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Competing Mechanisms in Markets for Lemons*

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Abstract

We study competitive equilibria in a market with adverse selection, where uninformed buyers post general direct trading mechanisms and informed sellers select one of them. We demonstrate that there exists a unique equilibrium allocation and characterize its properties. In equilibrium all buyers post the same mechanism and low quality sellers receive priority in any meeting with a buyer. Hence, in contrast to the existing approach to markets with adverse selection where contracts are assumed to be bilateral, our model predicts that sellers are pooled at the stage of selecting a mechanism but screened when the mechanism is played. Compared to the equilibrium with bilateral contracting, our equilibrium yields a higher surplus for most, but not all, parameter specifications.

1 Introduction

We study the properties of competitive markets with adverse selection. Uninformed principals post trading mechanisms, informed agents select a mechanism and a principal posting it. Each principal may be selected by several or no agents and we allow trading mechanisms to exploit this fact, by making agents compete among themselves and possibly extracting information over their types. As an illustration we can think of a trading mechanism as a form of auction. The approach we follow is natural in situations like procurement markets, where procurers meet several firms, privately informed about the quality of their projects. In this context, the terms of trade between a procurer and a contractor may not only depend on the contractor's type but also on the number and composition of the contractor's competitors. Other applications include the labor market in which firms meet a variety of workers, privately informed about their productivity, but also financial markets where traders have private information about the assets they own and buyers may meet several sellers.

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More formally, we allow principals to post general direct mechanisms that specify trading probabilities and transfers for agents, contingent not only on their own reported type but also on the reports of other agents meeting the same principal. We thereby depart from the traditional approach to competition in markets with adverse selection, which focuses on bilateral contracting between principals and agents. In particular, we consider an environment as in Akerlof (1970) with a measure of sellers, privately informed about the quality of their good, which can be high or low, and a measure of uninformed buyers. Each buyer posts a trading mechanism and each seller picks at random one of the buyers posting the preferred mechanism. A buyer therefore meets a random number of sellers, with probabilities that depend on the ratio between buyers posting a given mechanism and sellers selecting it. Hence, the equilibrium notion is that of competitive, directed search with urn-ball meetings.

We show that the consideration of multilateral meetings and general trading mechanisms has important implications for market outcomes. We find that in equilibrium all buyers post the same mechanism. This mechanism specifies that a low quality object is traded whenever such object is present in a meeting, meaning that low type sellers receive priority in every meeting. Hence different types of sellers are not separated via their choice of mechanism - they all select the same one - but rather within the mechanism. Such an equilibrium always exists and the equilibrium allocation is unique. The intuition for why sellers are pooled in their choice of mechanism is that it is always profitable for buyers to attract high type sellers on top of low type sellers, while retaining priority for the latter. The equilibrium outcome does not depend on the relative gains from trade for the high and low quality object. Even when the quality of the good matters a lot to buyers so that the gains from trade for the low quality object are arbitrarily small and those for the high quality object are arbitrarily large, high type sellers only get to trade in meetings where there are no low type sellers.

With regard to the properties of the equilibrium mechanism, the fact that in equilibrium high and low type sellers pick the same mechanism implies that buyers' payoffs are not equalized across trades with the two types, in contrast to the case of bilateral contracting. In particular, when competition for high type sellers is intense, buyers earn a zero payoff with them, while they earn a strictly positive payoff with low types. We further show that the equilibrium mechanism can be implemented via a sequence of second price auctions with increasing reserve prices.

We show that there are situations in which meetings between buyers and sellers do not necessarily lead to trade in equilibrium. To be more precise, trade always occurs when buyers do not care too much about quality and the competition among buyers for high quality sellers is not too intense, a situation we refer to as adverse selection being mild. When this is not the case and adverse selection is severe, high type sellers are rationed in equilibrium, meaning that in meetings where all sellers have a high quality object there is a strictly positive probability that no one trades. In such a situation, additional equilibria exist in which sellers partially sort themselves through their choice of mechanism: buyers post different mechanisms, attracting different ratios of high versus low type sellers. However, these equilibria yield the same allocation and payoffs as the equilibrium described above, where all sellers are pooled in their choice of a trading mechanism.

The extent to which adverse selection is or is not severe also matters for the welfare properties of the equilibrium. We show that, whenever adverse selection is mild and the gains from trade are higher for the low than the high quality object, the equilibrium allocation maximizes social surplus. The result follows directly from the facts that (i) the pooling of sellers on the choice of the same mechanism maximizes the number of meetings and that (ii) the equilibrium mechanism gives priority to the good with the larger gains from trade. On the other hand, with severe adverse selection social surplus is no longer maximal in equilibrium and, provided the share of high type sellers is large enough, the equilibrium allocation can be Pareto improved, subject to the constraints imposed by incentives and the meeting technology.

Finally, it is of interest to compare our findings with those of the situation typically considered in the literature, where contracting can only be bilateral and equilibria are such that sellers of different types select different contracts and buyers' payoff are equalized across trades (see the discussion at the end of this section). In our environment, the limitation to bilateral contracting is equivalent to the assumption that each buyer can at most meet one seller. We thus consider the case where the mechanisms available to buyers are restricted to the class of simple menus from which one seller, randomly selected among those meeting a buyer, can choose. As one should expect in the light of the results of earlier work (see for example Guerrieri et al., 2010), we find that in equilibrium buyers post two distinct prices, the higher one selected by high type sellers and the lower one only by low types. That is, sellers now sort themselves ex-ante, when they select which mechanism to trade, rather than being screened ex-post, within the mechanism, as in the equilibrium discussed in the previous paragraphs. It also follows that now buyers earn the same payoff with low and high type sellers.

When we compare welfare at the equilibrium with general direct mechanisms to the one with bilateral contracts we see that in the first one social surplus is strictly higher for many, but not all, parameter specifications. In particular, with mild adverse selection and higher gains from trade for the low quality good, the equilibrium with ex-post screening always yields a higher surplus since, as mentioned above, social surplus is maximal in that case. More surprisingly, this equilibrium also generates a higher total surplus when it is constrained inefficient and entails rationing, that is when adverse selection is severe. Despite this, we show that there exist parameter specifications under which surplus is higher at the sorting equilibrium with bilateral contracting. This happens when gains from trade are larger for the high quality good, while the intensity of the competition for high type sellers is sufficiently low so that adverse selection is still mild. Hence in that situation introducing restrictions on the set of available mechanisms proves welfare improving.

Related Literature: Since the work of Akerlof (1970) and Rothschild and Stiglitz (1976), the properties of market outcomes in the presence of adverse selection have been the subject of intense study, both in Walrasian models as well as in models where agents act strategically. In the latter agents compete among themselves over contracts which determine transfers of goods and prices, whereas in the former available contracts specify transfers of goods while prices are taken as given and set so as to clear the markets. Initiated by Gale (1992), Inderst and Mueller (1999) and, more recently, Guerrieri et al. (2010), the use of competitive, directed search models to study markets with adverse selection has generated several interesting insights.[1] This framework allows for a richer specification of the terms of contracts available for trade, which - in contrast to Walrasian models - also include the price to be paid. The interest in a search approach to markets has been enhanced by recent developments in financial markets, which have seen the growth of new trading arrangements, as the OTC markets, less centralized and with less transparency on trading conditions.

As already mentioned earlier, a key underlying assumption in the existing literature on markets with adverse selection is that contracting is bilateral, that is the terms of trade only rely on the report of the agent with whom the principal trades. Both in competitive search models and in Walrasian ones (on the latter, see Dubey and Geanakoplos (2002) and Bisin and Gottardi (2006)), an equilibrium always exists and is separating, with agents of different types trading different contracts. Principals are in turn indifferent between posting any of the contracts chosen by the agents, implying that in equilibrium there is no cross-subsidization among the types. As a consequence, the equilibrium outcome is inefficient - even taking incentive compatibility constraints into account - when the fraction of higher quality agents is sufficiently large.[2]

[1] For earlier work on directed search in markets without informational asymmetries see, for instance, Moen (1997), Burdett et al. (2001).

[2] There are also papers studying adverse selection in economies where meetings and trades take place over a sequence of periods, both in markets with random search, where offers of contracts are made only after meetings occur, and in markets la Akerlof (1970), where all trades occur at a single price (see for example Blouin and Serrano (2001), Janssen and Roy (2002), Carmargo and Lester (2014), Fuchs and Skrzypacz (2013) and Moreno and Wooders (2015)). It is interesting to notice that the equilibrium outcome is similar in these models: separation in that case obtains with sellers of different types trading at different prices, at different points in time.

Our paper is also closely related to the work on competing mechanisms in independent private value environments. Peters (1997) and Peters and Severinov (1997) assume the same meeting technology as in our paper and show for such environments that there exists an equilibrium where all buyers post the same second-price auction with a reserve price equal to their valuation.[3] Subsequent papers examine the features of the equilibrium and its welfare properties in more general search environments, allowing for example for meeting technologies where buyers face capacity constraints in their ability to meet sellers, while maintaining the independent private value assumption (e.g. Eeckhout and Kircher 2010, Albrecht et al. 2014, Cai et al. 2015). In particular, Eeckout and Kircher (2010) show in this setting the connection between the properties of the meeting technology and the existence of equilibria with ex-ante sorting (at the search stage) versus equilibria with ex-post screening (at the mechanism stage). The environment we consider allows to nest a situation with independent private values as a special case where the valuation of buyers does not depend on the quality of the good. Our characterization of competitive search equilibria shows that, as one moves away from this particular parameter specification, the equilibrium mechanism differs from the second-price auction identified in the literature recalled above, while the other features of the equilibrium are robust as long as adverse selection is mild. In contrast, when adverse selection is severe, our results show that the search equilibrium features rationing of high type sellers, the possibility of partial sorting at the search stage and that it is constrained inefficient, properties that never arise in independent private value environments.

The paper is organized as follows. The next section presents the economy, the space of mechanisms and defines the notion of competitive search equilibrium that is considered. Section 3 presents the main result, stating the existence of competitive search equilibria and the uniqueness of the equilibrium allocation, and the argument of the proof. The properties of the equilibrium trading mechanism are presented in Section 4, together with a welfare analysis of equilibria. Finally, the last section compares the equilibrium we found with multilateral meetings and general direct mechanisms to those of the equilibrium which obtains when each principal can meet at most one agent, showing that welfare is typically, though not always higher in the first one. Proofs are collected in the Appendix.

2 Environment

There is a measure b of uninformed buyers and a measure s of informed sellers. Each seller possesses one unit of an indivisible good with uncertain quality and each buyer wants to buy at most one

[3] Note that in this literature labels of buyers and sellers are typically reversed.

unit. The good's quality is identically and independently distributed across sellers. Quality can be either high or low and μ denotes the fraction of sellers that possess a high quality good. Let $\overline{\lambda}^p = \mu \frac{s}{b}$ denote the ratio of high type sellers to buyers and let $\underline{\lambda}^p = (1-\mu)\frac{s}{b}$ denote the ratio of low type sellers to buyers. The buyers' and sellers' valuation of the high (low) quality good are denoted by $\overline{v}$ ($\underline{v}$) and $\overline{c}$ ($\underline{c}$), respectively. We assume that both the buyers and the sellers value the high quality good more than the low quality good, i.e. $\overline{v} \geq \underline{v}, \overline{c} > \underline{c}$. For sellers this preference is assumed to be strict, while we allow buyers to have the same valuation for both types of good, i.e. $\overline{v} = \underline{v}$. When $\overline{v}$ is strictly greater than $\underline{v}$, the buyer's valuation depends on the seller's valuation of the object, a situation we refer to as the common value case. This is no longer true when $\overline{v} = \underline{v}$, which we refer to as the private value case. We further assume that there are always positive gains from trade, meaning that for both types of good the buyer's valuation strictly exceeds the seller's valuation, i.e. $\overline{v} > \overline{c}, \underline{v} > \underline{c}$.

Meeting Technology: Matching between buyers and sellers operates as follows. Buyers simultaneously post mechanisms that specify how trade takes place with the sellers with whom they meet. Sellers observe the posted mechanisms and select one of the mechanisms they like best. We refer to the collection of buyers posting the same mechanism and the collection of sellers selecting that mechanism as constituting a submarket. We assume that markets are anonymous. Anonymity is captured by the assumption that mechanisms cannot condition on the identity of sellers and that sellers cannot condition their choice on the identity of buyers but only on the mechanism they post (see for example Shimer, 2005). More specifically, we assume that, in any submarket, a seller meets one, randomly selected, of the buyers that are present and that buyers have no capacity constraints, that is they can meet all arriving sellers, no matter how many they are. As a result, the number of sellers that meet a particular buyer follows a Poisson distribution with a mean equal to the seller-buyer ratio in the submarket.[4] According to this meeting technology, referred to as urn-ball matching, sellers are sure to meet a buyer, while buyers may end up with many sellers or with no seller at all. Moreover, a buyer's probability of meeting a seller of a given type is fully determined by the ratio between sellers of that type and buyers in the submarket, while it does not depend on the presence of other types of sellers.[5] The latter property and the fact that buyers can meet multiple sellers are essential for the following analysis, while most other features of the meeting technology are not.

Under urn-ball matching, a buyer's probability of meeting k sellers in a market with seller-buyer

[4]This is one of the most commonly used specifications in the directed search literature, e.g. Peters and Severinov (1997), Albrecht et al. (2006), Kim and Kircher (2015).

[5]The class of meeting technologies that have this property, called 'invariance' (Lester et al., 2015a), includes the urn-ball matching technology as a special case.

ratio λ is given by

$$P_k(\lambda) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Since the presence of high type sellers does not affect the meeting chances of low type sellers and vice versa, the probability for a buyer to meet L low type sellers and H high type sellers in a market where the ratio between high (low) type sellers and buyers is $\overline{\lambda}$ ($\underline{\lambda}$) is given by

$$P_L(\underline{\lambda}) P_H(\overline{\lambda}) = \frac{(\underline{\lambda})^L}{L!} e^{-\underline{\lambda}} \frac{(\overline{\lambda})^H}{H!} e^{-\overline{\lambda}}.$$

Similarly, $P_L(\underline{\lambda}) P_H(\overline{\lambda})$ corresponds to the probability for a seller to be in a meeting with other L low type sellers and H high type sellers.

Mechanisms and payoffs: We restrict attention to direct mechanisms that do not condition on mechanisms posted by other buyers.[6] A mechanism m is defined by

$$m : \{(L, H)\}_{L \in \mathbb{N}, H \in \mathbb{N}} \rightarrow [0, 1]^2 \times \mathbb{R}^2,$$

where L is the number of low messages and H is the number of high messages in a meeting. Let $\underline{X}_m(L, H), \overline{X}_m(L, H)$ and $\underline{T}_m(L, H), \overline{T}_m(L, H)$ denote the trading probabilities and (unconditional) transfers specified by mechanism m for sellers reporting, respectively, L and H. We say a mechanisms m is feasible if

$$\underline{X}_m(L, H)L + \overline{X}_m(L, H)H \leq 1, \forall (L, H) \in \mathbb{N}^2. \tag{1}$$

That is, in each meeting the probability that a good is exchanged cannot exceed one. Let M denote the measurable set of feasible mechanisms.

We assume that, when matched with a buyer, a seller does not observe how many other sellers are matched with the same buyer nor their types.[7] Let $\overline{\lambda}$ denote the expected number of H reports and $\underline{\lambda}$ denote the expected number of L reports, which under truthful reporting simply correspond to the respective seller-buyer ratios for mechanism m. The expected trading probabilities for a

[6]In the equilibria we characterize in Theorem 3.1 below, each buyer knows the realized distribution of mechanisms posted by other buyers and therefore, as noted by Eeckhout and Kircher (2010), does not profit from conditioning on those mechanisms. Hence, the equilibria we characterize exist even when the restriction to mechanisms conditioning only on the sellers' reported types is relaxed, though there may be additional equilibria.

[7]The assumption that a seller cannot observe the number of competitors in a meeting facilitates notation considerably but is not essential for any of our results. Since traders are assumed to be risk neutral, sellers' payoffs are linear in trading probabilities and transfers. Therefore, if incentive compatibility constraints (2) and (3) below are satisfied, expected trading probabilities and transfers can always be decomposed in terms of trading probabilities and transfers in each possible meeting in a way such that incentive compatibility is satisfied also when sellers observe the number of competitors in a meeting.

seller when reporting L and H, respectively, are then given by

$$\begin{aligned}\underline{x}_m(\underline{\lambda},\overline{\lambda}) &= \sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty} P_L(\underline{\lambda})P_H(\overline{\lambda})\underline{X}(L+1,H),\\ \overline{x}_m(\underline{\lambda},\overline{\lambda}) &= \sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty} P_L(\underline{\lambda})P_H(\overline{\lambda})\overline{X}(L,H+1).\end{aligned}$$

Similarly, we can determine expected transfers $\underline{t}_m(\underline{\lambda},\overline{\lambda})$ and $\overline{t}_m(\underline{\lambda},\overline{\lambda})$. The expected payoff for low and high type sellers, relative to their endowment point,[8] when choosing mechanism m and revealing their type truthfully is given by

$$\begin{aligned}\underline{u}(m|\underline{\lambda},\overline{\lambda}) &= \underline{t}_m(\underline{\lambda},\overline{\lambda}) - \underline{x}_m(\underline{\lambda},\overline{\lambda})\underline{c},\\ \overline{u}(m|\underline{\lambda},\overline{\lambda}) &= \overline{t}_m(\underline{\lambda},\overline{\lambda}) - \overline{x}_m(\underline{\lambda},\overline{\lambda})\overline{c}.\end{aligned}$$

Truthful reporting is optimal if the following two inequalities hold

$$\overline{t}_m(\underline{\lambda},\overline{\lambda}) - \overline{x}_m(\underline{\lambda},\overline{\lambda})\underline{c} \leq \underline{t}_m(\underline{\lambda},\overline{\lambda}) - \underline{x}_m(\underline{\lambda},\overline{\lambda})\underline{c}, \tag{2}$$

$$\underline{t}_m(\underline{\lambda},\overline{\lambda}) - \underline{x}_m(\underline{\lambda},\overline{\lambda})\overline{c} \leq \overline{t}_m(\underline{\lambda},\overline{\lambda}) - \overline{x}_m(\underline{\lambda},\overline{\lambda})\overline{c}. \tag{3}$$

Note that, since incentive compatibility is defined in terms of expected trading probabilities and transfers, whether a given mechanism m is incentive compatible or not depends on the values of $\underline{\lambda}$ and $\overline{\lambda}$. Let $\mathcal{M}^{IC}$ denote the set of tuples $(m,\underline{\lambda},\overline{\lambda})$ such that $m \in M$ and incentive compatibility with respect to $\underline{\lambda},\overline{\lambda}$ is satisfied.

