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# Ambiguity Attitudes and Self-Confirming Equilibrium in Sequential Games\*

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## Abstract

We consider a game with sequential moves played by agents who are randomly drawn from large populations and matched. We assume that, when players are uncertain about the strategy distributions of the opponents, preferences over actions at any information set admit a smooth-ambiguity representation in the sense of Klibanoff, Marinacci, and Mukerji (*Econometrica*, 2005). This may induce dynamically inconsistent preferences and calls for an appropriate definition of sequential best response. We take this into account in our analysis of self-confirming equilibrium (SCE) and rationalizable SCE in sequential games with feedback played by agents with non-neutral ambiguity attitudes. Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (*Amer. Econ. Rev.*, 2015) show that the set of SCE's of a simultaneous-move game with feedback expands as ambiguity aversion increases. We show by example that SCE in a sequential game is not equivalent to SCE applied to the strategic form of such game, and that the previous monotonicity result does not extend to general sequential games. Still, we provide sufficient conditions under which the monotonicity result holds for (rationalizable) SCE.

KEYWORDS: Sequential games with feedback, smooth ambiguity, self-confirming equilibrium, rationalizable self-confirming equilibrium.

## 1 Introduction

**Self-confirming equilibrium** When a game is played recurrently and the learning dynamic has reached a rest point, each agent chooses a best reply to his subjective belief,

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which may be incorrect and yet confirmed by the evidence available to him. A profile of strategies and beliefs with this property is a self-confirming equilibrium (henceforth, SCE).<sup>1</sup> The standard definition of SCE assumes that agents are subjective expected utility maximizers, i.e., that they are ambiguity neutral.<sup>2</sup> Yet, a large body of empirical evidence supports the ambiguity aversion hypothesis. This is particularly relevant when agents only have scarce evidence of opponents' behavior and face therefore strategic uncertainty. Following Battigalli, Cerreia-Vioglio, Maccheroni and Marinacci (2015, henceforth BCMM), we analyze the SCE concept in games played by ambiguity averse agents. Unlike BCMM, who essentially restrict their attention to simultaneous-move games, we consider games with sequential moves, represented in extensive form. We also analyze a notion of rationalizable SCE that captures sophisticated strategic reasoning. In a sequential game, agents have evidence, at best, of how opponents play on the equilibrium path, but no evidence of how they would play off the path. Thus, sequential games constitute a natural context for self-confirming equilibrium analysis. In the rest of this introduction, we describe how our paper adds to the previous literature in general and to BCMM in particular.

**SCE and model uncertainty** BCMM analyze SCE in simultaneous-move population games played recurrently by agents with non-neutral attitudes toward ambiguity, which is the imperfect quantifiability of the relevant risks. Specifically, agents are assumed to have smooth-ambiguity preferences in the sense of Klibanoff, Marinacci, and Mukerji (2005, henceforth KMM). This decision model is flexible and analytically convenient for game theoretic applications. First, it separates ambiguity attitudes, a stable personal trait like risk attitudes, from the perception of uncertainty, which is a property of subjective beliefs affected by the game situation; second, it provides a parameterization of ambiguity aversion analogous to the parameterization of risk aversion, which simplifies comparative static exercises.

SCE is defined as follows: Let  $I$  denote the set of player roles (e.g., buyer and seller) and  $S_i$  the set of pure strategies of any agent playing in role  $i$ . A profile of strategy distributions  $\sigma^* = (\sigma_i^*)_{i \in I} \in \times_{i \in I} \Delta(S_i)$  is a SCE if, for each  $i \in I$  and each  $s_i^* \in S_i$  with  $\sigma_i^*(s_i^*) > 0$ , there is a belief  $\mu_i$  about the strategy distributions  $\sigma_{-i}$  of the opponents that justifies  $s_i^*$  as a KMM-best response and is consistent with the long-run distribution of ex-post observations for  $i$  generated by  $s_i^*$  and  $\sigma_{-i}^*$  (e.g., the  $(s_i^*, \sigma_{-i}^*)$ -induced distribution of terminal nodes). Since the distribution of observations may not reveal the true underlying distribution of strategies  $\sigma_{-i}^*$ , agents may be uncertain about it.

Assuming that the own-payoff relevant consequences are observed by each agent after each play, BCMM prove a monotonicity result: higher ambiguity aversion entails a larger set of equilibria. Intuitively, for each agent, the strategy played repeatedly in equilibrium yields known risks because the agent observes its long-run distribution of payoffs, while

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<sup>1</sup>This terminology is due to Fudenberg and Levine (1993).

<sup>2</sup>For a perspective on ambiguity attitudes that applies to this paper, see Cerreia-Vioglio et al. (2013) and the survey by Marinacci (2015).

deviations are untested and may be perceived as ambiguous; therefore, higher ambiguity aversion penalizes deviations relative to the equilibrium choice. This monotonicity result implies that greater ambiguity aversion entails less predictability of strategies in the long-run, because the set of possible steady states is larger.

**Our contribution I: lack of covert commitment** The scope of BCMM’s analysis is essentially limited to simultaneous-move games and, possibly, games played in strategic form—such as experimental games played in the lab with the so-called “strategy method”<sup>3</sup>—because it is well known that ambiguity aversion may make preferences over strategies dynamically inconsistent (e.g., Siniscalchi 2011). As a consequence, there would be incentives to make covert commitments if such commitment moves were available.<sup>4</sup> Thus, the fact that agents in sequential games cannot (irreversibly) choose strategies—because they just choose actions at decision nodes—must be faced and dealt with explicitly. We assume that agents are sophisticated: They understand their future contingent incentives and choose actions in early stages predicting that such incentives determine their actions in later stages, that is, they plan by “folding back” given their subjective conditional beliefs about the strategies of other players. Hence, they execute “unimprovable” strategies. Since unimprovable strategies may be different from best replies in the normal form of the game, the definition of equilibrium due to BCMM cannot be applied to sequential games. Note that this is unrelated to how agents react to completely unexpected moves. The SCE concept has two features: First, its confirmed-beliefs condition implies that agents cannot be surprised on the equilibrium path. Second, it does not rely on complete information nor does it capture strategic reasoning. In particular, SCE does not model how agents predict the reactions of others to unexpected moves. In this paper, we also put forward a notion of rationalizable SCE to analyze reactions to unexpected moves and strategic reasoning about such reactions.

