expectations.

[4] If $t = T$, then H reduces to $L_0 \left(\mathcal{G}_t \right)$, $\| \ \|_2$ to the modulus, and d_2 to the metric of convergence in probability. We will always consider these metrics on H and $L_0 \left(\mathcal{G}_t \right)$.

[5] Completeness here is in the metric sense. Financial completeness requires, instead, that all contingent claims are marketed, i.e., $M = H$. Financial completeness implies A.1, but the converse is false (HR, p. 592). Note that H and M are, respectively, denoted by P^+ and P in HR.

Theorem 1 (Hansen and Richard) *If a payoff market* (M, π) *satisfies assumptions A.1-A.3, then there exists a unique marketed payoff* $m^* \in M$ *such that*

$$\pi(y) = \mathrm{E}[m^* y \mid \mathcal{G}_t] \qquad \forall y \in M. \tag{1}$$

Furthermore, if also A.4 is satisfied, then $\|m^*\| > 0$.

The payoff m^* is the (*marketed*) *stochastic discount factor.* This theorem, which says that the price of a payoff is its discounted conditional expectation, has far reaching financial consequences, as HR showed (see Cochrane, 2005, for a textbook exposition).

2 Results

2.1 Market

Theorem 1 is what Ross (2005) calls a "Representation Theorem," a result that describes how a payoff pricing function π that satisfies some assumptions (here A.1-A.4) can be represented. In contrast, a "Fundamental Theorem of Finance" describes the portfolio formation and pricing rules that ensure that such assumptions hold for the resulting pricing function π. The aim of the present note is to establish a fundamental theorem of finance for conditional asset pricing.

Specifically, we assume that at time t, a finite number of primary assets, which will pay off at the subsequent time T, are traded. Because of the measurability gap, the price p^i of each asset $i \in I = \{1, ..., n\}$ is an element of $L_0(\mathcal{G}_t)$; in contrast, its uncertain time T payoff y^i is an element of H.

A *portfolio* x *at time* t is a vector $x = (x^1, x^2, ..., x^n) \in L_0(\mathcal{G}_t)^n$ that specifies, for each asset $i \in I$, the long or short position x^i at time t. The portfolio consisting of a single unit of asset i, and nothing else, is denoted by e^i.

An *asset market* is a quintet $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ where:

(i) $X \subseteq L_0(\mathcal{G}_t)^n$, with $e^i \in X$ for all $i \in I$, is the set of available portfolios;

(ii) $p : X \to L_0(\mathcal{G}_t)$, with $p(e^i) = p^i$ for all $i \in I$, is the *portfolio pricing function*;

(iii) $P : X \to H$, with $P(e^i) = y^i$ for all $i \in I$, is the *portfolio payoff function.*

The first point requires that at least one unit of each primary asset is available on the market at time t. The remaining two points just confirm that p^i and y^i are the price and payoff of each primary asset i.

2.2 Payoff reduction

Let $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ be an asset market. Setting

$$M = P(X)$$

formalizes the idea of HR p. 593 according to which, given a payoff market (M, π), the set M is the set of payoffs at time T from portfolios that can be purchased at time t. In fact, $P(X)$ is the collection of contingent claims y in H such that $y = P(x)$ for some x in X. In this perspective, it is natural to say that the price $\pi(y)$ of payoff y is the price $p(x)$ of purchasing a portfolio x which replicates y, that is, $\pi(P(x)) = p(x)$. More precisely, the *payoff pricing function* $\pi : M \to L_0(\mathcal{G}_t)$ of HR should satisfy the relation

$$p = \pi \circ P. \tag{2}$$

Diagrammatically, π should solve

$$\begin{array}{ccc} X & \xrightarrow{P} & M \\ {\scriptstyle p}\downarrow & \swarrow_{\pi} & \\ L_0(\mathcal{G}_t). & & \end{array}$$

Momentarily, we will see that a classical financial property, the law of one price, is equivalent to the existence (and uniqueness) of such π.

Summing up, a payoff market (M, π) is canonically associated with each asset market $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ in which the law of one price is satisfied.

2.3 Financial assumptions

We will make a few standard financial assumptions on the asset market.