Finally, it can be verified that the expected payoff for a buyer posting mechanism m, when sellers report truthfully and the expected number of high and low type sellers, respectively, is $\overline{\lambda}$ and $\underline{\lambda}$, is

$$\pi(m|\underline{\lambda},\overline{\lambda}) = \overline{\lambda}[\overline{x}_m(\underline{\lambda},\overline{\lambda})\overline{v} - \overline{t}_m(\underline{\lambda},\overline{\lambda})] + \underline{\lambda}[\underline{x}_m(\underline{\lambda},\overline{\lambda})\underline{v} - \underline{t}_m(\underline{\lambda},\overline{\lambda})].$$

Equilibrium: An allocation in this setting is defined by a measure β over M, where $\beta(m)$ denotes the measure of buyers that post mechanism m, and two maps $\underline{\lambda},\overline{\lambda} : M \to \mathbb{R}^+ \cup +\infty$ specifying, respectively, the ratio of low and high type sellers selecting mechanism m relative to the buyers posting that mechanism. Let M^β denote the support of β. We say an allocation is *feasible*

[8] For example, the endowment point of a low type seller is $\underline{c}$. His utility gain when participating in mechanism m is given by $\underline{t}_m(\underline{\lambda},\overline{\lambda}) + (1-\underline{x}_m(\underline{\lambda},\overline{\lambda}))\underline{c} - \underline{c} = \underline{u}(m|\underline{\lambda},\overline{\lambda})$.

if

$$\int_{M^\beta} d\beta(m) = b, \quad \int_{M^\beta} \underline{\lambda}(m) d\beta(m) = s(1-\mu), \quad \int_{M^\beta} \overline{\lambda}(m) d\beta(m) = s\mu. \tag{4}$$

We call an allocation *incentive compatible* if $(m, \underline{\lambda}(m), \overline{\lambda}(m)) \in \mathcal{M}^{IC}$ for all $m \in M^\beta$. We can show that we can restrict our attention to incentive compatible allocations w.l.o.g.:[9] for any non-incentive compatible mechanism (both in M^β and $M \backslash M^\beta$) there always exists an incentive compatible mechanism that yields the same payoff for buyers and sellers as the original mechanism.

An equilibrium specifies a feasible allocation such that, for all $m \in M^\beta$, the values $\underline{\lambda}(m), \overline{\lambda}(m)$ are consistent with buyers and sellers optimal choices. For all $m \notin M^\beta$, the maps $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ describe the beliefs of buyers over the expected number of low and high type sellers, respectively, that a deviating mechanism attracts. We require that in equilibrium buyers' beliefs are consistent with sellers' optimal choices. More specifically, a buyer believes that a deviating mechanism attracts some low (high) type sellers if and only if low (high) type sellers are indifferent between this mechanism and the one they choose in equilibrium, while the other type weakly prefers his equilibrium mechanism. This is captured the two following inequalities[10]

$$\underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) \leq \max_{m' \in M^\beta} \underline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \overline{\lambda}(m) > 0, \tag{5}$$

$$\overline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) \leq \max_{m' \in M^\beta} \overline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \underline{\lambda}(m) > 0. \tag{6}$$

We then impose the following conditions on out of equilibrium beliefs, $\underline{\lambda}(m), \overline{\lambda}(m), m \notin M^\beta$:

i) if (5,6) admit a unique solution, then $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ are given by that solution;

ii) if (5,6) admit no solution, we set $\underline{\lambda}(m)$ and/or $\overline{\lambda}(m)$ equal to $+\infty$ and $\pi(m|\underline{\lambda}(m), \overline{\lambda}(m)) = c$ for some $c \leq 0$;[11]

iii) if (5,6) admit multiple solutions, then $\underline{\lambda}(m), \overline{\lambda}(m)$ are given by the solution for which the buyer's payoff $\pi(m|\underline{\lambda}(m), \overline{\lambda}(m))$ is the highest.

Condition i) says that whenever there is a unique pair $\underline{\lambda}(m), \overline{\lambda}(m)$ satisfying conditions (5) and (6) for an out of equilibrium mechanism $m \notin M^\beta$, buyers' beliefs regarding the seller-buyer ratios

[9] For a formal proof see the Online Appendix, available at https://sites.google.com/site/austersarah/

[10] Noting the analogy between these conditions and the sellers' optimality conditions appearing in Definition 1, we see that (5),(6) indeed require the seller-buyer ratios to be consistent with sellers' optimal choices, also for out of equilibrium mechanisms, as if all mechanisms were effectively available to sellers. This is analogous to existing refinements in competitive environments with adverse selection such as Gale (1992), Dubey and Geanakoplos (2002) and Guerrieri et al. (2010), among others (see also Eeckhout and Kircher, 2010).

[11] More precisely, if $\underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) > \max_{m' \in M^\beta} \underline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m'))$ and $\overline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) > \max_{m' \in M^\beta} \overline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m'))$ for all $\underline{\lambda}(m), \overline{\lambda}(m) \in \mathbb{R}^+$, then $(\underline{\lambda}(m), \overline{\lambda}(m)) = (+\infty, +\infty)$. If only one of the two inequalities is violated, say the first one, then $\overline{\lambda}(m) = +\infty$, while $\underline{\lambda}(m)$ is determined by (6).

for such deviating mechanism are pinned down by these conditions. Since we allow for arbitrary direct mechanisms and buyers have no capacity constraints in their ability to meet sellers, there exist mechanisms for which a solution to (5,6) does not exist.[12] That is, there exist mechanisms such that for any finite pair $\underline{\lambda}(m), \overline{\lambda}(m)$, there is at least one type of seller that strictly prefers the deviating mechanism over any mechanism in the support of M^{β}. For example, a mechanism could specify a participation transfer that is paid to a seller independently of whether trade occurs or not. If that participation transfer is large enough, all sellers strictly prefer the deviating mechanism regardless of how many other sellers are expected to be present in a meeting. The payoff associated to such mechanism cannot be positive since the mechanism attracts infinitely many sellers, to each of whom the buyer has to promise a strictly positive transfer, while the gains from trade in the meetings involving the buyer are finite. Hence, condition ii) specifies that in such case the seller-buyer ratios $\underline{\lambda}(m), \overline{\lambda}(m)$ are set equal to infinity, while a buyer's associated payoff is non-positive.

Finally, if a solution to (5,6) exists, it is typically unique. If that should not be the case, we follow McAfee (1993) and others and assume, in condition iii), that buyers are 'optimistic', so that the pair $\underline{\lambda}(m), \overline{\lambda}(m)$ is given by their preferred solution. This specification makes deviations maximally profitable and may thus, in principle, restrict the set of equilibria.
We are now ready to define a competitive equilibrium.

Definition 1. *A competitive search equilibrium is a feasible and incentive compatible allocation given by a measure β with support M^{β} and two maps $\underline{\lambda}, \overline{\lambda}$ such that the following conditions hold:*

- *buyers' optimality: for all $m \in M$ such that $(m, \underline{\lambda}(m), \overline{\lambda}(m)) \in \mathcal{M}^{IC}$,*

$$\pi(m|\underline{\lambda}(m), \overline{\lambda}(m)) \leq \max_{m' \in M^{\beta}} \pi(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } m \in M^{\beta};$$

- *sellers' optimality: for all $m \in M^{\beta}$*

$$\begin{aligned} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) &\leq \max_{m' \in M^{\beta}} \underline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \overline{\lambda}(m) > 0, \\ \overline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) &\leq \max_{m' \in M^{\beta}} \overline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \underline{\lambda}(m) > 0; \end{aligned}$$

- *beliefs: for all $m \notin M^{\beta}$, $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ are determined by conditions i)-iii)*

[12] Note that this situation cannot arise in settings where the meeting technology is such that meetings can only be bilateral.

3 Competitive Search Equilibrium

We state now our main result, which characterizes the competitive search equilibria in the environment described in the previous section.

Theorem 3.1. *There exists a competitive search equilibrium with the following properties:*

i) All buyers post the same mechanism.

ii) Whenever a low type seller is present in a match, a low quality good is traded.

iii) The equilibrium is unique in terms of expected payoffs.

Theorem 3.1 states that there always exists an equilibrium in which sellers are pooled at the stage of selecting a mechanism and screened at the mechanism stage. That is, all buyers post identical mechanisms so that everybody trades in a single market and these mechanisms specify different trading probabilities for different types of sellers. In particular, the equilibrium mechanism always gives priority to low type sellers, meaning that a low quality good is traded whenever a low type seller is present in a meeting with a buyer. This not only implies that a low type seller's probability of trade strictly exceeds a high type seller's probability of trade,[13] but also that the equilibrium allocation maximizes trade of low quality goods in the economy. It is important to point out that this property of the equilibrium does not depend on the size of the relative gains from trade or the fraction of high type sellers in the population. That is, even when the gains from trade for the low quality good are arbitrarily small and those for the high quality good are arbitrarily large, in equilibrium high type sellers only trade when they are in meetings where there are no low type sellers. Theorem 3.1 also states that the equilibrium is unique in terms of payoffs. In particular, although there may also be equilibria where more than one mechanisms is posted, all those equilibria yield the same expected levels of trade and transfers.

The remainder of this section is devoted to proving the above result. The argument is constructive and proceeds through a series of lemmas and propositions that establish the stated properties of the equilibrium outcome. As a preliminary step, we show that, in order to characterize equilibrium payoffs, we can conveniently restrict our attention to the space of expected trading probabilities and transfers associated to mechanisms in M, clearly simpler than the original mechanism space. More precisely, the next proposition provides conditions on expected trading probabilities and transfers that any feasible and incentive compatible mechanism satisfies and, viceversa, that are generated by some feasible and incentive compatible mechanism.

[13] The probability of trade for a low type seller is in fact strictly larger than his probability of meeting no other low type seller, while the probability of trade for a high type seller is strictly smaller than his probability of meeting no low type seller.

Proposition 3.2. *For any* $(\underline{x}, \overline{x}, \underline{t}, \overline{t}) \in [0,1]^2 \times \mathbb{R}^2$ *and* $\overline{\lambda}, \underline{\lambda} \in [0, \infty)$*, there exists a feasible and incentive compatible mechanism* m *such that*

$$\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}, \quad \overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}, \quad \underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}, \quad \overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t},$$

if and only if

$$\overline{t} - \overline{x}\underline{c} \leq \underline{t} - \underline{x}\underline{c}, \tag{7}$$

$$\underline{t} - \underline{x}\overline{c} \leq \overline{t} - \overline{x}\overline{c}, \tag{8}$$

$$\overline{\lambda}\overline{x} \leq 1 - e^{-\overline{\lambda}}, \tag{9}$$

$$\underline{\lambda}\underline{x} \leq 1 - e^{-\underline{\lambda}}, \tag{10}$$

$$\overline{\lambda}\overline{x} + \underline{\lambda}\underline{x} \leq 1 - e^{-(\overline{\lambda}+\underline{\lambda})}. \tag{11}$$

Proof See Appendix A.1.

Conditions (7) and (8) are analogous to the sellers' incentive compatibility constraints (2) and (3). It is immediate to see that these two conditions imply $\underline{x} \geq \overline{x}$, that is the expected trading probability is higher for low than for high type sellers. The remaining three conditions correspond to the properties that the mechanism m associated to $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ is feasible according to (1) and that meetings take place according to the urn-ball technology. In particular, inequality (9) requires that a buyer's probability of trading with a high type seller is weakly smaller than a buyer's probability of meeting at least one high type seller. The expected probability of trading with a high type seller is given by the product of the expected number of high type sellers in a meeting, $\overline{\lambda}$, and their trading probability, $\overline{x}$, while the probability of meeting at least one high type seller is given by $\sum_{k=1}^{+\infty} P_k(\overline{\lambda}) = 1 - e^{-\overline{\lambda}}$. Similarly, inequality (10) requires that the probability that a buyer trades with a low type seller, $\underline{\lambda}\underline{x}$, is weakly smaller than the probability that a buyer meets at least one low type seller, $1 - e^{-\underline{\lambda}}$. Finally, inequality (11) requires that a buyer's probability of trading with any seller, $\overline{\lambda}\overline{x} + \underline{\lambda}\underline{x}$, is weakly smaller than the probability of meeting at least one seller, $1 - e^{-(\overline{\lambda}+\underline{\lambda})}$. It is useful to point out that condition (9) is redundant: it can be verified that $\underline{x} \geq \overline{x}$ together with condition (11) implies that a buyer's probability of trading a high quality object cannot exceed his probability of meeting a high type seller.

Following Eeckhout and Kircher (2010) and others, we next state an auxiliary optimization problem of a representative buyer who chooses a mechanism m, together with arrival rates $\overline{\lambda}(m)$ and $\underline{\lambda}(m)$, so as to maximize his payoff, taking as given the utility gain attained by low and high type sellers relative to their endowment point, denoted by $\underline{U}$ and $\overline{U}$. Given Proposition 3.2, rather than solving for a mechanism in the original mechanism space, we can equivalently solve for the expected values of trading probabilities and transfers associated to the mechanism, $\underline{x}, \overline{x}, \underline{t}, \overline{t}$, as long as they satisfy conditions (7-11). In the auxiliary problem, the choice of arrival rates $\underline{\lambda}, \overline{\lambda}$ associated to the mechanism is constrained by the conditions restricting equilibrium beliefs (5) and (6), which can be viewed as a form of participation constraints.[14] Letting $\underline{U} = \max_{m \in M^\beta} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)), \underline{U} = \max_{m \in M^\beta} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m))$, this amounts to the optimization problem

$$\max_{\underline{x}, \overline{x}, \underline{t}, \overline{t}, \overline{\lambda}, \underline{\lambda}} \overline{\lambda}(\overline{x}\overline{v} - \overline{t}) + \underline{\lambda}(\underline{x}\underline{v} - \underline{t}), \tag{P^{aux}}$$

subject to

$$\begin{aligned}
\overline{t} - \overline{x}\overline{c} &\leq \overline{U} \quad \text{holding with equality if } \overline{\lambda} > 0, \\
\underline{t} - \underline{x}\underline{c} &\leq \underline{U} \quad \text{holding with equality if } \underline{\lambda} > 0, \\
\overline{t} - \overline{x}\underline{c} &\leq \underline{t} - \underline{x}\underline{c}, \\
\underline{t} - \underline{x}\overline{c} &\leq \overline{t} - \overline{x}\overline{c}, \\
\underline{\lambda}\underline{x} &\leq \left(1 - e^{-\underline{\lambda}}\right), \\
\overline{\lambda}\overline{x} + \underline{\lambda}\underline{x} &\leq \left(1 - e^{-\overline{\lambda} - \underline{\lambda}}\right), \\
\overline{\lambda}, \underline{\lambda} &\geq 0.
\end{aligned}$$

If utilities $\underline{U}$ and $\overline{U}$ are such that the solutions of the buyer's auxiliary problem with respect to $\overline{\lambda}$ and $\underline{\lambda}$ are *consistent with the population parameters*, these solutions identify the mechanisms that are offered in equilibrium. By consistent we mean that to any solution $(\underline{x}^*, \overline{x}^*, \underline{t}^*, \overline{t}^*, \underline{\lambda}^*, \overline{\lambda}^*)$ of the buyer's auxiliary problem P^{aux} we can associate a value of β, indicating the measure of buyers posting the associated mechanism, so that the feasibility condition (4) is satisfied. More specifically, if the solution to the auxiliary problem is unique, consistency simply requires that the optimal arrival rates $\underline{\lambda}^*$ and $\overline{\lambda}^*$ coincide with the population parameters $\underline{\lambda}^p$ and $\overline{\lambda}^p$; in such case, there is a pooling equilibrium where all buyers post the same mechanism.[15] If the solution is not unique and the optimal values $\underline{\lambda}^*, \overline{\lambda}^*$ differ across the different solutions, consistency requires that the average value of arrival rates equals the population parameters, with weights equal to the fraction of buyers assigned to each solution; in such case, there is a separating equilibrium where

[14] Letting the representative buyer optimize directly over arrival rates implies that in cases where there are multiple solutions to (5,6), the buyer picks the preferred pair, which is consistent with condition iii) pinning down equilibrium beliefs. The auxiliary optimization problem will not allow the buyer to choose mechanisms for which the set of inequalities (5,6) does not have a solution. This comes without loss of generality because if $\underline{U}, \overline{U} > 0$, attracting infinitely many sellers always yields a strictly negative payoff and thus is never a solution of the auxiliary optimization problem. Lemma 3.3 will show that in equilibrium $\underline{U}, \overline{U} > 0$ is always satisfied.

[15] The same situation arises if we have multiple solutions of the auxiliary problem but for all of them we have the same values of $\underline{\lambda}^*, \overline{\lambda}^*$.

sellers sort according to their type across different mechanisms.[16]

Provided that the solutions to the buyer's auxiliary optimization problem P^{aux} are indeed consistent with the population parameter, we can find a set of feasible and incentive equilibrium mechanisms M^{β} such that each mechanism $m \in M^{\beta}$ corresponds to a solution of P^{aux}. By setting $\lambda(m) = \lambda^*$ for each of those mechanisms, the respective allocation not only satisfies the feasibility condition (4) but also all remaining equilibrium conditions. In particular, the two participation constraints imply that the seller's optimality condition is satisfied for all mechanisms posted in equilibrium and that there is no profitable deviation for buyers: given Proposition 3.2, for any $m \notin M^{\beta}$, the respective arrival rates, trading probabilities and expected transfers must belong to the constraint set of P^{aux} and thus yield a weakly smaller payoff than a solution of P^{aux}. Similarly, it is easy to see that any competitive search equilibrium, as specified in Definition 1, has to be such that the expected values of trading probabilities, transfers and arrival rates associated to mechanisms $m \in M^{\beta}$ solve the buyer's auxiliary optimization problem.[17] In the next section, we thus proceed to analyse the solutions of P^{aux}.

3.1 Solving the Buyer's Auxiliary Problem

It is useful to derive first some conditions on the sellers' market utilities $\underline{U}, \overline{U}$ that need to be satisfied in any equilibrium.

Lemma 3.3. *At a competitive search equilibrium, we have*

1.) $\underline{U} > \overline{U}$ *and* $\underline{U} - \overline{U} < \bar{c} - \underline{c}$*;*

2.) $\underline{U}, \overline{U} > 0$*;*

3.) $\underline{U} < \underline{v} - \underline{c}$ *and* $\overline{U} \leq \frac{\overline{v} - \bar{c}}{\overline{v} - \underline{c}} \underline{U}$*.*

Proof See Appendix A.2.

The properties stated in 1.) follow from the incentive compatibility conditions for low and high type sellers: in equilibrium the utility gain of low type sellers must be larger than that of high

[16] For example, suppose the buyer's auxiliary problem has two solutions with arrival rates, respectively, $\underline{\lambda}_1, \overline{\lambda}_1$ and $\underline{\lambda}_2, \overline{\lambda}_2$. If γ denotes the fraction of buyers posting in market 1, consistency requires $\gamma \underline{\lambda}_1 + (1-\gamma)\underline{\lambda}_2 = \underline{\lambda}^p$ and $\gamma \overline{\lambda}_1 + (1-\gamma)\overline{\lambda}_2 = \overline{\lambda}^p$.