Two questions naturally arise: First, how does SCE defined on the extensive form relate to SCE in the normal form of the same game? Second, does the comparative ambiguity aversion result extend to games with sequential moves? To elaborate on the first question, fix a sequential game  $\Gamma$  represented in extensive form with a set of paths (terminal nodes)  $Z$  and a specification of players’ feedback  $f = (f_i)_{i \in I}$  that describes what they can observe at the end of each play; formally, each  $f_i$  is a function defined on  $Z$ . Then, we can derive the normal (or strategic) form  $(G, F) = \mathcal{N}(\Gamma, f)$ , where  $G$  is given by the normal-form payoff functions, and each player  $i$ ’s feedback  $f_i$  about the path is replaced by a corresponding normal-form feedback  $F_i$  defined on the set  $S = \times_{i \in I} S_i$  of pure strategy profiles (e.g., each player only observes his monetary payoff, which is a function of  $s \in S$ ). Under subjective expected utility maximization, which is dynamically consistent, SCE in the extensive form

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<sup>3</sup>See the survey by Brandts and Charness (2011) and the references therein.

<sup>4</sup>Unlike *overt* commitment, covert commitment moves are not observed by other players. The strategic advantages of overt commitment are well known at least since Schelling (1960) and do not depend on dynamic inconsistency as traditionally defined in decision theory.

is realization equivalent to SCE in the normal form. With ambiguity aversion, instead, we show that there may be different, non-nested sets of equilibrium outcomes; in other words, working with the normal form is neither too permissive nor too restrictive, it is just wrong.

Given that the sets of SCE outcomes in the extensive and normal form do not coincide, we cannot rely on the monotonicity result of BCMM to argue that, in a game with sequential moves, the set of SCE outcomes expands as ambiguity aversion increases. Indeed, we show by example that monotonicity does not hold in general. It is still true that, on the equilibrium path, equilibrium actions entail known risks while deviations may be perceived as ambiguous. But, if a deviation is followed by other actions of the same player, folding-back planning may require that plans (i.e., predictions) about these actions change with ambiguity aversion, and dynamic inconsistency may lead to an increase in the value of deviations.

Despite this, we can prove the monotonicity result for two special cases: (i) games where, on each path, no player moves more than once, and (ii) pure strategy equilibria of games with no chance moves.<sup>5</sup> In case (i), the difficulties mentioned above cannot arise; in case (ii) we show that, loosely speaking, in equilibrium ambiguity aversion is not distinguishable from the risk aversion of expected utility maximizers, who are dynamically consistent. We also prove a related noteworthy result: The set of SCE outcomes with ambiguity neutral agents is always included in the set of SCE outcomes with ambiguity averse agents. This means that the standard version of the SCE concept, by ignoring ambiguity aversion, overestimates the predictability of long-run outcomes in recurrent interactions.

**Our contribution II: lack of overt commitment** Next, we turn to the well known issue of the impossibility of *overt* (i.e., observable) commitment and how this affects strategic reasoning and equilibrium. Rationality in dynamic games requires that agents choose subjective best replies to their updated beliefs at *all* information sets, including the unexpected ones. We call this condition “full unimprovability.” If SCE is not refined so as to capture strategic reasoning, requiring full unimprovability rather than simple unimprovability does not change the set of equilibrium outcomes. The reason is that in an SCE agents may hold wrong beliefs about the reactions of others to non-equilibrium moves, which are necessarily unexpected; hence, they can expect responses that are irrational given the actual payoff functions of co-players. Assuming instead that (some features of) such payoff functions are common knowledge, one can embed strategic reasoning into the SCE concept: if (a) players are rational, (b) their beliefs are confirmed, and (c) there is common belief of (a) and (b), then their strategies form a rationalizable SCE. This concept has been analyzed under the assumption of subjective expected utility maximization, that is, neutral ambiguity attitudes (e.g., Rubinstein and Wolinsky 1994, Dekel et al. 1999). Here we provide an extension for non-neutral ambiguity attitudes and we prove that versions of our monotonicity

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<sup>5</sup>We call such equilibria “symmetric” because in the population game scenario they represent situations where all agents in the same role play in the same way.

results hold for rationalizable SCE.

**Related literature** This is the first paper that analyzes SCE and rationalizable SCE with non-neutral ambiguity attitudes in sequential games. BCMM and Battigalli et al. (2016b) only consider games played in strategic form. The extensive form game from which the strategic form is derived only affects feedback: two strategy profiles that yield the same terminal node necessarily yield the same feedback to every player. Furthermore, Battigalli et al. (2016b) focus on “maxmin SCE” a notion of equilibrium with extreme ambiguity aversion, whereas here we apply the smooth ambiguity criterion of KMM. Hereafter we call “standard SCE” the self-confirming equilibrium concept with ambiguity neutrality, i.e., subjective expected utility maximization.

A version of the standard SCE concept was first put forward in the undergraduate thesis of Battigalli (1987)<sup>6</sup> and called “conjectural equilibrium” (*“equilibrio congetturale”*). The first definition in English appears in a working paper by Battigalli and Guaitoli (1988), who also define and apply a rationalizability refinement.<sup>7</sup> Fudenberg and Levine (1993) independently provided a definition of standard SCE under the maintained assumption that agents perfectly observe ex-post the path of play. Unlike Battigalli (1987) and Battigalli and Guaitoli (1988), the analysis of Fudenberg and Levine applies to large population games. This is important because it justifies the assumption that agents maximize their short-run expected utility even if they are patient. The reason is that in a large population game there are no incentives to affect the future behavior of current opponents, who are almost surely different from the future ones.<sup>8</sup> For further references and details see the survey by Battigalli et al. (1992) and the discussion in Section IV of BCMM.