B.1 (Frictionless) *The asset market is frictionless, that is, it has:*

(i) *free portfolio formation:* $wx + w'x' \in X$ *for all* $w, w' \in L_0(\mathcal{G}_t)$ *and all* $x, x' \in X$*;*

(ii) *no transaction costs at time* t*:* $p(x) = \sum_{i\in I} x^i p^i$ *for all* $x \in X$*;*

(iii) *no transaction costs at time* T*:* $P(x) = \sum_{i\in I} x^i y^i$ *for all* $x \in X$*.*

Free portfolio formation requires that all portfolios of primary assets are available. For example, it rules out short sales constraints and regulatory restrictions (see, e.g., p. 63 of Cochrane, 2005). Mathematically, it implies $X = L_0(\mathcal{G}_t)^n$.

The typical examples of transaction costs are bid/ask spreads and reduced liquidity of some assets. Mathematically, absence of transaction costs amounts to $L_0(\mathcal{G}_t)$-linearity of both functions p and P, that is, linearity with respects to the "random scalars" in $L_0(\mathcal{G}_t)$.

B.2 (Law of one price) *If* $x, x' \in X$ *and* $P(x) = P(x')$*, then* $p(x) = p(x')$*.*

The law of one price (LOP) is a classical market property that requires that portfolios with identical payoffs must be traded at the same price. It permits to price any replicable contingent claim y according to the formation cost $p(x)$ of a portfolio x that generates y as a payoff. That is, the LOP makes the payoff pricing function

$$\pi(y) = p(x) \qquad \text{if } y = P(x)$$

well defined. Furthermore, it is easy to see that π is the only solution of (2). Therefore, under the LOP the payoff market (M, π) that corresponds to the asset market $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ is well defined too.

B.3 (Numeraire) *There exists a* numeraire*, that is, a primary asset j such that $p^j > 0$.*

The numeraire makes it possible to price any asset i in relative terms by replacing p^i with p^i/p^j.

2.4 A fundamental theorem of finance

Lemma 1 *If an asset market satisfies assumptions B.1-B.3, then the corresponding payoff market satisfies assumptions A.1-A.4.*

Inspection of the proof shows that the absence of frictions is the financial underpinning of assumption A.1, that the LOP underlies assumptions A.2 and A.3, and that assumption A.4 corresponds to the existence of a numeraire. This is in perfect financial analogy with the unconditional case, though here the conditional setting requires novel mathematical arguments related to Hilbert modules.

This lemma, along with the representation theorem (Theorem 1), leads to the following fundamental theorem of finance for conditional asset pricing. Unlike the representation theorem, it relies on properties that have a direct financial interpretation.

Theorem 2 *A frictionless asset market $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ satisfies the LOP if and only if there exists a unique marketed payoff $m^* \in M$, such that*

$$p(x) = \mathrm{E}[m^* P(x) \mid \mathcal{G}_t] \qquad \forall x \in X. \tag{3}$$

Furthermore, if a numeraire exists, then $\|m^*\| > 0$.

This theorem says that the price of a portfolio is the discounted conditional expectation of its payoff. In turn, this subsumes the earlier discounted representation (1) of the payoff pricing function. In fact, by definition of M and π, if $y \in M$ then $y = P(x)$ for some $x \in X$, and

$$\pi(y) = p(x) = \mathrm{E}[m^* P(x) \mid \mathcal{G}_t] = \mathrm{E}[m^* y \mid \mathcal{G}_t]. \tag{4}$$

Note that, by (3), the possibly different portfolios x^* that replicate the stochastic discount factor m^* share a common price $p(x^*) = \mathrm{E}[(m^*)^2 \mid \mathcal{G}_t]$.

We close with a no arbitrage condition, which requires that there are no portfolios that always have positive payoffs and a negative price on some non zero probability event, over which "free lunches" would take place.

B.4 (No free lunch) *If* $x \in X$ *and* $P(x) \geq 0$*, then* $p(x) \geq 0$*.*

It is easy to show that, if the asset market is frictionless, the absence of free lunches implies the LOP, and so the existence of a stochastic discount factor m^*. More is true:

Proposition 1 *The following conditions are equivalent for a frictionless asset market:*

(i) the LOP is satisfied and $m^* \geq 0$*;*

(ii) the no free lunch condition is satisfied and $m^* \vee 0 \in M$*.*

Note that $m^* \vee 0$ is the payoff of a call option on m^* with zero strike price. The condition $m^* \vee 0 \in M$ thus requires that this option be replicable by a portfolio of primary assets. This proposition shows that such replicability is necessary for the positivity of the stochastic discount factor.