[17] Suppose not and let $\underline{U} = \max_{m \in M^{\beta}} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)), \underline{U} = \max_{m \in M^{\beta}} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m))$. Then there exists a tuple $(\underline{x}, \overline{x}, \underline{t}, \bar{t}, \overline{\lambda}, \underline{\lambda})$ that satisfies the constraint set of P^{aux} and yields a strictly higher payoff for the buyer than the associated expected trading probabilities, transfers and arrival rates of any mechanism $m \in M^{\beta}$. By Proposition 3.2 and the conditions on equilibrium beliefs, we know that there exists a feasible and incentive compatible mechanism $m' \notin M^{\beta}$ with associated expected trading probabilities, transfers $(\underline{x}, \overline{x}, \underline{t}, \bar{t})$ and beliefs given by $(\overline{\lambda}, \underline{\lambda})$, implying that posting m' is a profitable deviation for a buyer.

type sellers but not too large. Conditions 2.) and 3.) are obtained from the property that any pair $(\underline{x}, \underline{t})$ and $(\overline{x}, \overline{t})$ of expected trading probabilities and transfers associated to the mechanisms traded in equilibrium by, respectively, low and high type sellers, must solve the buyers' auxiliary problem P^{aux} for some $\underline{\lambda}, \overline{\lambda} \in (0, +\infty)$. This requires first that in equilibrium both types of sellers make strictly positive gains (condition 2.)): if not, buyers would want to attract infinitely many sellers since additional sellers would come at no cost but increase a buyer's probability of trade. A second implication is that in equilibrium the profits of buyers, conditional on trading both with low type sellers, $\underline{x}\underline{v} - \underline{t}$, and with high type sellers, $\overline{x}\overline{v} - \overline{t}$, must be non-negative, as otherwise it would not be optimal for buyers to choose $\underline{\lambda}, \overline{\lambda} > 0$. This yields condition 3.): the market utility of low type sellers cannot exceed the gains from trade of the good they own; the restriction on the market utility of high type sellers is more stringent because their trading probability $\overline{x}$ is bounded above by the incentive compatibility constraint of low type sellers.

Next, we notice that the participation constraints of the two types of sellers in P^{aux} can be set holding as equalities w.l.o.g., as buyers can always specify transfers and trading probabilities that make indifferent the type they do not wish to attract.[18] Solving then the participation constraints for $\underline{t}, \overline{t}$ and substituting into the objective function and the remaining constraints, the buyer's optimization problem can be rewritten in the following simpler form

$$\max_{\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda}} \quad \overline{\lambda}\left[\overline{x}(\overline{v} - \overline{c}) - \overline{U}\right] + \underline{\lambda}\left[\underline{x}(\underline{v} - \underline{c}) - \underline{U}\right], \qquad (P^{aux\prime})$$

subject to

$$\overline{x}(\overline{c} - \underline{c}) \leq \underline{U} - \overline{U}, \qquad (12)$$

$$\underline{x}(\overline{c} - \underline{c}) \geq \underline{U} - \overline{U}, \qquad (13)$$

$$\underline{\lambda}\underline{x} \leq 1 - e^{-\underline{\lambda}}, \qquad (14)$$

$$\overline{\lambda}\overline{x} + \underline{\lambda}\underline{x} \leq 1 - e^{-\overline{\lambda} - \underline{\lambda}}, \qquad (15)$$

$$\overline{\lambda}, \underline{\lambda} \geq 0. \qquad (16)$$

In what follows we proceed to establish a number of properties of the solutions of problem $P^{aux\prime}$, some of which will directly translate into the features of competitive search equilibria stated in Theorem 3.1. Using these properties, we can then construct the competitive search equilibrium and thereby demonstrate existence and uniqueness of the equilibrium.

[18] See the Online Appendix for the details of the argument.

We can first show that at any solution of $P^{aux'}$, we have $\underline{\lambda x} = 1 - e^{-\underline{\lambda}}$. This condition says that a buyer's probability of trading a low quality good is equal to his probability of meeting a low type seller, which immediately implies property ii) of Theorem 3.1: any mechanism posted in equilibrium must give priority to low type sellers.

Lemma 3.4. *At any solution of $P^{aux'}$ the low type feasibility constraint (14) is satisfied with equality.*

Proof See Appendix A.3.

In the proof of Lemma 3.4 we show that if the low type sellers' feasibility constraint (14) holds as strict inequality so that not all meetings where a low type seller is present end up with the trade of a low quality object, buyers have a profitable deviation. This deviation is reminiscent of cream skimming and consists in offering a mechanism that attracts fewer low types and more high types, but gives priority to low type sellers so that the overall probability of trading a low and high quality good remains unchanged for the buyer. Hence, the expected gains from trade associated to the deviating mechanism are the same as those of the original mechanism. The deviation is feasible and incentive compatible. Moreover, it is profitable because high type sellers obtain a strictly lower market utility than low type sellers and are thus less costly to attract.

The property that the low type sellers' feasibility constraint is satisfied with equality at any solution of $P^{aux'}$ implies that a low type seller's probability of trade strictly exceeds a high type seller's probability of trade within any mechanism offered in equilibrium. In other words, no pooling mechanism can solve the buyer's auxiliary optimization problem. As pointed out above, the property that optimal mechanisms give priority to low type sellers holds irrespectively of whether the gains from trade are larger for the low quality or the high quality object. The nature of the gains from trade, and hence the severity of the adverse selection, on the other hand matters for the role played by the incentive constraints in equilibrium:

Lemma 3.5. *If $\underline{v} - \underline{c} \leq \overline{v} - \overline{c}$, at any solution of $P^{aux'}$ with $\overline{\lambda} > 0$ the low type incentive constraint (12) is satisfied with equality.*

Proof See Appendix A.4.

The proof shows that when (12) is slack it is possible to propose another mechanism that keeps priority for low type sellers in place but attracts more high and fewer low type sellers. This increases the trading probability of both types of sellers and constitutes a profitable deviation if gains from trade are larger for the high quality object. Conversely, if the low type incentive constraint is binding, the described deviation is not admissible because increasing the high type sellers' trading probability would violate that constraint.

We then show that, in order to further characterize the properties of equilibria, it is useful to distinguish the cases where condition 3.) of Lemma 3.3, $\overline{U} \leq \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$, which can be rewritten as $0 \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c}) - \overline{U}$, holds as a strict inequality or as an equality. In the first case, the payoff of a buyer with each high type seller, $\overline{x}(\overline{v}-\overline{c}) - \overline{U}$, is strictly positive at the maximal incentive feasible trading probability of high type sellers, $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. We show now that in this case at any solution of $P^{aux'}$ the feasibility condition (15) is satisfied with equality. That is, a buyer's probability of trade equals his probability of meeting a seller or, equivalently, in every meeting a good is traded. Given Lemma 3.4, this implies that also meetings with only high type sellers always lead to trade. In addition, the solution of the buyer's auxiliary problem is unique:

Lemma 3.6. *If $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$, then (i) at any solution of $P^{aux'}$ the overall feasibility constraint (15) is satisfied with equality, and (ii) the solution of $P^{aux'}$ is unique.*

Proof See Appendix A.5.

The proof of claim i) is immediate. If $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$, a buyer offering a mechanism that satisfies the low type incentive constraint (12) with equality makes a strictly positive profit with high type sellers. When the overall feasibility constraint (15) is slack, the buyer has a profitable deviation, which consists in attracting additional high type sellers, while keeping the terms of trade of all sellers unchanged.

Claim ii) is then established by showing first that if $\underline{v} - \underline{c} > \overline{v} - \overline{c}$, problem $P^{aux'}$ is strictly convex and hence has a unique solution. If $\underline{v} - \underline{c} \leq \overline{v} - \overline{c}$, the problem is not strictly convex but uniqueness can still be established by using the property established in Lemma 3.5 that the low type incentive constraint (12) is binding. Uniqueness of the solution of $P^{aux'}$ has an important implication for the equilibrium: any equilibrium satisfying $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$ must be such that sellers are pooled in a single submarket since all buyers will offer the uniquely optimal mechanism.

Consider next the alternative case where market utilities are such that $\overline{U} = \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. Here at the maximal trading probability of high type sellers $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ buyers make zero profits with these sellers. Buyers are then indifferent between the number of high type sellers they attract and there are so typically multiple values of $\overline{\lambda}$ that solve the buyer's auxiliary problem. As the following lemma asserts, the solution of $P^{aux'}$ is still unique for the other variables of that problem:

Lemma 3.7. *If $\overline{U} = \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$, $P^{aux'}$ has a unique solution for $\underline{\lambda}$ and $\underline{x}$. Moreover, at a solution of $P^{aux'}$ with $\overline{\lambda} > 0$ we have $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$.*

Proof See Appendix A.6.

The uniqueness of the solution of $P^{aux'}$ with respect to $\underline{\lambda}$ and $\underline{x}$ means that each mechanism that is posted in equilibrium must attract the same queue length of low type sellers, and the trading probability for low type sellers must be the same across all posted mechanisms. The result follows again from the strict convexity of the choice problem with regard to these variables. On the other hand, the solution with respect to the fraction of high type sellers to buyers is indeterminate, while their trading probability is pinned down by incentive compatibility.

The established properties of the solutions of $P^{aux'}$ demonstrate some key features of the candidate equilibrium. We will now use these properties to construct the competitive search equilibrium, and thereby demonstrate its existence as well as the remaining properties i) and iii) in Theorem 3.1.

3.2 Constructing the Competitive Search Equilibrium

3.2.1 Positive Profits with High Type Sellers

In light of the results of the previous section, to establish the existence of competitive search equilibria and further characterize their properties, it is useful to examine separately the two cases where market utilities are such that sellers can or cannot make strictly positive profits with high type sellers. We begin by considering the first case: $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. Here we know from Lemmas 3.4 and 3.6 that both feasibility constraints (14) and (15) are binding at all solutions of $P^{aux'}$, and so we have:

$$\underline{\lambda}\underline{x} = 1 - e^{-\underline{\lambda}} \quad \text{and} \quad \overline{\lambda}\overline{x} = e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right). \tag{17}$$

A buyer's probability of trading a low quality good thus equals his probability of meeting a low type seller, $1 - e^{-\underline{\lambda}}$, while a buyer's probability of trading a high quality good is equal to the probability of meeting no low type seller and meeting some high type seller, $e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right)$. Also, it follows from Lemma 3.6 that the solution of problem $P^{aux'}$ is unique. In order to obtain an equilibrium, we thus need to find values of $\underline{U}$ and $\overline{U}$ satisfying $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$ such that the population ratios $\underline{\lambda}^p$ and $\overline{\lambda}^p$, together with the associated values of $\underline{x}, \overline{x}$ obtained from (17), solve problem $P^{aux'}$. The following proposition provides the parameter restrictions under which such values of $\underline{U}, \overline{U}$ exist. It also establishes their equilibrium values, thereby characterizing the equilibrium allocation in closed form.

Proposition 3.8. *If* $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, *there exists a competitive search equilibrium in which buyers make strictly positive profits with high type sellers and post the same mechanism, character-*

ized by

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1 - e^{-\underline{\lambda}^p}\right), \quad \overline{x} = e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right), \quad \underline{t} = \underline{xc} + \underline{U}, \quad \overline{t} = \overline{xc} + \overline{U},$$

with $\underline{U}$ *and* $\overline{U}$ *set as follows:*

a) if $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \leq 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$,

$$\underline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\overline{v} - \overline{c}) + e^{-\underline{\lambda}^p}[(\underline{v} - \underline{c}) - (\overline{v} - \overline{c})], \tag{18}$$
$$\overline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\overline{v} - \overline{c}); \tag{19}$$

b) if $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \in \left(1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}, \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}\right)$,

$$\underline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right)\frac{1 - e^{-\overline{\lambda}^p}}{1 - \frac{1}{\overline{\lambda}^p}(1 - e^{-\overline{\lambda}^p})}(\overline{v} - \underline{v}), \tag{20}$$
$$\overline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right)\left[\frac{1 - e^{-\overline{\lambda}^p}}{1 - \frac{1}{\overline{\lambda}^p}(1 - e^{-\overline{\lambda}^p})}(\overline{v} - \underline{v}) - (\overline{c} - \underline{c})\right]. \tag{21}$$

Proof See Appendix A.7.

Condition $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ is always satisfied in the case of independent private values ($\underline{v} = \overline{v}$), and holds more generally when the difference between the buyer's valuation of the high and low quality good is sufficiently small and the ratio of high type sellers to buyers $\overline{\lambda}^p$ is sufficiently large.[19] We refer so to this parameter region as characterizing a situation where adverse selection is 'mild'. The fact that buyers only moderately care about the quality of the good and the property that the ratio of high type sellers to buyers is relatively large limit the extent to which buyers compete for high type sellers. As a consequence, the utility gain of high type sellers compared to that of low type sellers is sufficiently small so that in equilibrium buyers make strictly positive profits with high type sellers. It then follows from Lemma 3.6 that when buyers can make strictly positive profits with high type sellers in equilibrium all meetings lead to trade.

3.2.2 Zero Profits with High Type Sellers

We turn next our attention to the alternative scenario where market utilities are such that buyers can make at most zero profits with high type sellers: $\overline{U} = \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. In this case we know from Lemmas 3.4 and 3.7 that the low type feasibility constraint (14) and the low type incentive constraint (12)

[19] To see this, note that the function $f(\lambda) = \frac{1}{\lambda}(1 - e^{-\lambda})$ is strictly decreasing and lies between zero and one on its domain $(0, +\infty)$: we have in fact $f'(\lambda) = -\frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda^2} < 0$ for all $\lambda > 0$, $\lim_{\lambda\to 0} f(\lambda) = 1$ and $\lim_{\lambda\to+\infty} f(\lambda) = 0$.

are binding. The trading probability of low type sellers $\underline{x}$ is thus determined by (17) as before, while the trading probability of high type sellers, for $\overline{\lambda} > 0$, is given by $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$.

Consider first the possibility of a pooling equilibrium, i.e. the case where the population ratios $\underline{\lambda}^p, \overline{\lambda}^p$ belong to a solution of the buyer's problem $P^{aux'}$. As shown in Lemma 3.7, there is a unique value of $\underline{\lambda}$ solving $P^{aux'}$. Since buyers cannot make positive profits with high type sellers, this value simply maximizes the buyer's payoff with low type sellers. At an equilibrium the market utility of low type sellers $\underline{U}$ then needs to be such that the optimal value of $\underline{\lambda}$ is the population ratio $\underline{\lambda}^p$. The value of $\overline{U}$ is pinned down by the assumed condition $\overline{U} = \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. The equilibrium exists if, after substituting these values of $\underline{U}, \overline{U}$ into $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ and, using $\underline{x}$ as determined by (17), the overall feasibility constraint (15) is satisfied at $(\underline{\lambda}, \overline{\lambda}) = (\underline{\lambda}^p, \overline{\lambda}^p)$. The parameter restriction under which this is possible is stated in the following proposition.

Proposition 3.9. *If $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, there exists a competitive search equilibrium in which all buyers make zero profits with high type sellers and post the same mechanism characterized by*

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1 - e^{-\underline{\lambda}^p}\right), \quad \overline{x} = e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}, \quad \underline{t} = \underline{x}\underline{c} + \underline{U}, \quad \overline{t} = \overline{x}\overline{c} + \overline{U},$$

with

$$\underline{U} = e^{-\underline{\lambda}^p}(\underline{v} - \underline{c}), \tag{22}$$

$$\overline{U} = e^{-\underline{\lambda}^p}\frac{(\underline{v} - \underline{c})(\overline{v} - \overline{c})}{\overline{v} - \underline{c}}. \tag{23}$$

Proof See Appendix A.8.

The condition under which the above equilibrium exists, $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, is the exact complement of the one stated in Proposition 3.8. Note first that this condition can only be satisfied if $\overline{v} > \underline{v}$, that is in a situation of common values. More precisely, the condition requires that the buyers' valuation of the high quality object is sufficiently larger than that for the low quality object and the ratio of high type sellers to buyers $\overline{\lambda}^p$ is sufficiently small, a situation we refer to as 'severe' adverse selection. The facts that buyers have a high value for quality and that there are relatively few high type sellers per buyer, together with property that meeting with high type sellers do not crowd out those with low type sellers, imply that now there is intense competition for high type sellers. In equilibrium this leads buyers to make zero profits in trades with these types, who extract so all the gains from trade that are realized.

It is important to point out that, if the condition $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ is satisfied with strict inequality, the probability that a buyer ends up trading a high quality object, given by $\overline{\lambda}^p e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, is strictly smaller than the probability that a buyer is in a match with high type sellers only, $e^{-\underline{\lambda}^p}(1-e^{-\overline{\lambda}^p})$.[20] In the present parameter configuration the equilibrium mechanism thus not only gives priority to low type sellers but also rations high type sellers in meetings where no low type seller is present. Rationing can be sustained in equilibrium because buyers make zero profits with high type sellers and are therefore indifferent with respect to how many high type sellers they attract with the mechanism they post. The severity of the adverse selection further implies that incentive constraints are tight, in fact the low type incentive constraint (12) always binds in equilibrium. Rationing helps to sustain these incentives by driving the high types' probability of trade below the maximal level that is feasible (given the priority of low types).

Remark: *It is interesting to notice that the parameter region with severe adverse selection includes as a limiting case the situation where there is free entry of buyers, captured in our environment by letting the measure of buyers tend to* $+\infty$*. This case is of interest since it corresponds to the situation considered in Rothschild and Stiglitz's (1976) classic model of adverse selection as well as in other models where there is free entry of uninformed traders (or equivalently, each uninformed trader can trade any number of contracts). As* $b \to +\infty$*, the ratio of high type sellers to buyers,* $\overline{\lambda}^p$*, tends to zero, implying that* $\frac{1}{\overline{\lambda}^p}(1-e^{-\overline{\lambda}^p})$ *tends to one. In this case buyers make zero profits in equilibrium in all their trades. From the expressions in Proposition 3.9 it can be seen that, as* $b \to +\infty$*, the transfer to low type sellers conditional on trading converges to* $\underline{v}$*, while their trading probability converges to one;*[21] *the transfer to high type sellers conditional on trading, on the other hand, converges to* $\overline{v}$*, while their trading probability converges to* $\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$*. These trading probabilities and transfers precisely correspond precisely to the ones of the separating candidate equilibrium found by Rothschild and Stiglitz (1976).*[22]

As we already noticed, the conditions under which Propositions (3.8) and (3.9) hold partition the parameter space into two regions. The two results together thus ensure that a competitive equilibrium always exists. The results also say that in each of those regions market utilities $\underline{U}$ and $\overline{U}$ are uniquely determined by the exogenous parameter values. This implies that, in the class of pooling equilibria, the equilibrium is unique.