Fudenberg and Levine (1993) note that standard SCE is not strategic-form invariant, arguing that the strategic—or normal—form is therefore insufficient to characterize SCE. But their comment rests on the maintained assumption that players always observe ex-post the path of play, which in a simultaneous-move game is just the actual profile of strategies (actions of the strategic form). Therefore, when they compare standard SCE in the extensive (i.e., sequential) and strategic form, they change what players can observe ex post about the behavior of others: only on-path actions in the extensive form, and complete strategy profiles in the strategic form. If feedback about the behavior of others is instead kept constant—as we do when we look at strategic-form feedback—standard SCE in the extensive form is equivalent to standard SCE in the strategic form (see Remark 3). On the other hand, we show that SCE with ambiguity aversion is *not* strategic-form invariant (see Example 4).

Standard SCE can be refined to make beliefs consistent with deductive strategic thinking based on common knowledge of the game (or some aspects of it). Rubinstein and Wolinsky (1994) put forward such a refinement for simultaneous-move games with complete

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<sup>6</sup>In Italian.

<sup>7</sup>Published in 1997 as a book chapter.

<sup>8</sup>Furthermore, incentives to experiment vanish in the long run.

information; Esponda (2013) extends the analysis to simultaneous-moves games with incomplete information and provides an explicit derivation from epistemic assumptions about how players reason about each other; Dekel et al. (1999) analyzed a similar refinement for sequential games under the assumption that players have perfect feedback about the path of play, Fudenberg and Kamada (2015, 2017) extend the analysis to allow for imperfect feedback about the path of play. These equilibrium concepts appear under different names such as “rationalizable conjectural equilibrium” and “rationalizable partition-confirmed equilibrium,” but they are all refinements of standard SCE—as defined here and in BCMM—that capture common initial belief in both rationality and belief-confirmation.<sup>9</sup> Battigalli (1987) and Battigalli and Guaitoli (1988) consider instead a different forward-induction refinement of standard SCE derived from rationality, confirmed beliefs and common strong belief in rationality.<sup>10</sup> In this paper we define and analyze rationalizable SCE with non-neutral ambiguity attitudes, by looking at the behavioral implications of (folding-back) rationality, and mutual initial belief in rationality of level  $k$  for each  $k = 1, 2, \dots, \infty$ . Thus—like Esponda (2013) and unlike the other papers on rationalizable SCE—we give an inductive definition based on iterated deletions of strategy profiles. In Section 8 we discuss the merits of this and alternative definitions.

As mentioned above, BCMM proved that in simultaneous-move games the SCE correspondence is monotone with respect to players’ degree of ambiguity aversion. Here we prove partial versions of this result for sequential games. A similar monotonicity result can be proved for comparative risk aversion in the particular case of *pure* standard SCE (see the lecture notes of Battigalli, 2017), and for comparative risk or ambiguity aversion in the case of rationalizable strategies (Battigalli et al., 2016a). However, the intuition and proof in the case of rationalizability are very different from the SCE case. Interestingly, we rely on Battigalli et al. (2016a) to prove (in some special cases) monotonicity of the rationalizable SCE correspondence (see Section 8).

Hanany et al. (2017) use the smooth ambiguity criterion of KMM to analyze sequential games with incomplete information. Their work differs from ours in two important ways. First, they analyze a notion of perfect Bayesian equilibrium, which assumes that players’ conjectures about the (type-dependent) strategies of other players are correct, whereas we analyze SCE, which allows for incorrect conjectures. Second, they differ from us in the way they model subjectively rational planning: We maintain that players’ beliefs satisfy the standard rules of conditional probability, which imply Bayes rule; given this, each player adopts an unimprovable strategy that—by the dynamic inconsistency of ambiguity averse preferences—may be suboptimal according to his initial preferences, or the interim preferences he holds at some information set. Hanany et al. instead assume that each player’s equilibrium strategy is sequentially optimal, that is, optimal according to his pref-

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<sup>9</sup>Common initial belief is common belief at the beginning of the game. In simultaneous-moves games, it coincides with the usual notion of common belief.

<sup>10</sup>The connection between forward-induction thinking, common strong belief in rationality, and strong rationalizability is established in Battigalli and Siniscalchi (2002).

ferences at each information set. Ambiguity aversion implies that this cannot always be reconciled with Bayesian updating. But they show it is without loss of generality to restrict attention to beliefs generated using a particular generalization of Bayesian updating, where more ambiguity aversion implies larger deviations from standard Bayesian updating. Eichberger et al. (2017) also analyze a notion of equilibrium with ambiguity aversion in sequential games. Like Hanany et al. (2017) they deviate from Bayesian updating; like us, they rely on folding-back planning; but they model ambiguity aversion with a form of Choquet-expected utility maximization.

**Outline** The rest of the paper is organized as follows. Sections 2 and 3 introduce the setup and the smooth ambiguity criterion; Section 4 defines and analyzes unimprovability; Section 5 defines our SCE concept; Section 6 presents comparative results for SCE; Section 7 introduces full unimprovability and shows that such strengthening by itself does not affect SCE outcomes; Section 8 relies on full unimprovability to define rationalizable SCE and provides comparative results for such concept; finally, Section 9 discusses the relevance of some assumptions and equilibrium concepts, and provides hints for the generalization of the analysis. The main text contains some intuitive arguments, but all formal proofs are collected in the Appendix.

## 2 Framework

We analyze an agent with non-neutral ambiguity attitudes who plays a game with sequential moves. We assume that the commitment technology of this agent is explicitly represented by the rules of the game. Therefore, the agent can control—i.e., irreversibly choose—only his (pure) actions at whatever information set is being reached. We also assume that he is sophisticated and therefore he takes this into account when he plans how to play the game.

Consider an agent who plays in role  $i \in I$  of a *finite extensive-form game*  $\Gamma$  with *perfect recall*. Let  $H_i$  denote the collection of **information sets** of  $i$  and let  $A_i(h)$  be the set of actions available at  $h \in H_i$ . We assume for expositional simplicity that  $|A_i(h)| \geq 2$  for each  $h \in H_i$ , where  $|X|$  denotes the cardinality of a finite set  $X$ . This means that we include in  $H_i$  only the information sets where  $i$  is active. Let  $\emptyset$  denote the root of the game; then  $\{\emptyset\} \in H_i$  if and only if  $i$  is a first mover.<sup>11</sup> We endow  $H_i$  with the weak (respectively, strict) precedence relation  $\preceq$  ( $\prec$ ) inherited from the game tree.<sup>12</sup> The set of **strategies**

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<sup>11</sup>Our preferred representation of games in extensive form starts from sequences of action profiles, that correspond to the nodes of the game tree (e.g., Chapters 6 and 11 of Osborne and Rubinstein, 1994) and allows for the representation of players' information also at nodes where they are not active, such the root for players who are not first-movers. This affects the way we draw pictures and describe examples, but it is otherwise irrelevant for the analysis of the paper.