Finally, the stronger property $m^* > 0$ is guaranteed, under the assumption of financial market completeness, by the no arbitrage condition of HR p. 594.

3 Proofs

3.1 Proof of Theorem 2

We split the proof in several steps which show the role of the various assumptions.

In the proof we rely on the fact that conditional L_2-spaces are normed $L_0(\mathcal{G}_t)$-modules, and we denote by 1_Ω the constant 1-valued random variable (the multiplicative unit of $L_0(\mathcal{G}_t)$).[6]

Lemma 2 *The market* $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ *has free portfolio formation if and only if* $X = L_0(\mathcal{G}_t)^n$*.*

The simple proof is omitted.

Proposition 2 *If* $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ *is a frictionless market, then A.1 is satisfied.*

[6] The definitions of $L_0(\mathcal{G}_t)$-module Y, $L_0(\mathcal{G}_t)$-submodule $M \subseteq Y$, $L_0(\mathcal{G}_t)$-norm $\|\cdot\| : Y \to L_0(\mathcal{G}_t)$, $L_0(\mathcal{G}_t)$-linear operator $f : Y \to Z$ (if Z is another $L_0(\mathcal{G}_t)$-module), are formally identical to those of vector space, linear subspace, norm, linear operator, where the real field $\mathbb{R}$ is replaced by $L_0(\mathcal{G}_t)$. See HR, Cerreia-Vioglio et al. (2014), and Cerreia-Vioglio et al. (2015).

Proof. It is easy to check that $X = L_0(\mathcal{G}_t)^n$ is an $L_0(\mathcal{G}_t)$-module with respect to the component-wise operations and $P : L_0(\mathcal{G}_t)^n \to H$ is an $L_0(\mathcal{G}_t)$-linear operator so that $M = P(X)$ is a submodule of H. As to completeness of M, observe that (H, d_2) is a complete metric space,[7] therefore it suffices to show that M is closed in (H, d_2). By the Stricker's Lemma (see Lemma 2.2 of Schachermayer, 1992),

$$M = \left\{ \sum_{i \in I} x^i y^i : x \in L_0(\mathcal{G}_t)^n \right\}$$

is closed in $(L_0(\mathcal{G}_T), d_0)$, where d_0 is the metric of convergence in probability. Assume now that $z_k \in M$ for all $k \in \mathbb{N}$ and $z_k \xrightarrow{d_2} z$ in H, then

$$d_2(z_k, z) \to 0 \qquad \text{as } k \to \infty$$

but, as shown by Cerreia-Vioglio et al. (2014) p. 20,

$$d_2(y, y') \geq d_0(y, y') \qquad \forall y, y' \in H$$

then $d_0(z_k, z) \to 0$ as $k \to \infty$ and $z_k \xrightarrow{d_0} z$ in $L_0(\mathcal{G}_T)$, thus by the Stricker's Lemma $z \in M$. As wanted. ■

Status: Proposition 2 proves that, if an asset market satisfies B.1, the corresponding payoff market satisfies A.1.

Proposition 3 *If $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ is a frictionless market and the LOP is satisfied, then A.2 is satisfied.*

Proof. The crucial observation has already been done: by the LOP, π is well defined. As to $L_0(\mathcal{G}_t)$-linearity, let $y, y' \in M$ and $w, w' \in L_0(\mathcal{G}_t)$. Arbitrarily choose $x, x' \in L_0(\mathcal{G}_t)^n$ such that $y = P(x)$ and $y' = P(x')$. Then

$$wy + w'y' = wP(x) + w'P(x') = P(wx + w'x')$$

because P is $L_0(\mathcal{G}_t)$-linear, but then

$$\pi(wy + w'y') = \pi(P(wx + w'x')) = p(wx + w'x') = wp(x) + w'p(x')$$

because p is $L_0(\mathcal{G}_t)$-linear. By definition of π, $p(x) = \pi(y)$ and $p(x') = \pi(y')$, and so

$$\pi(wy + w'y') = w\pi(y) + w'\pi(y')$$

as wanted. ■

Status: Propositions 2 and 3 prove that, if an asset market satisfies B.1 and B.2, the corresponding payoff market satisfies A.1 and A.2.

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[7] See the papers mentioned in the previous footnote.