To complete the proof of claim iii) of Theorem 3.1, it remains to consider the possibility that other kinds of equilibria exist. The fact that in equilibrium it is always the case that all posted mechanisms must attract the same queue length of low type sellers immediately implies that there

[20] That is, the overall feasibility constraint (15) is satisfied as a strict inequality.

[21] As $b \to +\infty$, a seller's probability of being alone in a meeting with a buyer converges to one.

[22] Note that in competitive search models, as well as in Walrasian models, the non-existence issue found by Rothschild and Stiglitz (1976) in a strategic setting does not arise.

cannot be an equilibrium in which sellers of different types trade in separate markets, as it happens in environments with pairwise meetings. Nevertheless, when $\overline{U} = \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$, due to the indeterminacy of the solution with respect to $\overline{\lambda}$ established in Lemma 3.7, there may be other equilibria in which buyers post different mechanisms, attracting different queue lengths of high type sellers. The uniqueness of the solution of $P^{aux'}$ with respect to $\underline{\lambda}$, $\underline{x}$ and $\overline{x}$ implies however that buyers' and sellers' payoffs are the same as at the pooling equilibrium characterized in Proposition 3.9.

Proposition 3.10. *If there is an equilibrium where buyers post different mechanisms, a payoff equivalent equilibrium also exists in which all buyers post the same mechanism.*

Proof See Appendix A.9.

The proof is immediate: whenever an equilibrium exists with different submarkets and an asymmetric assignment of high type sellers across them, we can construct a payoff equivalent pooling equilibrium where all buyers and sellers participate in a single market and so high type sellers are distributed symmetrically across buyers. Together with the previous results, Proposition 3.10 then implies that the competitive search equilibrium is unique in expected payoffs and trading probabilities. This establishes Theorem 3.1.

4 Properties of the Equilibrium

In this section we discuss the main properties of the mechanisms that are traded in equilibrium and the payoffs attained by buyers and sellers.

4.1 Cross Subsidization

It is immediate to verify that in the equilibrium we found the payoff of a buyer conditional on meeting a low type seller is strictly higher than his payoff conditional on meeting a high type seller, i.e.

$$\underline{x}(\underline{v} - \underline{c}) - \underline{U} > \overline{x}(\overline{v} - \overline{c}) - \overline{U}.$$

To see why this property must hold in equilibrium note that, if on the contrary buyers make more profits with high than with low type sellers, a profitable cream skimming deviation always exists. Replacing low type sellers with high type sellers while keeping their trading probabilities and transfers unchanged is in fact feasible since sellers with a high quality good trade with a lower probability than sellers with a low quality good and is profitable in that case. A similar argument applies if buyers make the same profits with high and low type sellers.[23]

[23] The profitable deviation in this case consists in attracting a slightly larger number of high types, which is again feasible due to $\underline{x} > \overline{x}$.

The property that buyers must make more profits when they trade with low type sellers holds for all parameter values, also when the gains from trade of the high quality good are large relative to those of the low quality good. In that case it implies that most of these gains are appropriated by the sellers and so that the high quality object is traded at a high price. Incentive compatibility can then only be satisfied if the trading probability of the high type sellers is sufficiently small. As a result, we saw that in equilibrium we have rationing of high type sellers and the larger the buyers' valuation of the high quality good is, the more severe rationing becomes. Note that this feature stands in contrast to a monopolistic auction setting, or one with random rather than directed search, where a larger value of $\overline{v}$ favours pooling offers and thus leads to a weakly larger trading probability of the high quality good. Moreover, in such case of severe adverse selection buyers, as we saw, earn zero profits with high type sellers and strictly positive profits with low types. The pooling equilibrium thus features cross-subsidization, with buyers' payoffs not equalized across trades with high and low types sellers.

The argument that a buyer must make higher profits with low type sellers than with high type sellers does not rely on the fact that there are only two types of sellers but extends to any pair of types of sellers. It therefore provides a clue for how our results may extend to an environment with $N \geq 3$ types. In particular, the fact that cream skimming deviations must not be profitable in equilibrium implies that a buyer's profit conditional on meeting a seller must be decreasing in the seller's type, even when there are more than two types of sellers. From this the optimality of proposing mechanisms that give priority to sellers of lower type in turn follows. It remains then to be seen whether the trading probabilities associated to such a priority mechanism can be combined with a schedule of transfers such that buyers' profits are indeed decreasing in the sellers' type and incentive compatibility is still satisfied. If this is not the case, we conjecture that the equilibrium mechanism rations some types of sellers on top of what the priority rule implies. In such situations, by arguments similar to that of Lemma 3.6,[24] buyers must make zero profits with all sellers that are rationed in equilibrium. This in turn implies that buyers also make zero profits with sellers of higher types so that no profitable cream skimming deviation exists. Thus, in an environment with $N \geq 3$ types of sellers, buyers trade according to a priority mechanism and there might be a cutoff type such that buyers make strictly positive profits with seller types below the cutoff and zero profits with the remaining sellers, who are rationed in equilibrium.

[24] Lemma 3.6 shows that whenever buyers can make positive profits with high type sellers, a mechanism that rations high type sellers cannot solve a buyer's problem. Viceversa, one can show that, if the optimal mechanism rations a certain type of seller, buyers must make at most zero profits with that type.

4.2 Partial Sorting

We will next demonstrate that in situations where adverse selection is severe, so that buyers make zero profits with high type sellers, other equilibria that feature partial sorting indeed exist, in addition to the pooling equilibrium characterized in Proposition 3.9, as anticipated in the previous section. More specifically, when $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ there exist equilibria in which buyers post different mechanisms, attracting different queue lengths of high type sellers. To show this, consider the case where two submarkets are active in equilibrium, labelled 1 and 2, with seller-buyer ratios $\overline{\lambda}_1, \underline{\lambda}_1$ and $\overline{\lambda}_2, \underline{\lambda}_2$, respectively. For simplicity set $\overline{\lambda}_2 = 0$ so that high type sellers only visit submarket 1. Let the trading probabilities in the two submarkets be denoted by $\underline{x}_1, \overline{x}_1$ and $\underline{x}_2, \overline{x}_2$ and let γ denote the fraction of buyers posting mechanism 1.

From Proposition 3.10 we know that any equilibrium with sorting must be payoff equivalent to the pooling equilibrium characterized in Proposition 3.9. Market utilities $\underline{U}$ and $\overline{U}$ of low and high type sellers are therefore determined by (22) and (23). Given these market utilities, the uniquely optimal values of $\underline{\lambda}$ and $\underline{x}$, respectively, are given by $\underline{\lambda}^p$ and $\frac{1}{\underline{\lambda}^p}(1-e^{-\underline{\lambda}^p})$. We thus have $\underline{\lambda}_i = \underline{\lambda}^p, \underline{x}_i = \frac{1}{\underline{\lambda}_i}(1-e^{-\underline{\lambda}_i}), i = 1, 2$. The optimal trading probability for high type sellers in submarket 1 is pinned down by the low type incentive constraint (12), as in the corresponding pooling equilibrium, and is therefore given by $\overline{x}_1 = e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$.[25] Next, consistency with the population parameters requires that the ratio between high type sellers and buyers in submarket 1 is equal to $\overline{\lambda}_1 = \frac{\mu s}{\gamma b} = \frac{1}{\gamma}\overline{\lambda}^p$. Furthermore, the value of $\overline{\lambda}_1$ must be such that the feasibility constraint (15) is satisfied, which, after substituting the values we obtained for $\overline{x}_1$ and $\overline{\lambda}_1$, requires[26]

$$\frac{1}{\frac{1}{\gamma}\overline{\lambda}^p}\left(1-e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}.$$

Under the stated parameter restriction, $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, the above condition is always satisfied for an interval of values of γ sufficiently close to one, that is, such that sufficiently many buyers and low type sellers participate in submarket 1, the one where both types of objects are traded.

In the situation described, two mechanisms coexist in equilibrium. In submarket 2 buyers post a simple mechanism (effectively a price) that only attracts low type sellers, while in submarket 1 buyers post a more complex mechanism (some form of auction) attracting both high and low type sellers. Evidently, other sorting equilibria exist, where two or more mechanisms are traded, attracting different ratios of high type sellers. However, all these equilibria are payoff equivalent to

[25] In contrast, in submarket 2 where $\overline{\lambda}_2 = 0$, the value of $\overline{x}_2$ can be chosen freely as long as (12) is satisfied. For example, we can set $\overline{x}_2 = \overline{x}_1$.

[26] We have in fact $\overline{\lambda}_1\overline{x}_1 = \frac{1}{\gamma}\overline{\lambda}^p e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}} \leq e^{-\underline{\lambda}^p}\left(1-e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right)$, which yields the expression in the text.

the pooling equilibrium described in Proposition 3.9.

4.3 Implementation

Propositions 3.8 and 3.9 characterize expected trading probabilities and expected transfer payments of the mechanism traded in equilibrium. What can we say regarding the properties of mechanisms that implement these values? As pointed out in the introduction, for the case of independent private values, earlier work (see in particular Peters, 1997) has shown the existence of a competitive search equilibrium where all buyers post the same mechanism and the equilibrium trading probabilities and transfers can be implemented through a second-price auction with a reserve price equal to the buyers' valuation.[27] The validity of this property in our environment can be seen from the equilibrium characterization in Proposition 3.8, part a): if $\underline{v} = \overline{v} = v$, the parameter restriction $\frac{1}{\overline{\lambda}^p}(1-e^{-\overline{\lambda}^p}) \leq 1 - \frac{\overline{v}-v}{\overline{c}-\underline{c}}$ is always satisfied and the equilibrium market utilities of low and high type sellers are then given by

$$\underline{U} = e^{-\underline{\lambda}^p}\left(e^{-\overline{\lambda}^p}(v-\underline{c}) + \left(1-e^{-\overline{\lambda}^p}\right)(\overline{c}-\underline{c})\right), \quad \overline{U} = e^{-\underline{\lambda}^p-\overline{\lambda}^p}(v-\overline{c}).$$

In a second-price auction with reserve price v, a high type seller has a positive payoff if and only if he is the only seller in a meeting, which happens with probability $e^{-\underline{\lambda}^p-\overline{\lambda}^p}$. In this event, his payoff equals the difference between the reserve price v and his valuation $\overline{c}$. Otherwise the seller either looses the auction or pays a price equal to his valuation. A low type seller has a positive payoff if and only if he is the only low type seller in a meeting, which happens with probability $e^{-\underline{\lambda}^p}$. In this event, with probability $e^{-\overline{\lambda}^p}$ he is the only seller and earns $v-\underline{c}$, while with the complementary probability $1-e^{-\overline{\lambda}^p}$ there are also some high type sellers and his payoff is $\overline{c}-\underline{c}$.

From Propositions 3.8 and 3.9 we can see to which extent this result extends to the common value case. It turns out that, even though in the first parameter region (where $\frac{1}{\overline{\lambda}^p}(1-e^{-\overline{\lambda}^p}) < \frac{\underline{v}-\underline{c}}{\overline{v}-\overline{c}}$) the trading probabilities are identical to those in the private value case, they and the associated transfers can never be implemented through a standard second-price auction with reserve price r, not even if coupled with a participation fee or transfer κ. To see this, note that the payoffs of low and high type sellers associated to such an auction, denoted by $m_{r,\kappa}$, with associated seller-buyer ratios $\underline{\lambda}^p, \overline{\lambda}^p$, are given by

$$\begin{aligned}\underline{u}(m_{r,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p) &= e^{-\underline{\lambda}^p}\left(e^{-\overline{\lambda}^p}(r-\underline{c}) + (1-e^{-\overline{\lambda}^p})(\overline{c}-\underline{c})\right) + \kappa,\\ \overline{u}(m_{r,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p) &= e^{-\underline{\lambda}^p-\overline{\lambda}^p}(r-\overline{c}) + \kappa.\end{aligned}$$

[27] It should be emphasized that, compared to us, some of these papers consider more general environments regarding the type space (e.g. Peters, 1997) and the meeting technology (e.g. Eeckhout and Kircher, 2010).

Setting these two values, respectively, equal to the market utilities in Proposition 3.8 yields two linearly independent equations in r and κ which, as can be verified, have no solution, except when $\underline{v} = \overline{v}$.

In contrast, the equilibrium trading probabilities and transfers can be implemented, for instance, via a sequential auction. In particular, buyers could first run a second-price auction with reserve price $r_1 \in (\underline{c}, \overline{c})$ and, if nobody wins the auction, potentially run another auction with reserve price $r_2 > \overline{c}$. In the first parameter region the probability of going to the second round conditional on nobody winning the first round is equal to one, whereas in the second parameter region this probability is strictly smaller than one. Evidently, such sequential auction can generate the trading probabilities stated in Propositions 3.8 and 3.9. Furthermore, it can be easily verified that there always exists a pair (r_1, r_2) (potentially coupled with a participation transfer or fee κ) such that the sellers' expected utilities associated to this mechanism are given by the market utilities specified in Propositions 3.8 and 3.9.[28]

4.4 Welfare

In the economy under consideration the level of total surplus coincides with the realized gains from trade. At an allocation where the trading probabilities are, respectively, $\overline{x}$ and $\underline{x}$ for the high and low type sellers it is then given by

$$b\left[\overline{\lambda}^p \overline{x}(\overline{v} - \overline{c}) + \underline{\lambda}^p \underline{x}(\underline{v} - \underline{c})\right].$$

The welfare properties of the equilibria we characterized depend on the values of the parameters of the economy. We first establish the following:

Proposition 4.1. *If $\underline{v} - \underline{c} \geq \overline{v} - \overline{c}$ and $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, the competitive search equilibrium maximizes total surplus among all feasible allocations.*

The first condition in the statement of Proposition 4.1 says that the gains from trade are higher when trade occurs with the low type than with the high type seller. Under this condition, total surplus $b\left[\overline{\lambda}^p \overline{x}(\overline{v} - \overline{c}) + \underline{\lambda}^p \underline{x}(\underline{v} - \underline{c})\right]$ is maximal if, subject to the meeting friction, total trade is maximal and low quality is traded whenever possible. In the competitive search equilibrium the

[28] Letting $m_{r_1,r_2,\kappa,\rho}$ denote the mechanism with sequential reserve prices r_1, r_2, participation transfer κ, and a conditional probability of going to the second round ρ, the low and high type seller's payoff are, respectively, given by

$$\begin{aligned} \underline{u}(m_{r_1,r_2,\kappa}|\underline{\lambda}^p, \overline{\lambda}^p) &= e^{-\underline{\lambda}^p - \overline{\lambda}^p}(r_1 - \underline{c}) + \kappa, \\ \overline{u}(m_{r_1,r_2,\kappa}|\underline{\lambda}^p, \overline{\lambda}^p) &= e^{-\underline{\lambda}^p - \overline{\lambda}^p}\rho(r_2 - \overline{c}) + \kappa. \end{aligned}$$

latter property is always satisfied by Lemma 3.4, which states that a buyer's probability of trading a low quality good is equal to a buyer's probability of meeting a low type seller. Turning to the first property, total trade is maximal if the allocation maximizes the total number of meetings subject to the friction and if every meeting leads to trade. The requirement that every meeting leads to trade is satisfied in equilibrium if the overall feasibility constraint (15) holds with equality. The second condition in Proposition 4.1, $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, says that we are in the parameter region of mild adverse selection, where (15) is indeed satisfied as equality. To complete the proof it remains then to show that also the total number of meetings is maximal. It is well known that under the urn-ball meeting technology the total number of meetings is maximal whenever the expected queue length is the same for all the mechanisms traded in equilibrium.[29] This is indeed the case in the competitive search equilibrium, since sellers are pooled in a single market.

Note that the two conditions of Proposition 4.1 are always satisfied in the pure private value case, $\underline{v} = \overline{v}$. The result that the competitive search equilibrium maximizes social surplus in private value environments is well established in the literature (see for example Eeckhout and Kircher 2010).[30] Proposition 4.1 extends this result to the more general case where the gains from trade are larger for the low quality object, provided the ratio of high type sellers to buyers is not too low. It is interesting to point out that the only constraint that is considered in the social surplus maximization problem is the meeting friction. Thus, incentive compatibility does not constrain attainable welfare.

In contrast, when the gains from trade of the low quality object are strictly smaller than those of the high quality object, that is when the common value component of the agents' private information as captured by the difference $\overline{v} - \underline{v}$ is sufficiently large, the allocation of the competitive search equilibrium no longer maximizes total surplus. In this case social surplus is higher the larger the trades of the high quality good, however incentive constraints clearly limit such trades. We show in the next proposition that, even taking the incentive constraints into account, surplus is no longer maximal at the search equilibrium. Moreover, if the fraction of high type sellers relative to low type sellers is sufficiently large there exists an allocation that satisfies the constraints imposed by the meeting friction and incentive compatibility and Pareto improves on the allocation of the competitive search equilibrium.

[29]See for example Eeckout and Kircher (2010). For completeness of the argument we provide a formal proof of this property in our environment in the Online Appendix.

[30]In fact Cai et al. (2015) demonstrate for the private value case that the pooling of sellers into as single market maximizes social surplus under any type distribution if and only if the meeting technology satisfies a property called 'love for variety'. According to this property, in the binary type case, the probability of meeting at least one low type seller is a concave function of the ratio of low type sellers to buyers and the ratio of high type sellers to buyers. The condition implies that social surplus can be increased by merging any two submarkets, irrespective of their composition.

Proposition 4.2. *Suppose that $\underline{v} - \underline{c} < \overline{v} - \overline{c}$. Then there exists a feasible and incentive compatible allocation that attains a higher level of social surplus than at the competitive equilibrium. Moreover, the allocation constitutes a Pareto improvement if one of the two following conditions are satisfied:*

(i) $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ *and* $\mu \geq \frac{\overline{c}-\underline{c}}{\overline{v}-\underline{v}}$;

(ii) $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ *and* $\mu \geq \frac{\overline{c}-\underline{c}}{\overline{v}-\underline{c}}$.

Proof See Appendix A.10.

The proof of the proposition shows that an increase in the trading probability of high type sellers, relative to their level of trade at the equilibrium, is both feasible and incentive compatible. Such an increase, possibly combined with a suitable reduction in the trading probability of the low type sellers, is in fact always feasible. In addition, the expected transfers to the high and low type sellers can always be suitably adjusted so as to ensure that incentive compatibility is satisfied. This change in the allocation always increases social surplus when the gains from trade are larger for the high quality good and improves sellers' utility. We then show that it constitutes a Pareto improvement, in the sense that also buyers gain, provided the fraction of high type sellers in the population of sellers is sufficiently high. In this case the additional gains made with high type sellers more than compensate the possible losses with low types. It is interesting to notice that the inefficiency of the search equilibrium obtains not only under condition (ii), when the equilibrium displays rationing, but also under (i), where the feasibility constraint is satisfied with equality in equilibrium and every meeting leads to trade.