<sup>12</sup>Perfect recall implies that, for all  $h, h' \in H_i$ , there are nodes (histories)  $x \in h$  and  $x' \in h'$  such that  $x$  precedes  $x'$  if and only if every node of  $h'$  is preceded by a node of  $h$ . With this, we can stipulate that, for all  $h, h' \in H_i$ ,  $h$  strictly **precedes**  $h'$ , written  $h \prec h'$ , if every node of  $h'$  is strictly preceded by a node



for player  $i$  is  $S_i = \times_{h \in H_i} A_i(h)$ . For every  $s_i \in S_i$  and  $h \in H_i$ , we let  $s_{i,h}$  denote the action specified by  $s_i$  at  $h$ ; thus,  $s_i = (s_{i,h})_{h \in H_i} \in \times_{h \in H_i} A_i(h)$ . We let  $S = \times_{j \in I} S_j$  denote the set of pure strategy profiles of the players.

We model randomization explicitly as the choice of a randomization device. Therefore it is important to allow for **chance moves** as the moves of a special player denoted by  $0 \notin I$ . With this,  $H_0$  denotes the collection of information sets of the chance player,  $A_0(h)$  is set of chance moves available at  $h \in H_0$ , and  $S_0 = \times_{h \in H_0} A_0(h)$  is the set of “strategies” of the pseudo-player 0. Throughout, we maintain for simplicity the assumption that the *probabilities of chance moves are commonly known*. Such probabilities are specified by a “behavioral strategy”  $\bar{\beta}_0 \in \times_{h \in H_0} \Delta(A_0(h))$ , with  $\bar{\beta}_0(a_0|h) > 0$  for every  $h \in H_0$  and  $a_0 \in A_0(h)$ ;  $\bar{\beta}_0$  induces the “mixed strategy”  $\bar{\sigma}_0 \in \Delta(S_0)$  such that

$$\bar{\sigma}_0(s_0) = \prod_{h \in H_0} \bar{\beta}_0(s_{0,h}|h) > 0$$

for every  $s_0 \in S_0$ . Thus, the outcome distributions respectively induced by  $\bar{\beta}_0$  and  $\bar{\sigma}_0$  coincide for every strategy profile of the true players (see Kuhn, 1953). We let  $I_0 = I \cup \{0\}$  denote the set of true player-roles *and* chance, and we let  $-i = I_0 \setminus \{i\}$ . In particular,  $S_{-i} = \times_{j \in I_0 \setminus \{i\}} S_j$  denotes the set of pure strategy profiles of the opponents including chance.

Let  $Z$  denote the set of terminal nodes of the game. Every profile  $(s_0, s)$  induces a complete path, hence a terminal node, through the **outcome function**

$$\zeta : S_0 \times S \rightarrow Z.$$

Since the definition of  $\zeta$  is standard, we take it for granted and then define some derived concepts using  $\zeta$ .

For conceptual clarity, we also include in the description of the extensive form  $\Gamma$  a **consequence function**

$$\gamma : Z \rightarrow C$$

which specifies the material consequence  $c = \gamma(z) \in C$  of each terminal node  $z \in Z$ . For example, we may have  $C \subseteq \mathbb{R}^I$  where  $c = (c_j)_{j \in I} \in C$  is a consumption allocation or a distribution of monetary payoffs to players. Thus, player  $i$ 's risk attitudes (preferences over objective lotteries of consequences) are represented by a **vN-M utility function**

$$v_i : C \rightarrow \mathbb{R}.$$

To ease notation, we write the **payoff function** of  $i$  as

$$u_i = v_i \circ \gamma : Z \rightarrow \mathbb{R}.$$

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of  $h$ . Perfect recall implies that each  $h \in H_i$  can have at most one immediate predecessor. The reflexive closure of  $<$  is  $\preceq$ , an antisymmetric, and transitive relation that makes  $H_i$  a directed forest. If  $\{\emptyset\} \in H_i$ , then  $(H_i, \preceq)$  is a directed tree.

It is convenient to specify the information about strategies implied by any information set  $h$ . First, for any node  $x$ , let

$$S_{I_0}(x) = \{(s_0, s) \in S_0 \times S : x \prec \zeta(s_0, s)\}$$

denote the set of pure strategy profiles reaching  $x$ . With this, for any subset of nodes  $h$ ,

$$S_{I_0}(h) = \bigcup_{x \in h} S_{I_0}(x)$$

is the set of strategy profiles  $(s_0, s)$  reaching  $h$ ,

$$S_{-i}(h) = \text{proj}_{S_{-i}} S_{I_0}(h) = \{s_{-i} \in S_{-i} : \exists (x, s_i) \in h \times S_i, x \prec \zeta(s_i, s_{-i})\}$$

is the set of pure strategy profiles of chance and opponents that allow for  $h$ , and

$$S_i(h) = \text{proj}_{S_i} S_{I_0}(h) = \{s_i \in S_i : \exists (x, s_{-i}) \in h \times S_{-i}, x \prec \zeta(s_i, s_{-i})\}$$

is the sets of  $i$ 's strategies allowing for  $h$ .

It is useful to keep in mind that perfect recall implies the following factorization:

$$\forall h \in H_i, S_{I_0}(h) = S_i(h) \times S_{-i}(h).$$

Furthermore, it also implies that

$$\forall g, h \in H_i, g \prec h \Rightarrow S_{-i}(h) \subseteq S_{-i}(g).$$

Intuitively,  $i$  obtains finer information about strategies as the play unfolds.