Proposition 4 *If* $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ *is a frictionless market and the LOP is satisfied, then A.3 is satisfied.*

The proof relies on the following lemma.

Lemma 3 $L_0(\mathcal{G}_t)^n$ *is a normed* $L_0(\mathcal{G}_t)$*-module with respect to the component-wise operations and the* $L_0(\mathcal{G}_t)^n$*-norm*

$$\|x\| = \sum_{i\in I} \left|x^i\right| \in L_0(\mathcal{G}_t)^+ \qquad \forall x \in L_0(\mathcal{G}_t)^n .$$

The corresponding distance is $d(x, x') = \mathrm{E}\left[\|x - x'\| \wedge 1_\Omega\right]$ *and the induced topology is the product of the topologies of convergence in probability of the* n *components of* $L_0(\mathcal{G}_t)^n$*.*

In particular, $\left(L_0(\mathcal{G}_t)^n, d\right)$ *is complete.*

Proof. We already observed that $L_0(\mathcal{G}_t)^n$ is an $L_0(\mathcal{G}_t)$-module and the verification that $\|\cdot\|$ is an $L_0(\mathcal{G}_t)$-norm is routine. Consider a sequence x_k in $L_0(\mathcal{G}_t)^n$. If $x_k \xrightarrow{d} x$ in $L_0(\mathcal{G}_t)^n$, then $\mathrm{E}\left[\left(\sum_{i\in I}|x_k^i - x^i|\right)\wedge 1_\Omega\right] \to 0$ as $k \to \infty$, but $\left|x_k^j - x^j\right| \wedge 1_\Omega \le \left(\sum_{i=1}^n |x_k^i - x^i|\right)\wedge 1_\Omega$ for all $j \in I$, therefore $\mathrm{E}\left[\left|x_k^j - x^j\right| \wedge 1_\Omega\right] \to 0$ as $k \to \infty$, and $x_k^j \xrightarrow{d_0} x^j$ in $L_0(\mathcal{G}_t)$ for all $j \in I$. Conversely, if $x_k^j \xrightarrow{d_0} x^j$ in $L_0(\mathcal{G}_t)$ for all $j \in I$, then

$$0 \le \mathrm{E}\left[\left(\sum_{i=1}^n \left|x_k^i - x^i\right|\right)\wedge 1_\Omega\right] \le \sum_{i=1}^n \mathrm{E}\left[\left|x_k^i - x^i\right| \wedge 1_\Omega\right] \to 0 \qquad \text{as } k \to \infty$$

where the second inequality holds because $\left(\sum_{i=1}^n |x_k^i - x^i|\right)\wedge 1_\Omega \le \sum_{i=1}^n \left(|x_k^i - x^i| \wedge 1_\Omega\right)$ (see, e.g., Chapter 2, Theorem 12.5, of Luxemburg and Zaanen, 1971). Therefore, $x_k \xrightarrow{d} x$.

Finally, a product of n complete metric spaces is complete. ■

Proof of Proposition 4. Note that the following triangle is commutative

$$\begin{array}{ccc} L_0(\mathcal{G}_t)^n & \xrightarrow{P} & M \\ {}_p\downarrow & \swarrow_\pi & \\ L_0(\mathcal{G}_t) & & \end{array}$$

that is, $p = \pi \circ P$. Since P is onto ($M = P(X)$), if it is also open, then continuity of π is implied by continuity of p. In fact, let O be an open subset of $L_0(\mathcal{G}_t)$, if p is continuous, then $p^{-1}(O)$ is open in $L_0(\mathcal{G}_t)^n$, if P is open, then $P\left(p^{-1}(O)\right)$ is open in M, but

$$P\left(p^{-1}(O)\right) = P\left((\pi \circ P)^{-1}(O)\right) = P\left(P^{-1}\left(\pi^{-1}(O)\right)\right) = \pi^{-1}(O)$$

where the last equality holds because of surjectivity of P and allows to conclude that $\pi^{-1}(O) = P\left(p^{-1}(O)\right)$ is open in M, that is π is continuous.