The inefficiency of the competitive search equilibrium with private information of the common value type is related to analogous results obtained for different market structures (see Gale (1992), Guerrieri et al. (2010) for the case of competitive search equilibria when meetings are restricted to be bilateral[31], Dubey and Geanakoplos (2002) and Bisin and Gottardi (2006) for competitive equilibria in the absence of meeting frictions). The common feature to all these results is that the equilibrium is separating, with different mechanisms traded by low and high type sellers and buyers being indifferent between trading with any of the two types. In contrast, in the environment considered here, the space of possible mechanisms exploits the richness of the possible meetings between buyers and sellers allowed by the meeting technology and the equilibrium is pooling with a single mechanism traded in equilibrium, though the implied trading probabilities of high and low type sellers are different. Also, as we noticed earlier, buyers' profits are not equalized across sellers' types. However, both in the environment considered here and in the works recalled above,

[31]See the next section for further discussion of the relationship with this line of work.

the source of the inefficiency is the low trading probability of high type sellers, and a welfare improvement is attained by bringing their probability of trade closer to that of low type sellers.[32]

The reason why at the competitive search equilibrium characterized in Section 2 there is no profitable deviation that allows to capture the additional gains from trade is that such deviation, similarly to the papers cited above, would attract too many low type sellers in order to be profitable. To see this, notice that, in order to increase the trading probability of high type sellers, a buyer would have to pay an additional information rent to low type sellers, which implies that all low type sellers would have strict incentives to switch to the deviating contract and makes the deviation non profitable.[33]

5 Sorting Versus Screening

An important benchmark for our analysis is provided by the results obtained in earlier work on competitive equilibria with directed search in markets with adverse selection (see Gale (1992), Inderst Muller (2002), Guerrieri et al. (2010)). These papers restrict attention to the case where meetings are bilateral, that is where each buyer can meet at most one seller, and show that, under this restriction, the competitive search equilibrium exhibits ex-ante sorting instead of ex-post screening. Restricting the matching technology to bilateral meetings limits the set of available mechanisms to the posting of menus, from which one seller, randomly selected among those matched with a buyer, can choose. Conditionally on being chosen, a seller's trading probabilities and transfers then depend on his own report but not on the report of others. A special case of such mechanism is a posted price.

In our environment we capture this restriction on the set of feasible mechanisms to bilateral menus by requiring expected trading probabilities to satisfy, respectively, $\overline{x}, \underline{x} \leq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1 - e^{-\underline{\lambda}-\overline{\lambda}}\right)$. The term $\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1 - e^{-\underline{\lambda}-\overline{\lambda}}\right)$ corresponds to the probability that, in a market where each buyer is visited on average by $\underline{\lambda}+\overline{\lambda}$ sellers, a seller is randomly selected by a buyer among all arriving sellers. The next proposition shows that, given this restriction on admissible mechanisms, the equilibrium outcome, as one should expect, displays ex-ante sorting instead of ex-post screening and buyers simply post a price (that is, choose menus with $\underline{x} = \overline{x}$ and $\underline{t} = \overline{t}$).

[32] As recalled above, this stands in contrast to a monopolistic setting, where for μ large enough the equilibrium contract features equal trading probabilities for both types.

[33] More precisely, if a buyer posts a mechanism m that yields a payoff for low type sellers strictly larger than $\underline{U}$ for any pair $\underline{\lambda}, \overline{\lambda}$, the buyer's belief is pinned down by condition (ii) of Definition 1, i.e. $\underline{\lambda}(m) = +\infty$. Given this belief, the deviating contract always yields a strictly negative payoff for the buyer. On the other hand, if the buyer post a mechanism for which a solution to (5,6) exists, the pair of seller-buyer ratios is such that the deviating mechanism yields a utility for low type sellers not higher than their market utility, implying that they cannot receive an additional information rent.

Proposition 5.1. *If the set of available mechanism is restricted to bilateral menus, a competitive search equilibrium exists and has the following properties:*

- *a fraction* $\gamma \in [0, 1)$ *of buyers post a price* p_h *and only attract high type sellers;*
- *the remaining fraction* $1 - \gamma \in (0, 1]$ *of buyers post a price* $p_l < p_h$ *and only attract low type sellers.*

Proof See the Online Appendix.

As we show in the proof of Proposition 5.1, under the stated restriction on the feasible mechanisms, buyers never find it optimal to attract both types of sellers. We further demonstrate that, if the gains from trade for the low quality good are considerably larger than the gains from trade for the high quality good, in equilibrium buyers strictly prefer to attract low type sellers, while high type sellers are excluded from trade. If this is not the case, a fraction of buyers post a price that only attracts high type sellers, while the remaining buyers post a price that attracts low type sellers and, given the implied arrival rates in each market, all buyers are indifferent between posting any of the two prices.

The analysis in Section 3 shows that, once we allow for more general mechanisms, (pure) sorting equilibria as the ones characterized in Proposition 5.1 do not exist. The intuition for why this is the case is simple. Consider a buyer who posts a price which only attracts low type sellers. If the set of available mechanisms includes mechanisms that give priority to one type of seller over another, a buyer could alternatively post a mechanism, analogous to an auction, that yields the same trading probability and expected transfer for low type sellers as the posted price, but also attracts some high type sellers. Since buyers have no capacity constraints in their ability to meet sellers, these additional meetings would not crowd out any meetings with low type sellers. Hence this mechanism would attract the same number of low type sellers, and the buyer would obtain the same expected payoff from them; on top of that, the buyer can make some positive profits with high type sellers.

Given these rather different properties of the competitive search equilibrium when mechanisms are restricted to bilateral menus, where sellers sort ex-ante across separate submarkets, compared to those of the equilibrium characterized in our Theorem 3.1, where sellers are screened ex-post by the mechanism, it is then of interest to compare the two in terms of welfare. If the gains from trade for the low quality good exceed the gains from trade for the high quality good, the latter, as we showed in Proposition 4.1, maximizes social surplus. Hence the equilibrium with ex-post screening always dominates the one with ex-ante sorting in terms of total surplus. On the other hand, if adverse selection is severe and the gains from trade are larger for the high quality good, the equilibrium with ex-post screening can be constrained inefficient, as shown in Proposition 4.2, not only because the mechanism traded in equilibrium gives priority to the good with the lower gains from trade but also because there can be rationing on top of the meeting friction. One might conjecture that the equilibrium with ex-ante sorting may do better in such cases. However, we find that this is typically not the case,[34] as the next numerical example illustrates.

Example 1: *Let* $\underline{\lambda}^p = \overline{\lambda}^p = 1$ *and* $\underline{c} = 0, \overline{c} = 1, \underline{v} = 1, \overline{v} = 3$. *Thus the gains from trade for the high quality good are twice as large as those for the low quality good and there are twice as many buyers as sellers, half of whom have a high quality good. Under this specification, we have* $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, *that is we are in the parameter region with severe adverse selection. The equilibrium with general direct mechanisms is then as characterized in Proposition 3.9 and features rationing. In particular, a buyer's probability to trade, respectively, a high and low quality good is given by*

$$\begin{aligned} \overline{\lambda}^p \overline{x} &= \overline{\lambda}^p e^{-\underline{\lambda}^p} \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}} \approx 0.123, \\ \underline{\lambda}^p \underline{x} &= 1 - e^{-\underline{\lambda}^p} \approx 0.632. \end{aligned}$$

while a buyer's probability to meet some high type seller without meeting a low type seller is equal to $e^{-\underline{\lambda}^p}(1 - e^{-\overline{\lambda}^p}) \approx 0.233$. *Hence, in meetings without low type sellers trade occurs only slightly more than half of the time.*

In the equilibrium where mechanisms are restricted to bilateral menus a fraction $\gamma \approx 0.120$ *of buyers post the high price, while the remaining fraction of buyers post a low price. A buyer's probability to trade a high and low quality good is now*

$$\begin{aligned} \overline{\lambda}^p \overline{x} &= \gamma \left(1 - e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right) \approx 0.120, \\ \underline{\lambda}^p \underline{x} &= (1 - \gamma) \left(1 - e^{-\frac{1}{1-\gamma}\underline{\lambda}^p}\right) \approx 0.598. \end{aligned}$$

In the above example we see that in the equilibrium with ex-ante sorting the probability that a buyer meets a seller decreases, when compared to the equilibrium with ex-post screening, from 86.5% to 71.8%. This is due to the meeting technology, as allocating buyers and sellers over two submarkets with different seller-buyer ratios implies that there is a higher chance that sellers end up misallocated across buyers. The probability of trading a low quality good decreases as well and

[34] That is, in all numerical simulations we considered such that parameters fall into the region with severe adverse selection, the social surplus in the equilibrium with ex-post screening exceeds that in the equilibrium with ex-ante sorting.

this should be clear since with ex-ante sorting low type sellers distribute themselves only across a fraction of buyers rather than across all of them.

What is perhaps more surprising is that, in the situation considered in Example 1, in the equilibrium with ex-ante sorting also the probability of trading a high quality good decreases slightly from 12.3% to 12.0%, which implies that social surplus is unambiguously lower. To gain some understanding of why this happens, notice that in such an equilibrium the incentive constraints for low type sellers require that the seller-buyer ratio in the high quality market is higher than in the low quality market. This implies that a buyer's probability of meeting a high type seller strictly exceeds that of meeting a low type seller. Since in equilibrium buyers have to be indifferent between attracting high and low type sellers, it follows that, conditional on meeting a seller, a buyer has to make lower profits with high than with low type sellers. This in turn implies that most of the gains from trade of the high quality good have to go to high type sellers, which can only be incentive compatible if a high type seller's trading probability is sufficiently low, the more so the higher these gains from trade are, similarly to the case of ex-post screening.

At the same time, we should point out that we can also find environments in which the equilibrium with ex-ante sorting exhibits a strictly higher social surplus than the equilibrium with ex-post screening. Notably, this reversal does not arise when the latter equilibrium features rationing (as the previous example shows), but rather when parameters fall in the region where every meeting leads to trade. The next proposition demonstrates that, provided the gains from trade for the high quality good exceed those for the low quality good and the measure of high type sellers is sufficiently large (so that competition among buyers for high type sellers is limited), the equilibrium with ex-ante sorting yields a strictly larger social surplus compared to the equilibrium with ex-post screening.

Proposition 5.2. *Assume $\underline{v} - \underline{c} < \overline{v} - \overline{c}$. If the measure of high type sellers tends to $+\infty$, social surplus is strictly greater when buyers are restricted to bilateral menus compared to when they can post general direct mechanisms.*

Proof See Appendix A.11.

The above result can be explained as follows. For the equilibrium with general mechanisms, the conditions $\underline{v} - \underline{c} < \overline{v} - \overline{c}$ and $\overline{\lambda}^p \to +\infty$ imply that we are in the parameter region with mild adverse selection, where the equilibrium features no rationing. The property that low type sellers are given priority in every meeting implies that buyers trade the low quality good with probability $1 - e^{-\underline{\lambda}^p}$, the probability with which they meet at least one low type seller. As $\overline{\lambda}^p \to +\infty$, a buyer's probability of meeting some high type seller tends to one, which, given the no rationing property, implies that in the limit buyers trade a high quality good with the residual probability, $e^{-\underline{\lambda}^p}$. Social surplus thus tends to $b[e^{-\underline{\lambda}^p}(\overline{v}-\overline{c})+(1-e^{-\underline{\lambda}^p})(\underline{v}-\underline{c})]$.

In the equilibrium with ex-ante sorting social surplus is strictly higher because, as $\overline{\lambda}^p \to +\infty$, the fraction of buyers attracting high type sellers, γ, tends to one. In this equilibrium the trading probability of sellers in fact converges to zero in both submarkets, but the relative probability of trade in the high quality submarket compared to the low quality submarket is sufficiently small so that the incentive compatibility constraint of low type sellers is satisfied. As a consequence, the probability that a buyer trades tends to one in both submarkets and the measure of buyers posting the high price tends to b. In the limit, social surplus in the equilibrium with ex-ante sorting is thus given by $b(\overline{v}-\overline{c})$, which is equal to the first best level and strictly exceeds social surplus in the equilibrium with ex-post screening. Due to the property that in the latter equilibrium low type sellers are given priority over high type sellers, the difference in social surplus between the two types of equilibria is largest when also the measure of low type sellers is large. These properties are also illustrated in the following example.

Example 2: *Let $\overline{\lambda}^p = 8$, $\underline{\lambda}^p = 2$ and $\underline{c} = 0, \overline{c} < 1.5, \underline{v} = 2.5, \overline{v} = 4$. Again the gains of trade for the high quality good are strictly greater than those for the low quality good. However, compared to Example 1, for every buyer there are eight high type sellers and two low type sellers. Under this specification, we have $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, we are thus in the parameter region where the equilibrium with general direct mechanisms features no rationing. In this equilibrium, a buyer's probability to trade, respectively, a high and a low quality good is given by*

$$
\begin{aligned}
\overline{\lambda}^p\overline{x} &= e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \approx 0.135, \\
\underline{\lambda}^p\underline{x} &= 1-e^{-\underline{\lambda}^p} \approx 0.865.
\end{aligned}
$$

In the equilibrium where mechanisms are restricted to bilateral menus the fraction of buyers posting the high price is $\gamma \approx 0.392$ and the probability a buyer trades, respectively a high and low quality good is given by

$$
\begin{aligned}
\overline{\lambda}^p\overline{x} &= \gamma\left(1-e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right) \approx 0.392, \\
\underline{\lambda}^p\underline{x} &= (1-\gamma)\left(1-e^{-\frac{1}{1-\gamma}\underline{\lambda}^p}\right) \approx 0.585.
\end{aligned}
$$

In Example 2 the equilibrium with ex-post screening has the feature that, due to the high seller-buyer ratio, almost all buyers are matched but only 13.5% of them end up purchasing a high quality good. This is due to the fact that there are twice as many low type sellers as buyers, so that the probability that a buyer meets some low type seller is relatively high (86.5%). Thus, although the majority of sellers have a high quality good, the feature that low type sellers are given priority in any match, together with a large seller-buyer ratio, implies that high quality is traded relatively rarely. In the equilibrium with ex-ante sorting, on the other hand, the probability that a buyer trades is slightly lower $(97,7\%)$ but the probability of trading a high quality good is considerably higher (almost 40%). Whether this leads to an increase in surplus or not depends on the seller's valuation of the high quality good. If $\overline{c}$ is sufficiently small (in our numerical example $\overline{c} < 1.28$), the effect of the increased probability of trade of the high quality object outweighs the effect of the reduced overall probability of trade and surplus is larger in the sorting equilibrium.

To sum up, the analysis in this section shows that the features of the meeting technology and hence of the possible trading mechanisms between buyers and sellers have important and nontrivial implications for the properties of equilibrium allocations, and in particular welfare. While for most parameter specifications the equilibrium with general direct mechanisms, featuring ex-post screening, yields a higher level of social surplus than the equilibrium where mechanisms are restricted to menus and where we have ex-ante sorting, this is not always the case. Hence, there are some situations in which policies imposing restrictions on the set of available mechanisms are beneficial, but in several other situations improving policies are those that encourage meeting technologies without capacity constraints, imposing no restrictions on the set of mechanisms.[35]

References

[1] Akerlof, G.A., 1970. The Market for "Lemons": Quality Uncertainty and the Market Mechanism. The Quarterly Journal of Economics, 84(3), 488-500.

[2] Albrecht, J.W., P.A. Gautier and S.B. Vroman, 2006. Equilibrium Directed Search with Multiple Applications. Review of Economic Studies, 73, 869-891.

[3] Albrecht, J.W., P.A. Gautier and S.B. Vroman, 2014. Efficient Entry in Competing Auctions. American Economic Review, 104(10), 3288-3296.

[4] Blouin, M.R. and R. Serrano, 2001. A Decentralized Market with Common Values Uncertainty: Non-Steady States. Review of Economic Studie 68, 323-346.

[5] Burdett, S., S. Shi and R. Wright, 2001. Pricing and Matching with Frictions. Journal of Political Economy, 109, 1060-85.

[35] Lester et al. (2015b) study a related issue in an environment with random search and imperfect competition. In particular, they examine how the features of the meeting technology affect traders' market power and hence the consequences for the welfare properties of equilibria in the presence of adverse selection.

[6] Cai, X., P. Gautier and R. Wolthoff, 2015. Inclusive versus Exclusive Markets: Search Frictions and Competing Mechanisms. Working Paper.

[7] Camargo, B. and B. Lester, 2014. Trading Dynamics in Decentralized Markets with Adverse Selection. Journal of Economic Theory, 153, 534-568.

[8] Dubey, P. and J. Geanakoplos, 2002. Competitive Pooling: Rothschild-Stiglitz reconsidered. Quarterly Journal of Economics, 107(4), 1529-1570.

[9] Eeckhout, J. and P. Kircher, 2010. Sorting vs Screening - Search Frictions and Competing Mechanisms. Journal of Economic Theory, 145, 1354-1385.

[10] Fuchs, W. and A. Skrzypacz, 2013. Costs and Benefits of Dynamic Trading in a Lemons Market. Working Paper.

[11] Gale, D., 1992. A Walrasian Theory of Markets with Adverse Selection. Review of Economic Studies, 59, 229-55.

[12] Guerrieri, V., R. Shimer and R. Wright, 2010. Adverse Selection in Competitive Search Equilibrium. Econometrica, 78(6), 1823-1862.

[13] Inderst, R. and H. Mueller, 1999. Competitive Search Markets with Adverse Selection. Working Paper.

[14] Janssen, M. and R. Roy, 2002. Dynamic trading in a durable good market with asymmetric information. International Economic Review 43(1), 257-282.

[15] Kim, K. and P. Kircher, 2015. Efficient Competition through Cheap Talk: The Case of Competing Auctions. Econometrica 83(5), 1849-1875.

[16] Lester, B., L. Visschers and R. Wolthoff, 2015a. Meeting Technologies and Optimal Trading Mechanisms in Competitive Search Markets. Journal of Economic Theory, 155, 1-15.

[17] Lester, B., S. Shourideh, V. Venkateswaran and A. Zetlin-Jones, 2015b. Screening and Adverse Selection in Frictional Markets. Working Paper.

[18] McAfee, P., 1993. Mechanisms Design by Competing Sellers. Econometrica, 61, 1281-1312.

[19] Moen, E., 1997. Competitive Search Equilibrium. Journal of Political Economy, 105, 385-411.

[20] Moreno, D. and J. Wooders, 2015. Dynamic Markets for Lemons: Performance, Liquidity, and Policy Intervention. Theoretical Economics, *forthcoming.*

[21] Peters, M., 1997. A Competitive Distribution of Auctions. The Review of Economic Studies, 64(1), 97-123.

[22] Peters, P. and S. Severinov, 1997. Competition Among Sellers Who Offer Auctions Instead of Prices. Journal of Economic Theory, 75(1), 75(1), 141-179.

[23] Rothschild, M. and J. Stiglitz, 1976. Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information. Quarterly Journal of Economics, 90, 629-649.

[24] Shimer, R., 2005. The Assignment of Workers to Jobs in an Economy with Coordination Frictions. Journal of Political Economy, 2005, 113 (5), 996-1025.