Players are drawn at random from large populations of agents, each playing a pure strategy. The **distribution of pure strategies in population**  $j \in I$  is denoted by  $\sigma_j \in \Delta(S_j)$ , and it is unknown to the other players. For any profile of distributions  $(\sigma_j)_{j \in I}$ , we let<sup>13</sup>

$$\sigma_{-i} = \bar{\sigma}_0 \times (\times_{j \in I \setminus \{i\}} \sigma_j) \in \Delta(S_{-i})$$

denote the product distribution over strategies of chance and of  $i$ 's opponents. From the viewpoint of an agent in role  $i$ ,  $\sigma_{-i}(s_{-i})$  is the unknown *objective* probability of facing opponents and chance playing pure strategy profile  $s_{-i}$ .<sup>14</sup> We let

$$\Sigma_{-i} = \left\{ \sigma_{-i} \in \Delta(S_{-i}) : \exists (\sigma_j)_{j \in I \setminus \{i\}} \in \times_{j \in I \setminus \{i\}} \Delta(S_j), \sigma_{-i} = \bar{\sigma}_0 \times (\times_{j \in I \setminus \{i\}} \sigma_j) \right\}$$

<sup>13</sup>We use symbol  $\times$  to denote both the Cartesian product of sets and the product of measures.

<sup>14</sup>Statistical independence follows from random matching: For each  $i \in I$ , let  $P_i$  denote the set of agents playing in role  $i$ , and let  $\varsigma_i : P_i \rightarrow S_i$  denote the (measurable) strategy map of population  $i$ . If agents are drawn at random from their populations, that is, according to a *uniform* distribution on  $\times_{i \in I} P_i$ , then the induced distribution on  $S$  given  $(\varsigma_i)_{i \in I}$  is a product measure. Taking into account that player  $i$  has a known deterministic plan, knows chance probabilities, and knows that opponents are randomly matched, we obtain the product formula in the text.

denote the set of these product distributions. We endow  $\Sigma_{-i}$  with the topology inherited from the Euclidean topology on  $\mathbb{R}^{S_{-i}}$ , which makes it compact, and with the corresponding Borel sigma algebra  $\mathcal{B}(\Sigma_{-i})$ .

At each point of the game, the agent playing in role  $i$  has some **belief**  $\mu_i \in \Delta(\Sigma_{-i})$ . Note that by the definition of  $\Sigma_{-i}$ , the marginal of  $\mu_i$  on  $\Sigma_0$  always assigns probability 1 to  $\bar{\sigma}_0$ . The belief  $\mu_i$  that  $i$  holds at the beginning of the game is  $i$ 's **prior**. For each  $\mu_i \in \Delta(\Sigma_{-i})$ , we let  $p_{\mu_i} \in \Delta(S_{-i})$  denote the **predictive probabilities** implied by  $\mu_i$ : for each  $s_{-i} \in S_{-i}$ ,

$$p_{\mu_i}(s_{-i}) = \int_{\Sigma_{-i}} \sigma_{-i}(s_{-i}) \mu_i(d\sigma_{-i}).$$

We summarize our notation in the following table and illustrate it with an example. We will refer to this example repeatedly.

Primitive Elements	Terminology
$i, j \in I$ ( $I_0 = I \cup \{0\}$ )	<b>players</b> ( $0 \notin I$ denotes <b>chance</b> )
$h \in H_i$	<b>information sets</b> of $i$
$\preceq$ ( $\prec$ )	(strict) <b>precedence</b> relation of $\Gamma$
$(H_i, \preceq)$	directed <b>forest of information sets</b> of $i$
$z \in Z$	<b>terminal</b> histories/nodes
$\gamma : Z \rightarrow C$	<b>consequence function</b>
$v_i : C \rightarrow \mathbb{R}$	<b>vN-M utility</b> function of $i$
$a_i \in A_i(h)$	$i$ 's <b>actions</b> at $h \in H_i$

Derived Elements	Terminology
$s_i \in S_i = \times_{h \in H_i} A_i(h)$	<b>strategies</b> of $i$
$s \in S$ ( $s_{-i} \in S_{-i}$ )	<b>strategy profiles</b> (of $-i = I_0 \setminus \{i\}$ )
$S_{I_0}(h)$	<b>strategy profiles</b> (including 0) <b>reaching</b> $h$
$S_i(h) = \text{proj}_{S_i} S_{I_0}(h)$	<b>strategies of <math>i</math> allowing for <math>h</math></b>
$S_{-i}(h) = \text{proj}_{S_{-i}} S_{I_0}(h)$	<b>strategy profiles of <math>-i</math> (including 0) allowing for <math>h</math></b>
$\sigma_j \in \Delta(S_j)$	<b>strategy distributions</b> on $S_j$
$\sigma_{-i} \in \Sigma_{-i} \subset \Delta(S_{-i})$	<b>product distributions</b> on $S_{-i}$ , with $\text{marg}_{\Sigma_0} \sigma_{-i} = \bar{\sigma}_0$
$\mu_i \in \Delta(\Sigma_{-i})$	<b>beliefs</b> of $i$
$p_{\mu_i} \in \Delta(S_{-i})$	<b>predictive probabilities</b> implied by $\mu_i$
$\zeta : S_0 \times S \rightarrow Z$	<b>outcome function</b>
$u_i = v_i \circ \gamma : Z \rightarrow \mathbb{R}$	<b>payoff function</b> of $i$

**Example 1** The game depicted in Figure 1 is a two-person, common-interest, multistage game where  $I = \{1, 2\}$  and 0 is chance. If we identify nodes with histories,<sup>15</sup> information

<sup>15</sup>See, e.g., Chapter 11 in Osborne and Rubinstein (1994).