By Lemma 3 and Proposition 2, $\left(L_0(\mathcal{G}_t)^n, d\right)$ and (M, d_2) are complete metric spaces. Moreover, a distance induced by an $L_0(\mathcal{G}_t)$-norm always satisfies properties (i) and (ii) of the definition of an F-space (Definition II.1.10 of Dunford and Schwartz, 1958), since property

(iii) is completeness, we conclude that $(L_0(\mathcal{G}_t)^n, d)$ and (M, d_2) are F-spaces when regarded as vector spaces.[8] Since P is $L_0(\mathcal{G}_t)$-linear and onto, it is a fortiori linear and onto. Next we show that P is continuous, and hence open by the Interior Mapping Principle (Theorem II.2.1 of Dunford and Schwartz, 1958). Assume $x_k \xrightarrow{d} x$ in $L_0(\mathcal{G}_t)^n$, this means that $x_k^i \xrightarrow{d_0} x^i$ in $L_0(\mathcal{G}_t)$ for all $i \in I$, but then

$$
\begin{aligned}
0 &\leq \|P(x_k) - P(x)\|_2 = \left\| \sum_{i \in I} x_k^i y^i - \sum_{i \in I} x^i y^i \right\|_2 = \left\| \sum_{i \in I} \left(x_k^i - x^i\right) y^i \right\|_2 \\
&\leq \sum_{i \in I} \left\| \left(x_k^i - x^i\right) y^i \right\|_2 = \sum_{i \in I} \left|x_k^i - x^i\right| \left\|y^i\right\|_2
\end{aligned}
$$

and the latter term converges to 0 in probability in $L_0(\mathcal{G}_t)$ because x_k^i converges to x^i in probability in $L_0(\mathcal{G}_t)$ for all $i \in I$. But this implies that $\|P(x_k) - P(x)\|_2$ converges to 0 in probability in $L_0(\mathcal{G}_t)$, that is, $P(x_k) \xrightarrow{d_2} P(x)$, so that P is continuous.

A similar argument (obtained by replacing y^i with p^i and $\|\cdot\|_2$ with $|\cdot|$) shows that p is continuous too and concludes the proof. ■

Status: Propositions 2, 3, and 4 prove that, if an asset market satisfies B.1 and B.2, the corresponding payoff market satisfies A.1, A.2, and A.3.

At this point, we have that if a frictionless asset market $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ satisfies the LOP, then the corresponding payoff market (M, π) satisfies A.1, A.2, and A.3. By Theorem 1, there exists a unique payoff $m^* \in M$ such that

$$
\pi(y) = \mathrm{E}[m^* y \mid \mathcal{G}_t] \qquad \forall y \in M.
$$

But then, for each $x \in X$,

$$
p(x) = \pi(P(x)) = \mathrm{E}[m^* P(x) \mid \mathcal{G}_t]
$$

because $P(x) \in M$. Conversely, it is immediate to see that (3) implies the LOP.

Status: We have proved the first part of Theorem 2.

The proofs of both Lemma 1 and Theorem 2 are concluded by the following simple proposition.

Proposition 5 *Let $\left((p^i)_{i\in I}, (y^i)_{i\in I}, X, p, P\right)$ be a frictionless market and the LOP be satisfied. If a numeraire exists in the market, then A.4 is satisfied and $\|m^*\| > 0$.*

Proof. Let j be a numeraire and consider the portfolio $e^j \in L_0(\mathcal{G}_t)^n$, then $\pi(P(e^j)) = p(e^j) = p^j > 0$. This yields A.4, which, by Theorem 1, implies that $\|m^*\| > 0$. ■

[8] Note that an $L_0(\mathcal{G}_t)$-module is a vector space with the identification $\alpha x \equiv (\alpha 1_\Omega) x$ for all $\alpha \in \mathbb{R}$.

3.2 Proof of Proposition 1

We only prove that (ii) implies (i) since the converse is obvious. Set $v = (-m^*) \vee 0$ and notice that $v \geq 0$ and $v \in M$ because $v = m^* \vee 0 - m^*$ and $m^* \in M$. By the no free lunch condition, $\pi(v) = \mathrm{E}[-(m^*)^2 1_{(m^* \leq 0)} \mid \mathcal{G}_t] \geq 0$. On the other hand $-(m^*)^2 1_{(m^* \leq 0)} \leq 0$ implies $\mathrm{E}[-(m^*)^2 1_{(m^* \leq 0)} \mid \mathcal{G}_t] \leq 0$. Passing to the unconditional expectation, we then have $\mathrm{E}[(m^*)^2 1_{(m^* \leq 0)}] = 0$, which implies $(m^*)^2 1_{(m^* \leq 0)} = 0$ and so $m^* \geq 0$.

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