A Appendix

A.1 Proof of Proposition 3.2

If: We first show that for any vector $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ satisfying conditions (7)-(11), there exists a feasible and incentive compatible mechanism m such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}, \overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$ and $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}, \overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. Consider the following mechanism

$$\begin{aligned} \underline{X}_m(L,H) = &\ \underline{\rho}\tfrac{1}{L+\alpha H}, \qquad \underline{T}_m(L,H) = \underline{t}, \quad L \geq 1, \\ \overline{X}_m(L,H) = &\ \overline{\rho}\tfrac{\alpha}{L+\alpha H}, \qquad \overline{T}_m(L,H) = \overline{t}, \quad H \geq 1, \end{aligned}$$

for some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$. For the case $\alpha = 0$, let $\overline{X}_m(0,H) = \overline{\rho}\frac{1}{H}$.

This mechanism trivially satisfies $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}$ and $\overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. We now show that there always exists some tuple $(\alpha, \underline{\rho}, \overline{\rho})$ such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}$ and $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$. Ex-ante trading probabilities are given by

$$\begin{aligned} \underline{x}_m(\underline{\lambda}, \overline{\lambda}) &= \underline{\rho} \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{1}{L+1+\alpha H}, \\ \overline{x}_m(\underline{\lambda}, \overline{\lambda}) &= \overline{\rho} \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{\alpha}{L+\alpha(H+1)}. \end{aligned}$$

Define the function

$$f(\alpha)=\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+\alpha H}.$$

Note that $f'(\alpha)<0$. The function's range is given by $\left[\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right),\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right)\right]$. To see this, consider first the case $\alpha=0$:

$$f(0)=\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1}=\sum_{L=0}^{+\infty}P_L(\underline{\lambda})\frac{1}{L+1}\underbrace{\sum_{H=0}^{+\infty}P_H(\overline{\lambda})}_{=1}=\frac{1}{\underline{\lambda}}\sum_{L=0}^{+\infty}\frac{\underline{\lambda}^{L+1}}{(L+1)!}e^{-\underline{\lambda}}=\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right).$$

Consider next the case $\alpha=1$:

$$f(1)=\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+H}=\sum_{N=0}^{+\infty}P_N(\underline{\lambda}+\overline{\lambda})\frac{1}{N+1}=\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right).$$

Next, define the function

$$g(\alpha)=\begin{cases}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{\alpha}{L+\alpha(H+1)} & \text{if } \alpha>0,\\ \sum_{H=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{H+1} & \text{if } \alpha=0.\end{cases}$$

Note that $g'(\alpha)>0$ and that g is continuous at $\alpha=0$, i.e. $\lim_{\alpha\to 0}g(\alpha)=g(0)$. At $\alpha=1$, g is equal to f. At $\alpha=0$, we have

$$g(0)=\sum_{H=0}^{+\infty}P_H(\overline{\lambda})P_0(\underline{\lambda})\frac{1}{H+1}=e^{-\underline{\lambda}}\sum_{H=0}^{+\infty}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\frac{1}{H+1}=\frac{e^{-\underline{\lambda}}}{\overline{\lambda}}\sum_{H=0}^{+\infty}\frac{\overline{\lambda}^{H+1}}{(H+1)!}e^{-\overline{\lambda}}=e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right).$$

The range of g is consequently $\left[e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right),\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)\right]$.

With this we can show that for any $\underline{x}$ and $\overline{x}$ satisfying conditions (9)-(11) we can find some $\alpha,\underline{\rho},\overline{\rho}\in[0,1]$ such that $\underline{\rho}f(\alpha)=\underline{x}$ and $\overline{\rho}g(\alpha)=\overline{x}$. Given that $\underline{x},\overline{x}\geq 0$ and and $\underline{\rho},\overline{\rho}\in[0,1]$, this can be satisfied if there exists an α such that $f(\alpha)\geq\underline{x}$ and $g(\alpha)\geq\overline{x}$. The first inequality requires that α is not too large, while the second requires that α is not too small. Consider first the case in which $\underline{x}\leq\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$. Here $f(\alpha)\geq\underline{x}$ is satisfied for all $\alpha\in[0,1]$. Conditions (7),(8) and (11) together imply $\overline{x}\leq\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$, from which it follows that $g(\alpha)\geq\overline{x}$ can be satisfied (e.g. $\alpha=1$). Consider now the case $\underline{x}\geq\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$ and let $\tilde{\alpha}$ be such that $f(\tilde{\alpha})=\underline{x}$. We can show

$$\begin{aligned}
\overline{\lambda}g(\tilde{\alpha})+\underline{\lambda}f(\tilde{\alpha}) &= \overline{\lambda}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{\tilde{\alpha}}{L+\tilde{\alpha}(H+1)}+\underline{\lambda}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+\tilde{\alpha}H},\\
&= \overline{\lambda}\sum_{H=1}^{+\infty}\sum_{L=0}^{+\infty}\frac{\overline{\lambda}^{H-1}}{(H-1)!}\frac{\underline{\lambda}^{L}}{L!}e^{-\underline{\lambda}-\overline{\lambda}}\frac{\tilde{\alpha}}{L+\tilde{\alpha}H}\frac{H}{H}+\underline{\lambda}\sum_{H=0}^{+\infty}\sum_{L=1}^{+\infty}\frac{\overline{\lambda}^{H}}{H!}\frac{\underline{\lambda}^{L-1}}{(L-1)!}\frac{1}{L+\tilde{\alpha}H}\frac{L}{L},\\
&= \sum_{H=1}^{+\infty}\sum_{L=1}^{+\infty}\frac{\overline{\lambda}^{H}}{H!}\frac{\underline{\lambda}^{L}}{L!}e^{-\underline{\lambda}-\overline{\lambda}}\left(\underbrace{\frac{\tilde{\alpha}H}{L+\tilde{\alpha}H}+\frac{L}{L+\tilde{\alpha}H}}_{=1}\right)+\sum_{H=1}^{+\infty}\frac{\overline{\lambda}^{H}}{H!}e^{-\underline{\lambda}-\overline{\lambda}}+\sum_{L=1}^{+\infty}\frac{\underline{\lambda}^{L}}{L!}e^{-\underline{\lambda}-\overline{\lambda}},\\
&= \left(1-e^{-\overline{\lambda}}-e^{-\underline{\lambda}}+e^{-\underline{\lambda}-\overline{\lambda}}\right)+\left(e^{-\underline{\lambda}}-e^{-\underline{\lambda}-\overline{\lambda}}\right)+\left(e^{-\overline{\lambda}}-e^{-\underline{\lambda}-\overline{\lambda}}\right),\\
&= 1-e^{-\underline{\lambda}-\overline{\lambda}}.
\end{aligned}$$

With this,

$$g(\tilde{\alpha})=\frac{1}{\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}-\underline{\lambda},f(\tilde{\alpha})\right)=\frac{1}{\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}-\underline{\lambda}\underline{x}\right)\geq\overline{x},$$

where the last inequality follows from condition (11). Thus, there exists some $\overline{\rho}\in[0,1]$ such that $\overline{\rho}g(\tilde{\alpha})=\overline{x}$. Together this implies that for any $\underline{x}$ and $\overline{x}$ satisfying conditions (9)-(11), there exists some $\alpha,\underline{\rho},\overline{\rho}\in[0,1]$ such that $\underline{x}_m(\underline{\lambda},\overline{\lambda})=\underline{x}$ and $\overline{x}_m(\underline{\lambda},\overline{\lambda})=\overline{x}$.

Finally we need to check feasibility and incentive compatibility of the proposed mechanism. Feasibility follows from

$$\underline{x}(L,H)L+\overline{x}(L,H)H=\underline{\rho}\frac{1}{L+\alpha H}L+\overline{\rho}\frac{\alpha}{L+\alpha H}H\leq\frac{1}{L+\alpha H}L+\frac{\alpha}{L+\alpha H}H=1.$$

Incentive compatibility is trivially satisfied given that $\underline{x}_m(\underline{\lambda},\overline{\lambda})=\underline{x},\overline{x}_m(\underline{\lambda},\overline{\lambda})=\overline{x}$ and $\underline{t}_m(\underline{\lambda},\overline{\lambda})=\underline{t},\overline{t}_m(\underline{\lambda},\overline{\lambda})=\overline{t}$.

Only if: We now want to show that for any feasible and incentive compatible mechanism m, expected trading probabilities and prices satisfy conditions (7)-(11). Let $\underline{x}=\underline{x}_m(\underline{\lambda},\overline{\lambda}),\overline{x}=\overline{x}_m(\underline{\lambda},\overline{\lambda})$ and $\underline{t}=\underline{t}_m(\underline{\lambda},\overline{\lambda}),\overline{t}=\overline{t}_m(\underline{\lambda},\overline{\lambda})$. Incentive compatibility of m then trivially implies (7) and (8). Feasibility will imply the remaining conditions. To see this, note first that $\underline{X}_m(L,H)L+\overline{X}_m(L,H)H\leq 1,\forall L,H$ requires $\underline{X}_m(L,H)\leq\frac{1}{L}$, which in turn implies

$$\underline{x}_m(\underline{\lambda},\overline{\lambda})=\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\underline{X}_m(L+1,H)\leq\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1}=\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right).$$

Analogously it can be shown that $\overline{X}_m(L,H) \leq \frac{1}{H}$ implies $\overline{x}_m(\underline{\lambda},\overline{\lambda}) \leq \frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)$. From the perspective of a buyer the probability of trading a low quality good is given by

$$\sum_{L=1}^{+\infty}\sum_{H=0}^{+\infty}\frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\underline{X}_m(L,H)L = \underline{\lambda}\sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty}\frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\underline{X}_m(L+1,H) = \underline{\lambda}\underline{x}_m(\underline{\lambda},\overline{\lambda}).$$

Similarly, the probability for a buyer to trade a high quality good can be shown to equal $\overline{\lambda}\overline{x}_m(\underline{\lambda},\overline{\lambda})$. Feasibility then implies

$$\begin{aligned}
\overline{\lambda}\overline{x}_m(\underline{\lambda},\overline{\lambda}) + \underline{\lambda}\underline{x}_m(\underline{\lambda},\overline{\lambda}) &= \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\left(\underline{X}_m(L,H)L + \overline{X}_m(L,H)H\right), \\
&\leq \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\cdot 1 - P_0(\overline{\lambda})P_0(\underline{\lambda}), \\
&= 1 - e^{-\underline{\lambda}-\overline{\lambda}}.
\end{aligned}$$

□

A.2 Proof of Lemma 3.3

Let $(\underline{x},\underline{t})$ and $(\overline{x},\overline{t})$ be pairs of expected trading probabilities and transfers associated to (possibly different) mechanisms chosen by low and high type sellers in a given equilibrium. These values must then also be part of a solution of P^{aux}. Market utilities are then $\underline{U} = \underline{t} - \underline{x}\underline{c}$ and $\overline{U} = \overline{t} - \overline{x}\overline{c}$. The following properties must hold:

1a. $\underline{U} > \overline{U}$: the low type incentive constraint (12) requires $\underline{t} - \underline{x}\underline{c} \geq \overline{t} - \overline{x}\underline{c}$, which can be rewritten as $\overline{x}(\overline{c}-\underline{c}) \leq \underline{U} - \overline{U}$. Since $\overline{x} \geq 0$, this inequality can only be satisfied if $\underline{U} \geq \overline{U}$. Suppose now that $\underline{U} = \overline{U}$ so that $\overline{x} = 0$. Since, under any solution of P^{aux}, buyers must make weakly positive profits with both types of seller,[36] we must have $\overline{t} = 0$ and hence $\underline{U} = \overline{U} = 0$. A buyer's expected profit with low type sellers when $\underline{U} = \underline{t} - \underline{x}\underline{c} = 0$ is $\underline{\lambda}\underline{x}(\underline{v}-\underline{c})$. The maximal value of this last expression at an admissible solution of P^{aux} is attained when $\underline{\lambda}\underline{x} = (1-e^{-\underline{\lambda}})$, that is, if the buyer trades with low type sellers whenever possible. However, since $1-e^{-\underline{\lambda}}$ is strictly increasing in $\underline{\lambda}$, no finite value of $\underline{\lambda}$ can solve P^{aux}, implying that $\underline{U} = \overline{U} = 0$ cannot be admissible equilibrium values.

1b. $\underline{U} - \overline{U} < \overline{c} - \underline{c}$: incentive compatibility for the high type sellers (condition (13)) requires $\overline{t} - \overline{x}\overline{c} \geq \underline{t} - \underline{x}\overline{c}$, or $\underline{x}(\overline{c}-\underline{c}) \geq \underline{U} - \overline{U}$. Since by condition (10) we have $\underline{x} \leq \frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right) < 1$[37], the inequality $\underline{x}(\overline{c}-\underline{c}) \geq \underline{U} - \overline{U}$ can only be satisfied if $\underline{U} - \overline{U} < \overline{c} - \underline{c}$.

[36] If buyers make losses with one type of seller, they can always set the respective ratio, $\underline{\lambda}$ or $\overline{\lambda}$, equal to zero.
[37] $\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right) < 1$ is a general property for all $\underline{\lambda} \in (0,+\infty)$

2. $\underline{U},\overline{U} > 0$: $\underline{U} > 0$ follows directly from $\underline{U} > \overline{U}$ and $\overline{U} \geq 0$. It thus remains to show that $\overline{U} > 0$. Towards a contradiction, suppose $\overline{U} = 0$. In this case, by a symmetric argument to the one in 1a. above, the expected profit a buyer makes trading with high type sellers is $\overline{\lambda}\overline{x}(\overline{v} - \overline{c})$. Since $\underline{U} > \overline{U}$, there always exists a strictly positive value of $\overline{x}$ that satisfies the low type incentive constraint (12) and the overall feasibility constraint (15). Given that $\overline{\lambda}$ can be set equal to zero, a strictly positive value of $\overline{x}$ is also weakly optimal. Thus, w.l.o.g. suppose $\overline{x} > 0$ and consider an increase of $\underline{\lambda}$ together with a decrease of $\overline{x}$ so as to keep $\overline{\lambda}\overline{x}$ unchanged. Adjusting $\overline{t}$ in order to keep the utility of high type sellers constant, while keeping the remaining contracting parameters unchanged, we obtain a tuple $(\underline{\lambda}, \overline{\lambda}', \underline{x}, \overline{x}', \underline{t}, \overline{t}')$ that is incentive compatible and satisfies the overall feasibility constraint (15) with strict inequality, i.e. $\underline{\lambda}\underline{x} + \overline{\lambda}'\overline{x}' < 1 - e^{-\underline{\lambda}-\overline{\lambda}'}$. The buyer can then increase his payoff by deviating to $\left(\underline{\lambda}, \overline{\lambda}' + \varepsilon, \underline{x}, \overline{x}', \underline{t}, \overline{t}'\right)$, with $\varepsilon > 0$, while still satisfying all constraints of P^{aux}, thus a contradiction.

3a. $\underline{U} < \underline{v} - \underline{c}$: Suppose not, $\underline{U} \geq \underline{v} - \underline{c}$. Since, as shown in 1b. above, $\underline{x} < 1$ whenever $\underline{\lambda} > 0$, this implies that $\underline{x}(\underline{v} - \underline{c}) - \overline{U} < 0$, that is, a buyer's payoff with each low type seller is strictly negative. As a consequence, at any solution of P^{aux} we have $\underline{\lambda} = 0$. This in turn implies that the low types' market utility $\underline{U}$ must equal zero and therefore $\underline{U} < \underline{v} - \underline{c}$. A contradiction.

3b. $\overline{U} \leq \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$: Suppose not, $\overline{U} > \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. This implies, since - as argued above - the incentive compatibility constraint for the low type sellers can be written as $\overline{x} \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, that the payoff of a buyer with each high type seller is negative:

$$\overline{x}(\overline{v} - \overline{c}) - \overline{U} \leq \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}(\overline{v} - \overline{c}) - \overline{U} = \frac{(\overline{v} - \overline{c})\underline{U} - (\overline{v} - \underline{c})\overline{U}}{\overline{c} - \underline{c}} < 0.$$

All solutions of P^{aux} must therefore satisfy $\overline{\lambda} = 0$, which in turn implies $\overline{U} = 0$ and therefore $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$. A contradiction.

□

A.3 Proof of Lemma 3.4

Suppose not and let $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ be a solution of $P^{aux'}$ with $\underline{x}\underline{\lambda} < 1 - e^{-\underline{\lambda}}$. Note that the strict inequality requires $\underline{\lambda} > 0$. Consider then an alternative tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{\lambda}' < \underline{\lambda}$, $\overline{\lambda}' = \overline{\lambda} + (\underline{\lambda} - \underline{\lambda}')$, $\underline{x}' = \frac{\underline{\lambda}}{\underline{\lambda}'}\underline{x}$ and $\overline{x}' = \frac{\overline{\lambda}}{\overline{\lambda}'}\overline{x}$. If $\underline{\lambda}'$ is sufficiently close to $\underline{\lambda}$, $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies constraint (14). Since $\underline{x}'\underline{\lambda}' = \underline{x}\underline{\lambda}$, $\overline{x}'\overline{\lambda}' = \overline{x}\overline{\lambda}$ and $\underline{\lambda}' + \overline{\lambda}' = \underline{\lambda} + \overline{\lambda}$, it also satisfies the overall feasibility constraint (15). Lastly, since $\underline{x}' > \underline{x}$ and $\overline{x}' < \overline{x}$, the mechanism associated to this alternative tuple always satisfies the incentive compatibility constraints (12) and (13). Consider now the buyer's payoff associated to $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$:

$$
\begin{aligned}
& \underline{\lambda}'\underline{x}'(\underline{v}-\underline{c}) + \overline{\lambda}'\overline{x}'(\overline{v}-\overline{c}) - \underline{\lambda}'\underline{U} - \overline{\lambda}'\overline{U}, \\
= \;& \underline{\lambda x}(\underline{v}-\underline{c}) + \overline{\lambda x}(\overline{v}-\overline{c}) - \underline{\lambda}'\underline{U} - (\overline{\lambda} + (\underline{\lambda} - \underline{\lambda}'))\overline{U}, \\
= \;& \underline{\lambda x}(\underline{v}-\underline{c}) + \overline{\lambda x}(\overline{v}-\overline{c}) - \underline{\lambda U} - \overline{\lambda U} + (\underline{\lambda} - \underline{\lambda}')(\underline{U} - \overline{U}).
\end{aligned}
$$

Since the last term of the last line is strictly positive, $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ yields a strictly higher payoff than $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$, a contradiction. □

A.4 Proof of Lemma 3.5

When $\overline{U} = \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$, the claim follows directly from the condition that profits with each type of seller must be non-negative and the fact that under the assumption $\overline{U} = \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$ we have $\overline{x}(\overline{v}-\overline{c}) \leq 0$ for all $\overline{x} \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, where the inequality holds as equality if and only if $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$.