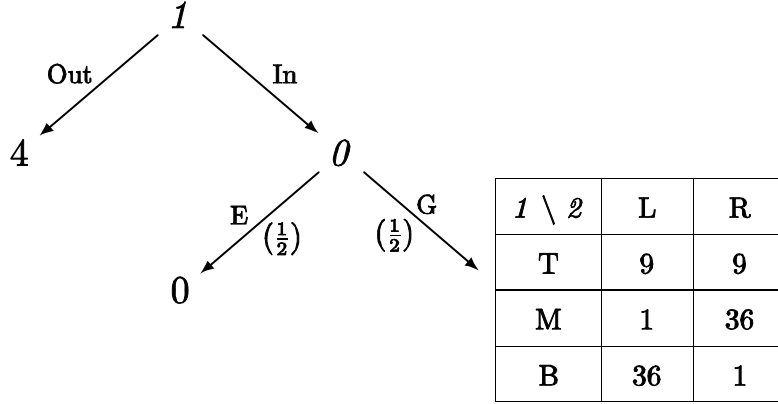


Figure 1: Leading Example

sets, actions sets and terminal histories/nodes are as follows:

$$\begin{aligned}
 H_0 &= \{\text{In}\}, \quad H_1 = \{\{\emptyset\}, \{(\text{In}, \text{G})\}\}, \quad H_2 = \{(\text{In}, \text{G})\}, \\
 A_0(\{\text{In}\}) &= \{\text{E}, \text{G}\}, \quad A_2(\{(\text{In}, \text{G})\}) = \{\text{L}, \text{R}\}, \\
 A_1(\{\emptyset\}) &= \{\text{In}, \text{Out}\}, \quad A_1(\{(\text{In}, \text{G})\}) = \{\text{T}, \text{M}, \text{B}\}, \\
 Z &= \{\text{Out}, (\text{In}, \text{E})\} \cup \{(\text{In}, \text{G})\} \times \{\text{T}, \text{M}, \text{B}\} \times \{\text{L}, \text{R}\}.
 \end{aligned}$$

Numbers at terminal histories/nodes, including the boxes in the matrix subgame, give the common payoff of players 1 and 2. The probabilities of chance moves are  $\bar{\sigma}_0(\text{E}) = \bar{\sigma}_0(\text{G}) = \frac{1}{2}$ .

Assume that the agent in role 1 has the following belief about the behavior of player 2:<sup>16</sup>

$$\mu_1(\sigma_2) = \begin{cases} \frac{1}{2} & \text{if } \sigma_2 \in \{\delta_{\text{L}}, \delta_{\text{R}}\}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\delta_x$  denotes the Dirac measure supported by  $x$ . Intuitively, he thinks that all the agents playing in role 2 “attended the same school” and hence are doing the same thing, but he does not know what. The induced predictive probabilities are  $p_{\mu_1}(\text{L}) = p_{\mu_1}(\text{R}) = \frac{1}{2}$ .

▲

<sup>16</sup>In the examples, we will always specify beliefs about the true opponents only, taking for granted that  $\bar{\sigma}_0$  is commonly known.

### 3 Smooth-ambiguity preferences over actions

In this section, we take the perspective of an agent, or decision maker, playing in role  $i$ , henceforth  $DM_i$ , with given beliefs about the behavior of agents in different roles and a contingent plan, or strategy. Specifically, he has a plan  $s_i$  specifying the action  $s_{i,h} \in A_i(h)$  he expects to take (but he is not committed to take) at each information set  $h \in H_i$ ; he has no randomization technology beyond what is already explicitly represented in the extensive form of the game (see Section 2), and we assume that he is certain about his contingent behavior, i.e., he has a deterministic contingent plan.

**Conditional distributions and conditional objective expected utility** Let  $\Sigma_{-i}(h)$  denote the set of distributions that assign positive probability to  $S_{-i}(h)$ , that is,

$$\Sigma_{-i}(h) = \{\sigma_{-i} \in \Sigma_{-i} : \sigma_{-i}(S_{-i}(h)) > 0\}.$$

For every  $h \in H_i$  and  $\sigma_{-i} \in \Sigma_{-i}(h)$ , we can compute the objective conditional distribution on the opponents' strategy profiles consistent with  $h$ :

$$\forall s_{-i} \in S_{-i}(h), \sigma_{-i}(s_{-i}|h) = \frac{\sigma_{-i}(s_{-i})}{\sigma_{-i}(S_{-i}(h))} \quad (1)$$

(note that whether  $\sigma_{-i} \in \Sigma_{-i}(h)$  or not depends only  $(\sigma_j)_{j \in I \setminus \{i\}}$  because  $\sigma_0 = \bar{\sigma}_0$  is strictly positive). With this, we can define the **strategic-form Bernoulli conditional expected utility** function

$$\begin{aligned} U_i(\cdot, \cdot|h) : S_i(h) \times \Sigma_{-i}(h) &\rightarrow \mathbb{R}, \\ (s_i, \sigma_{-i}|h) &\mapsto \sum_{s_{-i} \in S_{-i}(h)} \sigma_{-i}(s_{-i}|h) u_i(\zeta(s_i, s_{-i})). \end{aligned}$$

In words, if  $i$  is certain that  $\sigma_{-i}$  is the true objective probability model, then upon observing  $h$  (he believes that) his conditional objective expected utility from following strategy  $s_i \in S_i(h)$  is  $U_i(s_i, \sigma_{-i}|h)$ .<sup>17</sup>

**Plans and replacements** Plan  $s_i$  yields a continuation on the information sets in  $H_i$  following any given  $h \in H_i$  (that is, the projection of  $s_i$  onto  $\times_{\{h' \in H_i : h \preceq h'\}} A_i(h')$ ).  $DM_i$  expects to continue according to this plan, but he knows (by perfect recall) that he has already chosen the actions leading to  $h$ , possibly violating  $s_i$ , and he considers the consequences of choosing action  $a_i \in A_i(h)$ , again possibly violating  $s_i$ . It is convenient to define

<sup>17</sup>One can show that this coincides with the more familiar formula

$$U_i(s_i, \sigma_{-i}|h) = \sum_{x \in h} \mathbb{P}_{s_i, \sigma_{-i}}(x|h) \sum_{z \in Z} \mathbb{P}_{s_i, \sigma_{-i}}(z|x) u_i(z)$$

where  $\mathbb{P}_{s_i, \sigma_{-i}}(\cdot|\cdot)$  denotes the probability of reaching a node conditional on an information set, or an earlier node, given by  $\sigma_{-i}$ .