Consider next the case where $\overline{U} < \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$ and $\underline{v}-\underline{c} \leq \overline{v}-\overline{c}$. We argue by contradiction: suppose $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ is a solution of $P^{aux'}$ (with $\overline{\lambda} > 0$) and (12) is satisfied as inequality: $\overline{x} < \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. As shown in claim i) of Lemma 3.6 below in this case both feasibility constraints (14) and (15) hold as equality. We can then solve these constraints for $\underline{x}$ and $\overline{x}$ in terms of $\underline{\lambda}, \overline{\lambda}$. Substituting the expressions obtained, the buyer's payoff can be written as a function of $\underline{\lambda}$ and $\overline{\lambda}$ only:

$$
\hat{\pi}(\underline{\lambda}, \overline{\lambda}) := \left(1 - e^{-\underline{\lambda}}\right)(\underline{v}-\underline{c}) + e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right)(\overline{v}-\overline{c}) - \underline{\lambda U} - \overline{\lambda U}. \tag{24}
$$

At a solution of $P^{aux'}$ $\underline{\lambda}, \overline{\lambda}$ must then maximize $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$ subject to incentive compatibility constraints (12) and (13), which can, respectively, be rewritten as

$$
e^{-\underline{\lambda}}(1 - e^{-\overline{\lambda}}) \leq \overline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}, \tag{25}
$$

$$
(1 - e^{-\underline{\lambda}}) \geq \underline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}. \tag{26}
$$

Suppose first that $\underline{\lambda} > 0$ and consider an alternative tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{\lambda}' = \underline{\lambda} - \Delta, \Delta > 0$, $\overline{\lambda}' = \overline{\lambda} + \Delta$, $\underline{x}' = \frac{1}{\underline{\lambda}'}\left(1 - e^{-\underline{\lambda}'}\right)$ and $\overline{x}' = e^{-\underline{\lambda}'}\frac{1}{\overline{\lambda}'}(1 - e^{-\overline{\lambda}'})$. In the alternative tuple some low types are replaced with high types, while the ratio between buyers and all types of sellers is kept unchanged, and the feasibility constraints still hold as equality. As a consequence we have $\underline{x}' > \underline{x}$ and $\overline{x}' > \overline{x}$, implying that $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies all constraints of problem $P^{aux'}$ as long as Δ is sufficiently small. The difference between the buyer's payoff associated to $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ and that associated to $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ is then:

$$\hat{\pi}(\underline{\lambda}', \overline{\lambda}') - \hat{\pi}(\underline{\lambda}, \overline{\lambda}) = \left(e^{-\underline{\lambda}+\Delta} - e^{-\underline{\lambda}}\right)[(\overline{v} - \overline{c}) - (\underline{v} - \underline{c})] + \Delta(\underline{U} - \overline{U}) > 0.$$

The first term is the difference in the probability of meeting no low type seller (in which case a high quality good is traded), multiplied by the difference between the gains from trade of the high and low quality good, while the second term is the difference in rent paid to sellers: since the alternative mechanism on average replaces Δ low type sellers with high type sellers, the reduction in rent is $\Delta(\underline{U} - \overline{U})$. Under the assumption $\underline{v} - \underline{c} \leq \overline{v} - \overline{c}$, the sum of the two terms is strictly positive. Hence, $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ yields a strictly larger payoff than $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$, a contradiction.

For the case $\underline{\lambda} = 0$, notice that $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$ as well as the correspondence determining the set of feasible values of $\overline{\lambda}$, defined by the constraint $e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right) \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, are continuous in $\underline{\lambda}$ at $\underline{\lambda} = 0$. Consider the constrained optimization problem where we require $\underline{\lambda} \geq \varepsilon > 0$. By the previous argument (12) holds as equality at a solution of this constrained problem for all $\varepsilon > 0$, and by the continuity property the solution of the constrained problem converges to the solution of the original problem where we only require $\underline{\lambda} \geq 0$. Since (12) holds as equality along all points in the sequence, it does so in the limit. □

A.5 Proof of Lemma 3.6

i) We show first that, when $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$, at all solutions of $P^{aux'}$ the overall feasibility condition (15) is satisfied as equality.

Let $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ be a tuple that solves $P^{aux'}$. If $\overline{\lambda} = 0$, constraint (15) is satisfied as equality by Lemma 3.4. Consider then the case $\overline{\lambda} > 0$. Towards a contradiction, suppose that (15) is satisfied as a strict inequality at a solution of $P^{aux'}$. Under this assumption, if the buyer's profit with each high type seller, $\overline{x}(\overline{v} - \overline{c}) - \overline{U}$, is strictly positive, the buyer can strictly increase his payoff by increasing $\overline{\lambda}$. For this not to be a profitable deviation, we must therefore have $\overline{x}(\overline{v} - \overline{c}) \leq \overline{U}$. Noting that the stated property $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$ is equivalent to $\frac{\overline{U}}{\overline{v}-\overline{c}} < \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, the inequality $\overline{x}(\overline{v} - \overline{c}) \leq \overline{U}$ implies $\overline{x} < \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. That is, the low type incentive constraint (12) is satisfied as a strict inequality. By slackness of the constraints (12) and (15), an increase in $\overline{x}$ is then both feasible and incentive compatible. Since increasing $\overline{x}$ strictly increases the buyer's payoff with each high type seller and since $\overline{\lambda} > 0$, the buyer has a profitable deviation, thus a contradiction.

ii) Having shown that, when $\overline{U} < \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$, both feasibility constraints (14) and (15) are satisfied as equality at a solution of $P^{aux'}$, the values of $\underline{\lambda}, \overline{\lambda}$ belonging to a solution of $P^{aux'}$ maximize $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$, defined in the proof of Lemma 3.5 above, subject to the incentive constraints (25) and (26).

a) Consider first the case $\underline{v} - \underline{c} > \overline{v} - \overline{c}$. Under this condition, $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$ is strictly concave. To see this, note that $\frac{\partial^2 \hat{\pi}}{\partial \overline{\lambda}^2} = -e^{-\underline{\lambda}} e^{-\overline{\lambda}}(\overline{v} - \overline{c}) < 0$ and that the determinant of the Hessian is given by $e^{-2\underline{\lambda}} e^{-\overline{\lambda}}(\overline{v} - \overline{c})[(\underline{v} - \underline{c}) - (\overline{v} - \overline{c})] > 0$. The Hessian is thus negative definite.
We can show that, as a consequence, there cannot exist two solutions $(\underline{x}_1, \overline{x}_1, \underline{\lambda}_1, \overline{\lambda}_1)$ and $(\underline{x}_2, \overline{x}_2, \underline{\lambda}_2, \overline{\lambda}_2)$. Strict concavity of $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$ would in fact imply that a strictly larger payoff is attained at $\underline{\lambda} = \alpha \underline{\lambda}_1 + (1-\alpha)\underline{\lambda}_2, \overline{\lambda} = \alpha \overline{\lambda}_1 + (1-\alpha)\overline{\lambda}_2$, $\alpha \in (0,1)$. The pair $(\underline{\lambda}, \overline{\lambda})$ satisfies the incentive constraints since $\frac{1}{\underline{\lambda}}(1 - e^{-\underline{\lambda}}) \geq \text{Min}\left\{\frac{1}{\underline{\lambda}_1}(1 - e^{-\underline{\lambda}_1}), \frac{1}{\underline{\lambda}_2}(1 - e^{-\underline{\lambda}_2})\right\}$ and $e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}(1 - e^{-\overline{\lambda}}) \leq \text{Max}\left\{e^{-\underline{\lambda}_1}\frac{1}{\overline{\lambda}_1}(1 - e^{-\overline{\lambda}_1}), e^{-\underline{\lambda}_2}\frac{1}{\overline{\lambda}_2}(1 - e^{-\overline{\lambda}_2})\right\}$.[38] Hence $\left(\underline{\lambda}, \overline{\lambda}, \underline{x}, \overline{x}\right)$ is an admissible solution and yields a higher payoff, a contradiction.

b) Consider next the case $\underline{v} - \underline{c} \leq \overline{v} - \overline{c}$. From Lemma 3.5 it follows that, at all solutions with $\overline{\lambda} > 0$, the incentive constraint (12) holding as equality can be solved for $\underline{\lambda}$, yielding:

$$\underline{\lambda} = \ln\left(\frac{1 - e^{-\overline{\lambda}}}{\overline{\lambda}} \frac{\overline{c} - \underline{c}}{\underline{U} - \overline{U}}\right).$$

The condition $\underline{\lambda} \geq 0$ requires that $\overline{\lambda} \leq L$, where L is defined by $\frac{1}{L}(1 - e^{-L}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. For $\overline{\lambda} \in (0, L]$, the buyer's payoff can then be written as a function of $\overline{\lambda}$ only:

$$\tilde{\pi}(\overline{\lambda}) = \overline{\lambda}\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}(\overline{v} - \overline{c}) + \left(1 - \frac{\overline{\lambda}}{1 - e^{-\overline{\lambda}}}\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}\right)(\underline{v} - \underline{c}) - \overline{\lambda}\overline{U} - \ln\left(\frac{1 - e^{-\overline{\lambda}}}{\overline{\lambda}}\frac{\overline{c} - \underline{c}}{\underline{U} - \overline{U}}\right)\underline{U}. \quad (27)$$

Since $\underline{x} > \overline{x} =$ for $\underline{\lambda}, \overline{\lambda} > 0$ (by Lemma 3.4), the fact that the low type incentive constraint (12) holds as equality immediately implies that the high type incentive constraint (13) is slack. At a solution of $P^{aux'}$ with $\overline{\lambda} > 0$, $\overline{\lambda}$ is thus the unconstrained maximizer of $\tilde{\pi}$ on $(0, L]$. We show next that the function $\tilde{\pi}$ is strictly concave in $\overline{\lambda}$. Note that:

$$\frac{\partial^2 \tilde{\pi}}{\partial \overline{\lambda}^2} = e^{\overline{\lambda}}\frac{2(e^{\overline{\lambda}} - 1) - \overline{\lambda}(e^{\overline{\lambda}} + 1)}{(e^{\overline{\lambda}} - 1)^3}(\underline{v} - \underline{c})\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} + \frac{\overline{\lambda}^2 e^{\overline{\lambda}} - (e^{\overline{\lambda}} - 1)^2}{\overline{\lambda}^2(e^{\overline{\lambda}} - 1)^2}\underline{U}. \quad (28)$$

The numerator of the first term of the derivative is equal to zero at $\overline{\lambda} = 0$ and strictly

[38] It can be verified that $e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}(1 - e^{-\overline{\lambda}})$ is convex in $\underline{\lambda}$ and $\overline{\lambda}$.

decreasing for all $\overline{\lambda} > 0$:

$$\frac{\partial\left(2(e^{\overline{\lambda}}-1)-\overline{\lambda}(e^{\overline{\lambda}}+1)\right)}{\partial\overline{\lambda}} = -\overline{\lambda}e^{\overline{\lambda}}\underbrace{\left(1-\frac{1}{\overline{\lambda}}(1-e^{-\overline{\lambda}})\right)}_{>0} < 0,$$

so that this first term is strictly negative. As we show next, the numerator of the second term is also strictly negative: $\overline{\lambda}^2 e^{\overline{\lambda}} < (e^{\overline{\lambda}}-1)^2$. To see this, notice first that the inequality can be rewritten as $1-e^{\overline{\lambda}}+\overline{\lambda}e^{\frac{\overline{\lambda}}{2}} < 0$. The term $1-e^{\overline{\lambda}}+\overline{\lambda}e^{\frac{\overline{\lambda}}{2}}$ is equal to zero at $\overline{\lambda}=0$ and is strictly decreasing in $\overline{\lambda}$ for all $\overline{\lambda} > 0$:

$$\frac{\partial\left(1-e^{\overline{\lambda}}+\overline{\lambda}e^{\frac{\overline{\lambda}}{2}}\right)}{\partial\overline{\lambda}} = -e^{\overline{\lambda}}\left[1-e^{-\frac{\overline{\lambda}}{2}}-\frac{\overline{\lambda}}{2}e^{-\frac{\overline{\lambda}}{2}}\right] < 0,$$

where the term in the square bracket is the probability of at least two arrivals when the arrival rate is $\frac{\overline{\lambda}}{2}$ and therefore is strictly positive. This establishes $\frac{\partial^2\tilde{\pi}}{\partial\overline{\lambda}^2} < 0$, that is $\tilde{\pi}(\overline{\lambda})$ is strictly concave on the domain $(0, L]$.

Given the strict concavity of $\tilde{\pi}(\overline{\lambda})$ on $(0, L]$, $P^{aux'}$ can at most have two solutions, one at $\overline{\lambda} > 0$ and possibly one at $\overline{\lambda} = 0$. In what follows, we show there cannot be a solution with $\overline{\lambda} = 0$, which establishes the claim that the solution of $P^{aux'}$ is unique. To this end, we need to characterize the properties of possible solutions with $\overline{\lambda} = 0$. Recall that the above expression of $\tilde{\pi}(\overline{\lambda})$ was only valid for $\overline{\lambda} > 0$, hence when $\overline{\lambda} = 0$ we need to consider $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$. Ignoring for a moment incentive constraints, the value of $\underline{\lambda}$ that maximizes $\hat{\pi}(\underline{\lambda}, 0) = (1-e^{-\underline{\lambda}})(\underline{v}-\underline{c}) - \underline{\lambda}\underline{U}$, is $\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$. Given $\overline{\lambda} = 0$, the high type incentive constraint (12) can always be satisfied by picking some value of $\overline{x}$ weakly smaller than $\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. On the other hand, the high type incentive constraint (13) is satisfied at $\underline{\lambda} = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$ if and only if

$$\frac{1}{\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)}\left(1-\frac{\underline{U}}{\underline{v}-\underline{c}}\right) \geq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}. \tag{29}$$

– Suppose first that this inequality is not satisfied, that is the value of $\underline{\lambda}$ maximizing $\hat{\pi}(\underline{\lambda}, 0)$ is too large. The optimal value of $\underline{\lambda}$ is then given by L, defined earlier by the implicit condition $\frac{1}{L}(1-e^{-L}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. The buyer's payoff associated to this value of $\underline{\lambda}$ is $\hat{\pi}(L, 0) = L\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\underline{v}-\underline{c})-\underline{U}\right)$. Since $\underline{v}-\underline{c} \leq \overline{v}-\overline{c}$ and $\underline{U} > \overline{U}$, $\hat{\pi}(L, 0)$ is strictly smaller than the corresponding expression when all low types are swapped with high types, $\hat{\pi}(0, L) = L\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c})-\overline{U}\right)$. Since the pair $(0, L)$ is an admissible solution for

$(\underline{\lambda}, \overline{\lambda})$, we cannot have $\overline{\lambda} = 0$ at a solution of $P^{aux'}$, hence a contradiction.

– Suppose next that (29) is satisfied and so that $\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$ is an admissible value for $\underline{\lambda}$. Towards a contradiction, suppose that we have $\underline{\lambda} = \ln(\frac{\underline{v}-\underline{c}}{\underline{U}})$ and $\overline{\lambda} = 0$ at a solution of $P^{aux'}$. We need to distinguish two cases:

  * Suppose first that $\frac{\underline{U}}{\underline{v}-\underline{c}} > \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Consider the tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{\lambda}' = \underline{\lambda}$ and $\underline{x}' = \frac{1}{\underline{\lambda}}(1 - e^{-\underline{\lambda}})$, so that the payoff with low type sellers remains unchanged, and let $\overline{x}' = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Given these restrictions, the tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies the incentive compatibility constraints and, if $\overline{\lambda}' > 0$, yields a strictly positive payoff with high type sellers:

$$\overline{\lambda}' \left[\overline{x}'(\overline{v} - \overline{c}) - \overline{U}\right] = \overline{\lambda} \left[\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}(\overline{v} - \overline{c}) - \overline{U}\right].$$

  A strictly positive value of $\overline{\lambda}'$ is feasible (that is, consistent with (15), using the values specified above for $\underline{\lambda}'$, $\underline{x}', \overline{x}'$) if

$$\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} \leq \frac{\underline{U}}{\underline{v} - \underline{c}} \frac{1}{\overline{\lambda}'} \left(1 - e^{-\overline{\lambda}'}\right),$$

  for some $\overline{\lambda}' > 0$. In the case under consideration $(\frac{\underline{U}}{\underline{v}-\underline{c}} > \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}})$, this is indeed the case, implying that there is no solution of $P^{aux'}$ with $\overline{\lambda} = 0$.

  * Consider next the case $\frac{\underline{U}}{\underline{v}-\underline{c}} \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Let $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ be specified again with $\underline{\lambda}' = \underline{\lambda}$, $\underline{x}' = \frac{1}{\underline{\lambda}}(1-e^{-\underline{\lambda}})$, while the value of $\overline{x}'$ is now set so that (15) is satisfied with equality:

$$\overline{x}' = e^{-\underline{\lambda}} \frac{1}{\overline{\lambda}'} \left(1 - e^{-\overline{\lambda}'}\right) = \frac{\underline{U}}{\underline{v} - \underline{c}} \frac{1}{\overline{\lambda}'} \left(1 - e^{-\overline{\lambda}'}\right) < \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}.$$

  The above inequality implies that the low type incentive constraint (12) is satisfied for all $\overline{\lambda}' > 0$.[39] The difference in payoff between the two mechanisms is given by:

$$\begin{aligned} & \hat{\pi}\left(\ln\left(\frac{\underline{v} - \underline{c}}{\underline{U}}\right), 0\right) - \hat{\pi}\left(\ln\left(\frac{\underline{v} - \underline{c}}{\underline{U}}\right), \overline{\lambda}'\right), \\ = \quad & e^{-\underline{\lambda}} \left(1 - e^{-\overline{\lambda}'}\right)(\overline{v} - \overline{c}) - \overline{\lambda}'\overline{U} = \overline{\lambda}' \left[\frac{1}{\overline{\lambda}'}\left(1 - e^{-\overline{\lambda}'}\right) \frac{\overline{v} - \overline{c}}{\underline{v} - \underline{c}} \underline{U} - \overline{U}\right]. \end{aligned}$$

  Since $\underline{U} > \overline{U}$ and $\overline{v} - \overline{c} \geq \underline{v} - \underline{c}$ in the case under consideration (b), the above expression is strictly positive for $\overline{\lambda}'$ small enough.[40]

We have thus shown there can be no solution of $P^{aux'}$ with $\overline{\lambda} = 0$ and therefore that the

[39]The other incentive compatibility constraint (13) is also satisfied because the low type sellers' trading probability remains unchanged.

[40]Recall that $\lim_{x \to 0} \frac{1}{x}(1 - e^{-x}) = 1$.

solution of $P^{aux'}$ is unique.