the **replacement** plan  $(s_{i|h}, a_i)$  obtained by replacing  $s_i$  with the already chosen actions at information sets preceding  $h$  and with action  $a_i$  at  $h$ :

$$(s_{i|h}, a_i)_{h'} = \begin{cases} a_i & \text{if } h' = h, \\ \alpha_i(h', h) & \text{if } h' \prec h, \\ s_{i,h'} & \text{otherwise,} \end{cases}$$

where  $\alpha_i(h', h)$  is the action chosen at  $h' \prec h$  in order to reach  $h$ .<sup>18</sup> Finally, we let  $s_{i|h}$  denote the replacement plan obtained when action  $s_{i,h}$  is played at  $h$ :

$$(s_{i|h})_{h'} = \begin{cases} \alpha_i(h', h) & \text{if } h' \prec h, \\ s_{i,h'} & \text{otherwise.} \end{cases}$$

**Action values** We assume that  $DM_i$ 's preferences over actions, given his beliefs and plan, satisfy the smooth-ambiguity model of KMM: On top of the vN-M utility function  $v_i : C \rightarrow \mathbb{R}$  specified by game  $\Gamma$  (hence, the payoff function  $u_i = v_i \circ \gamma$ ), we assume that there is a continuous and *strictly increasing second-order utility* function

$$\phi_i : \mathbb{V}_i \rightarrow \mathbb{R},$$

where

$$\mathbb{V}_i = \left[ \min_z v_i(\gamma(z)), \max_z v_i(\gamma(z)) \right]$$

is the convex hull of the range of  $v_i$ . For every given  $h \in H_i$ ,  $\mu_i \in \Delta(\Sigma_{-i}(h))$ , and  $s_i \in S_i$ ,  $DM_i$  assigns values to actions  $a_i \in A_i(h)$  as follows:<sup>19</sup>

$$V_i(a_i|h; s_i, \mu_i, \phi_i) = \phi_i^{-1} \left( \int_{\Sigma_{-i}(h)} \phi_i(U_i((s_{i|h}, a_i), \sigma_{-i}|h)) \mu_i(d\sigma_{-i}) \right). \quad (2)$$

Condition  $\mu_i(\Sigma_{-i}(h)) = 1$  suggests that, in eq. (2), we interpret  $\mu_i$  as the conditional belief of  $DM_i$  upon observing  $h$ . We postpone the definition of conditional beliefs to Section 4. If  $\mu_i$  assigns probability 1 to some  $\sigma_{-i} \in \Sigma_{-i}(h)$  (that is,  $\mu_i = \delta_{\sigma_{-i}} \in \Delta(\Sigma_{-i}(h))$ ), then

$$V_i(a_i|h; s_i, \mu_i, \phi_i) = \phi_i^{-1}(\phi_i(U_i((s_{i|h}, a_i), \sigma_{-i}|h))) = U_i((s_{i|h}, a_i), \sigma_{-i}|h).$$

Thus, ambiguity attitudes are immaterial when  $DM_i$  is certain about the true probability model, because in this case he does not perceive any ambiguity. Note also that (2) boils down to the classical subjective expected utility formula if  $\phi_i$  is linear (ambiguity neutrality), hence equivalent to the identity function  $\text{Id}_{\mathbb{V}_i}$ . On the other hand, ambiguity aversion is characterized by the concavity of  $\phi_i$ . We emphasize in our notation only the dependence of values of  $i$ 's actions on parameter  $\phi_i$ , not on the vN-M utility function  $v_i$ , because we are going to consider different possible shapes of  $\phi_i$  (in particular, linear and concave) with a fixed  $v_i$ .

<sup>18</sup>By perfect recall,  $\alpha_i$  is well defined.

<sup>19</sup>For an axiomatization of the smooth ambiguity criterion, see Cerreia-Vioglio et al. (2013a)

**Example 2** In the game of Figure 1,

$$\begin{aligned} U_1(\text{In.B}, \bar{\sigma}_0 \times \delta_L | \{(\text{In}, \text{G})\}) &= 36, \\ U_1(\text{In.B}, \bar{\sigma}_0 \times \delta_R | \{(\text{In}, \text{G})\}) &= 1, \end{aligned}$$

and

$$\begin{aligned} U_1(\text{In.B}, \bar{\sigma}_0 \times \delta_L | \{\emptyset\}) &= \frac{1}{2} \cdot 36 = 18, \\ U_1(\text{In.B}, \bar{\sigma}_0 \times \delta_R | \{\emptyset\}) &= \frac{1}{2} \cdot 1 = \frac{1}{2}. \end{aligned}$$

Assume  $\phi_1(u_i) = \sqrt{u_i}$  and consider the (marginal) belief of Example 1, that is

$$\mu_1(\sigma_2) = \begin{cases} \frac{1}{2} & \text{if } \sigma_2 \in \{\delta_L, \delta_R\}, \\ 0 & \text{otherwise;} \end{cases}$$

then

$$V_1(\text{B} | \{(\text{In}, \text{G})\}; s_1, \mu_1, \phi_1) = \left( \frac{1}{2} \sqrt{36} + \frac{1}{2} \sqrt{1} \right)^2 = (3.5)^2 = 12.25$$

for each  $s_1 \in S_1(\{(\text{In}, \text{G})\})$ . ▲

## 4 Conditional beliefs and unimprovability

A prior belief  $\mu_i \in \Delta(\Sigma_{-i})$  over opponents' strategy distributions induces a joint belief  $\pi_{\mu_i} \in \Delta(S_{-i} \times \Sigma_{-i})$  determined by the following equation:

$$\forall (s_{-i}, E_{-i}) \in S_{-i} \times \mathcal{B}(\Sigma_{-i}), \pi_{\mu_i}(\{s_{-i}\} \times E_{-i}) = \int_{E_{-i}} \sigma_{-i}(s_{-i}) \mu_i(d\sigma_{-i}). \quad (3)$$