$\square$

A.6 Proof of Lemma 3.7

When $\overline{U} = \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$, we have $\overline{\lambda}[\overline{x}(\overline{v}-\overline{c})-\overline{U}] = 0$ for all solutions of $P^{aux'}$ and $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ for all solutions of $P^{aux'}$ with $\overline{\lambda} > 0$. We can then show that $\underline{\lambda}$ is the same at all solutions of $P^{aux'}$. Suppose not. By Lemma 3.4, the buyer's payoff from low type sellers is given by $(1-e^{-\underline{\lambda}})(\underline{v}-\underline{c}) - \underline{\lambda}\underline{U}$. This term is strictly concave in $\underline{\lambda}$ and attains its maximum at $\underline{\lambda} = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$. This implies that any solution of $P^{aux'}$ must satisfy $\underline{\lambda} \leq \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$: if in fact the buyer proposes a mechanism with $\underline{\lambda} > \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$, decreasing $\underline{\lambda}$ and adjusting $\underline{x}$ so that (14) still holds as equality strictly increases the buyer's payoff and weakly relaxes all constraints. Now suppose there exists two values $\underline{\lambda}_1$ and $\underline{\lambda}_2$ with $\underline{\lambda}_1 \neq \underline{\lambda}_2$ that belong to some solution of $P^{aux'}$. W.l.o.g. assume $\underline{\lambda}_1 < \underline{\lambda}_2$, which implies $(1-e^{-\underline{\lambda}_1})(\underline{v}-\underline{c}) - \underline{\lambda}_1\underline{U} < (1-e^{-\underline{\lambda}_2})(\underline{v}-\underline{c}) - \underline{\lambda}_2\underline{U}$. Since both solutions must yield the same payoff for the buyer, we must then have $\overline{\lambda}_1[\overline{x}_1(\overline{v}-\overline{c})-\overline{U}] > \overline{\lambda}_2[\overline{x}_2(\overline{v}-\overline{c})-\overline{U}]$. However, we know that $\overline{\lambda}[\overline{x}(\overline{v}-\overline{c})-\overline{U}] = 0$ for all solutions of $P^{aux'}$ and therefore have contradiction.

Finally, the property that $\underline{\lambda}$ is the same for all solutions of $P^{aux'}$ together with the result in Lemma 3.4 immediately imply that $\underline{x}$ must have the same value for all solutions of $P^{aux'}$. $\square$

A.7 Proof of Proposition 3.8

Given the assumed condition $\overline{U} < \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}$, using (17) to substitute for $\underline{x}$ and $\overline{x}$, $P^{aux'}$ can be written solely in terms of the variables $\underline{\lambda}, \overline{\lambda}$, as already done in the proof of Lemma 3.6. At a solution of $P^{aux'}$, the values of $\underline{\lambda}, \overline{\lambda}$ must then maximize $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$, specified in (24), subject to the incentive constraints, as rewritten in (25) and (26).

a) Suppose first the incentive constraints are not binding. The first order conditions for an interior solution with respect to $\underline{\lambda}$ and $\overline{\lambda}$ are:

$$e^{-\underline{\lambda}}(\underline{v}-\underline{c}) - e^{-\underline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)(\overline{v}-\overline{c}) - \underline{U} = 0, \tag{30}$$

$$e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\overline{c}) - \overline{U} = 0. \tag{31}$$

The solution of this system of equations is equal to $\underline{\lambda}^p, \overline{\lambda}^p$ if and only if market utilities $\underline{U}$ and $\overline{U}$ have the values specified in (18) and (19). Incentive compatibility is then satisfied if (25) and (26) hold at $\underline{\lambda}^p, \overline{\lambda}^p$ and at the value of $\underline{U}$ and $\overline{U}$ given in (18) and (19), that is if the following inequalities hold:

$$\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \leq 1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} \quad \text{and} \quad \frac{1-e^{-\underline{\lambda}^p}}{\underline{\lambda}^p e^{-\underline{\lambda}^p}} \geq 1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}.$$

The first condition is the one appearing in the statement of part a) of the proposition. On the other hand, the second inequality is always satisfied, since $1-e^{-\underline{\lambda}^p}$ equals the probability of at least one arrival, and this strictly exceeds the probability of exactly one arrival, $\underline{\lambda}^p e^{-\underline{\lambda}^p}$.

It remains to verify that buyers make strictly positive profits with high type sellers and that condition $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$ holds. Note that the first property implies the second one. Consider then the buyers' payoff with high type sellers

$$\overline{x}(\overline{v}-\overline{c})-\overline{U} = e^{-\overline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}-\overline{\lambda}^p e^{-\overline{\lambda}^p}\right)(\overline{v}-\overline{c}).$$

Since $1-e^{-\overline{\lambda}^p} > \overline{\lambda}^p e^{-\overline{\lambda}^p}$, as argued above, the buyers' payoff with high type sellers at the equilibrium mechanism is strictly positive.

b) Suppose next at least one of the incentive constraints (25), (26) is binding. Letting γ_l and γ_h denote the respective Lagrange multipliers of these constraints, the population parameters $\underline{\lambda}^p, \overline{\lambda}^p$ are optimal if they solve the first-order conditions:

$$\begin{aligned} e^{-\underline{\lambda}^p}(\underline{v}-\underline{c})-e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)(\overline{v}-\overline{c})-\underline{U}+\gamma_l e^{-\underline{\lambda}^p}(1-e^{-\overline{\lambda}^p})-\gamma_h\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}-e^{-\underline{\lambda}^p}\right) &= 0, \\ e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\overline{v}-\overline{c})-\overline{U}+\gamma_l\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}-e^{-\underline{\lambda}^p-\overline{\lambda}^p}\right) &= 0. \end{aligned}$$

Consider first the possibility that the high type incentive constraint (26) is binding, i.e. $\gamma_h > 0, \gamma_l = 0$. In this case $\overline{U}$ has the same value as in case a), given by (19), and $\frac{1}{\underline{\lambda}^p}(1-e^{-\underline{\lambda}^p}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Since $e^{-\underline{\lambda}^p} < \frac{1}{\underline{\lambda}^p}(1-e^{-\underline{\lambda}^p}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, the term multiplying γ_h is positive. For γ_h to be strictly positive the following condition must then hold:

$$e^{-\underline{\lambda}^p}(\underline{v}-\underline{c})-e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)(\overline{v}-\overline{c})-\underline{U} > 0.$$

With $\overline{U}$ determined by (19) and $\underline{U}$ such that $\frac{1}{\underline{\lambda}^p}(1-e^{-\underline{\lambda}^p}) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, this inequality can be rewritten as

$$1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} > \frac{1-e^{-\underline{\lambda}^p}}{e^{-\underline{\lambda}^p}\underline{\lambda}^p}.$$

Since $1 - e^{-\underline{\lambda}^p} > \underline{\lambda}^p e^{-\underline{\lambda}^p}$, the above inequality cannot be satisfied and so we must have $\gamma_h = 0$.

Consider then the case where the low type incentive constraint (25) binds: $e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. As shown in the proof of part ii.b) of Lemma 3.6, solving this equation for $\underline{\lambda}$ and substituting the result into the expression of the buyer's payoff in (24) yields the function $\tilde{\pi}(\overline{\lambda})$, specified in (27). The solution of the buyer's auxiliary problem is then determined by the first-order condition $\tilde{\pi}'(\overline{\lambda}) = 0$. Substituting the population parameter $\overline{\lambda}^p$ into this equality we get the following condition:

$$(\overline{v} - \underline{c})\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} - e^{\overline{\lambda}^p}\frac{e^{\overline{\lambda}^p} - \overline{\lambda}^p - 1}{(e^{\overline{\lambda}^p} - 1)^2}(\underline{v} - \underline{c})\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} + \frac{e^{\overline{\lambda}^p} - \overline{\lambda}^p e^{\overline{\lambda}^p} - 1}{\overline{\lambda}^p(e^{\overline{\lambda}^p} - 1)}\underline{U} = 0. \tag{32}$$

Solving this equation, together with $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, for $\underline{U}$ and $\overline{U}$ yields the values of the market utilities specified in (20) and (21). It is then immediate to verify that these values satisfy the condition $\overline{U} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$ if and only if $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. This condition implies that at the maximal incentive feasible trading probability for high type sellers buyers make strictly positive profits with high type sellers. Since the low type incentive constraint is binding, this is indeed the case. □

A.8 Proof of Proposition 3.9

With $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ and $\underline{x}$ determined by (17), $P^{aux'}$ corresponds to maximizing $(1 - e^{-\underline{\lambda}})(\underline{v} - \underline{c}) - \underline{\lambda U}$ over $\underline{\lambda}$ subject to the incentive constraint of high type sellers, as rewritten in (26). This constraint is not binding, which follows from the fact that the low type incentive constraint (12) is satisfied with equality at any solution with $\overline{\lambda} > 0$ (as stated in Lemma 3.7), in which case $\underline{x} > \overline{x}$ by Lemma 3.4, and that $\overline{\lambda}^p > 0$ must be a solution of $P^{aux'}$. Given that the high type incentive constraint (13) is not binding, the optimal value of $\underline{\lambda}$ is then characterized by the first-order condition

$$e^{-\underline{\lambda}}(\underline{v} - \underline{c}) - \underline{U} = 0.$$

It is easy to see that the population ratio $\underline{\lambda}^p$ solves problem $P^{aux'}$ if and only if $\underline{U}$ is as in (22). The value of $\overline{U}$ in the candidate equilibrium is then pinned down by the assumed condition $\overline{U} = \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$ and is so given by (23).

With $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, $P^{aux'}$ is solved by all values of $\overline{\lambda}$ that satisfy the overall feasibility constraint (15). Substituting $\overline{\lambda}^p$, together with $\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$ evaluated at the market utilities in (22) and (23), and $\underline{x}$ as determined by (17) and $\underline{\lambda}^p$, this constraint reduces to

$$\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}.$$

□

A.9 Proof of Proposition 3.10

Suppose there is an equilibrium with $\overline{U} = \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\underline{U}$, where high type sellers are distributed asymmetrically across at least two submarkets. Consistency with the population parameter values requires then that a submarket exists in which the ratio of high type sellers to buyers exceeds the population ratio $\overline{\lambda}^p$, or equivalently, that there exists a solution of $P^{aux'}$ with $\overline{\lambda} > \overline{\lambda}^p$. Since $\overline{\lambda}$ must satisfy the overall feasibility constraint (15), a smaller ratio of high type sellers to buyers must be feasible as well. To see this, notice that, for $\overline{\lambda} > 0$ and $\underline{\lambda}$ as determined by (17), constraint (15) can be written as

$$\overline{x} \leq e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{\overline{\lambda}}\right).$$

Since the right hand side is strictly decreasing in $\overline{\lambda}$, as argued in footnote 19, reducing the value of $\overline{\lambda}$ relaxes constraint (15). Hence, for the market utilities $\underline{U}$ and $\overline{U}$ another solution of $P^{aux'}$ is given by the same values of $\underline{\lambda}$ and $\underline{x}$, $\overline{x}$ and by $\overline{\lambda}^p$. This solution is consistent with the population parameter values and constitutes a pooling equilibrium. □

A.10 Proof of Proposition 4.2

Consider first the case $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. From Proposition 3.8 it follows that under this restriction, at a pooling equilibrium the feasibility constraint (11) is binding. Consider an increase in the trading probability of the high type seller by $\Delta\overline{x}$, while adjusting the trading probability of the low type seller so that the feasibility constraint is still satisfied as equality:

$$\overline{\lambda}^p\Delta\overline{x} + \underline{\lambda}^p\Delta, \underline{x} = 0 \quad \Leftrightarrow \quad \Delta\underline{x} = -\frac{\mu}{1-\mu}\Delta\overline{x}.$$

Let us modify the expected transfer to the high type seller so that his utility remains unchanged

$$\Delta\overline{t} = \Delta\overline{x}\overline{c},$$

and the expected transfer to the low type seller so that his incentive compatibility constraint is satisfied

$$\Delta\underline{t} - \Delta\underline{x}\underline{c} = \Delta\overline{t} - \Delta\overline{x}\underline{c}.$$

Substituting the previous equations into the above yields

$$\Delta\underline{t} = \frac{1}{1-\mu}\left((1-\mu)\overline{c} - \underline{c}\right)\Delta\overline{x}.$$

Note that these changes make high type sellers indifferent and strictly improve the utility of low type sellers:

$$\Delta\underline{t} - \Delta\underline{x}\underline{c} = \left[\frac{1}{1-\mu}\left((1-\mu)\overline{c} - \underline{c}\right) + \frac{\mu}{1-\mu}\underline{c}\right]\Delta\overline{x} = \overline{c} - \underline{c} > 0.$$

It is immediate to verify that the changes considered always allow to increase total surplus (while satisfying incentive compatibility and the feasibility constraints imposed by the matching technology). To verify that they also constitute a Pareto improvement, we need to show that they also make buyers weakly better off. This happens if

$$\begin{aligned} & \overline{\lambda}^p\left[\Delta\overline{x}\overline{v} - \Delta\overline{t}\right] + \underline{\lambda}^p\left[\Delta\underline{x}\underline{v} - \Delta\underline{t}\right] \geq 0, \\ \Leftrightarrow \quad & \frac{\underline{\lambda}^p}{1-\mu}\left[\mu(\overline{v} - \underline{v}) - (\overline{c} - \underline{c})\right]\Delta\overline{x} \geq 0. \end{aligned}$$

The above inequality is satisfied whenever $\mu \geq \frac{\overline{c}-\underline{c}}{\overline{v}-\underline{v}}$.

Consider now the case $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. From Proposition 2.6 it follows that at a competitive equilibrium the feasibility constraint (11) is slack. Consider an increase in the trading probability of the high type seller $\Delta\overline{x}$, small enough that (11) is not violated. Modify then the expected transfer to the high type seller so that his utility is kept constant

$$\Delta\overline{t} = \Delta\overline{x}\overline{c}.$$

The trading probability of the low type seller is kept unchanged and the expected transfer to the low type seller is adjusted to ensure that his incentive compatibility constraint is satisfied

$$\Delta\underline{t} - \underbrace{\Delta\underline{x}\underline{c}}_{=0} = \Delta\overline{t} - \Delta\overline{x}\underline{c} \quad \Leftrightarrow \quad \Delta\underline{t} = (\overline{c} - \underline{c})\Delta\overline{x}.$$

These changes again make the high type sellers indifferent, strictly improve the low type sellers (since $\Delta\underline{t} > 0$), and always increase total surplus. They also make buyers weakly better off and thus constitute a Pareto improvement if

$$\begin{aligned} & \overline{\lambda}^p\left[\Delta\overline{x}\overline{v} - \Delta\overline{t}\right] + \underline{\lambda}^p\left[\Delta\underline{x}\underline{v} - \Delta\underline{t}\right] \geq 0, \\ \Leftrightarrow \quad & \frac{\underline{\lambda}^p}{1-\mu}\left[\mu(\overline{v} - \underline{c}) - (\overline{c} - \underline{c})\right]\Delta\overline{x} \geq 0, \end{aligned}$$

which is satisfied whenever $\mu \geq \frac{\bar{c}-\underline{c}}{\bar{v}-\underline{c}}$. $\square$

A.11 Proof of Proposition 5.2

Let W^{GM} and W^{PP} denote social surplus in the equilibrium under general mechanisms and the equilibrium under bilateral menus, respectively. We are interested in the limiting case of $\mu s \to +\infty$, while μs and b are kept finite, implying that $\overline{\lambda}^p$ tends to $+\infty$ and $\underline{\lambda}^p$ is finite.

Consider first the case of general mechanisms. Given the assumption $\underline{v} - \underline{c} < \overline{v} - \overline{c}$, the condition $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \in \left(1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}, \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}\right)$, is always satisfied, meaning that the limiting case falls into parameter region b) of Proposition 3.8. To see this, note that $\lim_{\overline{\lambda}^p \to +\infty} \dfrac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) = 0$ and $1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} < 0$, $\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}} > 0$ for $\underline{v} - \underline{c} < \overline{v} - \overline{c}$. The limit of social surplus is thus given by

$$\begin{aligned}\lim_{\overline{\lambda}^p \to +\infty} W^{GM} &= \lim_{\overline{\lambda}^p \to +\infty} b\left[\left(1 - e^{-\underline{\lambda}^p}\right)(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right)(\overline{v} - \overline{c})\right] \\ &= b\left[\left(1 - e^{-\underline{\lambda}^p}\right)(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p}(\overline{v} - \overline{c})\right].\end{aligned}$$

Consider next the case of bilateral contracting. We can first show that as $\overline{\lambda}^p \to +\infty$, the equilibrium fraction of buyers going to the high quality market, γ, tends to one. A buyer's profit in the low and high quality market, respectively, is given by

$$\left(1 - e^{-\frac{\underline{\lambda}^p}{1-\gamma}} - \frac{\underline{\lambda}^p}{1-\gamma} e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right)(\underline{v} - \underline{c}),$$
$$\left(1 - e^{-\frac{\overline{\lambda}^p}{1-\gamma}}\right)(\overline{v} - \underline{c}) - \frac{\overline{\lambda}^p}{\gamma} e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v} - \underline{c}).$$

Suppose γ does not tend to one. Then

$$\lim_{\overline{\lambda}^p \to +\infty}\left(\frac{\overline{\lambda}^p}{\gamma} e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right) = +\infty \quad \Rightarrow \quad \lim_{\overline{\lambda}^p \to +\infty} \overline{\pi} = -\infty,$$

implying that the indifference condition for buyers cannot be satisfied. For the limit of $\frac{\overline{\lambda}^p}{\gamma} e^{-\frac{\underline{\lambda}^p}{1-\gamma}}$ to be finite we thus need γ to be a function of $\overline{\lambda}^p$ with $\lim_{\overline{\lambda}^p \to +\infty} \gamma(\overline{\lambda}^p) = 1$ such that $\lim_{\overline{\lambda}^p \to +\infty}\left(\dfrac{\overline{\lambda}^p}{\gamma(\overline{\lambda}^p)} e^{-\frac{\underline{\lambda}^p}{1-\gamma(\overline{\lambda}^p)}}\right) = l \in \mathbb{R}$. The indifference condition for buyers then requires $\underline{v} - \underline{c} = \overline{v} - \underline{c} - l \Leftrightarrow l = \overline{v} - \underline{v}$. With this, the limit of social surplus in the equilibrium with bilateral menus is given by

$$\begin{aligned}\lim_{\overline{\lambda}^p \to +\infty} W^{PP} &= \lim_{\overline{\lambda}^p \to +\infty} b\left[(1-\gamma(\overline{\lambda}^p))\left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma(\overline{\lambda}^p)}}\right)(\underline{v}-\underline{c})+\gamma(\overline{\lambda}^p)\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma(\overline{\lambda}^p)}}\right)(\overline{v}-\overline{c})\right] \\ &= b(\overline{v}-\overline{c}),\end{aligned}$$

which is strictly greater than $\lim_{\overline{\lambda}^p \to +\infty} W^{GM} = b\left[\left(1-e^{-\underline{\lambda}^p}\right)(\underline{v}-\underline{c})+e^{-\underline{\lambda}^p}(\overline{v}-\overline{c})\right]$. □