Note that<sup>20</sup>

$$\pi_{\mu_i}(S_{-i} \times E_{-i}) = \mu_i(E_{-i}), \quad (4)$$

and

$$\pi_{\mu_i}(\{s_{-i}\} \times \Sigma_{-i}) = p_{\mu_i}(s_{-i}).$$

Each information set  $h \in H_i$  corresponds to the conditioning event  $S_{-i}(h) \times \Sigma_{-i}(h)$ . Let

$$H_i(\mu_i) = \{h \in H_i : \pi_{\mu_i}(S_{-i}(h) \times \Sigma_{-i}(h)) > 0\} = \{h \in H_i : p_{\mu_i}(S_{-i}(h)) > 0\}$$

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<sup>20</sup>One may use beliefs over the product space  $S_{-i} \times \Sigma_{-i}$  as the primitive object. Of course, the structural assumption that agents know the probabilities of chance moves and that they are randomly matched with opponents drawn large populations implies that only beliefs that admit the representation (3) for some  $\mu_i \in \Delta(\Sigma_{-i})$  are admissible. Then we can derive the belief over distributions using (4). For a decision theoretic approach akin to the one discussed in this footnote, with models in place of population distributions, see Cerreia-Vioglio et al. (2013b).

denote the subset of information sets of player  $i$  that he believes he can reach with positive probability. If  $h \in H_i(\mu_i)$ , then we can derive the conditional probability of every measurable set  $E_{-i} \in \mathcal{B}(\Sigma_{-i})$  of strategy distributions:

$$\begin{aligned}
\mu_i(E_{-i}|h) &= \pi_{\mu_i}(S_{-i} \times E_{-i}|S_{-i}(h) \times \Sigma_{-i}(h)) \\
&= \frac{\pi_{\mu_i}((S_{-i} \times E_{-i}) \cap (S_{-i}(h) \times \Sigma_{-i}(h)))}{\pi_{\mu_i}(S_{-i}(h) \times \Sigma_{-i}(h))} \\
&= \frac{\pi_{\mu_i}(S_{-i}(h) \times (\Sigma_{-i}(h) \cap E_{-i}))}{\sum_{s_{-i} \in S_{-i}(h)} \int_{\Sigma_{-i}(h)} \sigma_{-i}(s_{-i}) \mu_i(d\sigma_{-i})} \\
&= \frac{\sum_{s_{-i} \in S_{-i}(h)} \int_{\Sigma_{-i}(h) \cap E_{-i}} \sigma_{-i}(s_{-i}) \mu_i(d\sigma_{-i})}{\sum_{s_{-i} \in S_{-i}(h)} p_{\mu_i}(s_{-i})} \\
&= \frac{\int_{E_{-i}} \sigma_{-i}(S_{-i}(h)) \mu_i(d\sigma_{-i})}{p_{\mu_i}(S_{-i}(h))}. \tag{5}
\end{aligned}$$

For example, if  $\mu_i$  has *finite support*:

$$\mu_i(\sigma_{-i}|h) = \frac{\mu_i(\sigma_{-i}) \cdot \sigma_{-i}(S_{-i}(h))}{\sum_{\sigma'_{-i} \in \text{Supp}\mu_i} \mu_i(\sigma'_{-i}) \cdot \sigma'_{-i}(S_{-i}(h))}.$$

With this, we let

$$V_i(a_i|h; s_i, \mu_i, \phi_i) = \phi_i^{-1} \left( \int_{\Sigma_{-i}(h)} \phi_i(U_i((s_i|h, a_i), \sigma_{-i}|h)) \mu_i(d\sigma_{-i}|h) \right) \tag{6}$$

whenever  $h \in H_i(\mu_i)$ , and we derive well defined preferences over actions (given  $\mu_i$  and  $s_i$ ) only for information sets that are possible according to  $\mu_i$ . This is what we need for our baseline definition of unimprovability, which is instead silent about choices at information sets deemed unreachable.

**Definition 1** *A strategy  $s_i$  is  $(\mu_i, \phi_i)$ -unimprovable if*

$$\forall h \in H_i(\mu_i), s_{i,h} \in \arg \max_{a_i \in A_i(h)} V_i(a_i|h; s_i, \mu_i, \phi_i).$$

Since the game is finite, we can interpret unimprovability as “folding-back optimality”: given  $(\mu_i, \phi_i)$ ,  $DM_i$  derives a contingent plan that prescribes a choice for each information set he deems reachable. He starts from information sets  $h \in H_i(\mu_i)$  with no followers in  $H_i(\mu_i)$ , and to each one of them he assigns an action  $s_{i,h}$  that maximizes  $V_i(a_i|h; s_i, \mu_i, \phi_i)$ . Then he folds back, considering the information sets  $h \in H_i(\mu_i)$  such that every follower has no further followers; for every such follower, viz.  $h'$ ,  $DM_i$  predicts that the previously



selected maximizing action  $s_{i,h'}$  will be chosen; and so on until all the reachable information sets in  $H_i(\mu_i)$  have been covered backwards. The folding-back procedure—possibly breaking some ties arbitrarily—computes an unimprovable strategy for every  $(\mu_i, \phi_i)$ .

Of course, we could define beliefs, and thus impose optimality requirements, also at information sets in  $H_i \setminus H(\mu_i)$ . For the time being, we are not interested in doing so: the moves of  $DM_i$  at each  $h \in H_i \setminus H(\mu_i)$  will be immaterial for the equilibrium outcomes (by ex post perfect recall and confirmed beliefs, see Section 7), and impossible to predict for the opponents as long as we do not assume that they know the payoff function of  $i$  (cf. Section 8).

From the point of view of an external observer, or of agents in roles different from  $i$ , it is impossible to distinguish between two strategies of  $DM_i$  that yield the same outcomes independently of the opponents' behavior. This leads to the following notion of equivalence, which will play an important role in comparing self-confirming equilibria for different levels of ambiguity aversion (see Section 6).

**Definition 2** (*Kuhn, 1953*) *Two (possibly degenerate) strategy distributions  $\sigma_i^*$  and  $\sigma_i$  are realization-equivalent if they induce the same distribution on terminal nodes, that is,*

$$\forall (z, s_{-i}) \in Z \times S_{-i}, \quad \sum_{s_i: \zeta(s_i, s_{-i})=z} \sigma_i^*(s_i) = \sum_{s_i: \zeta(s_i, s_{-i})=z} \sigma_i(s_i).$$

*The set of strategy distributions realization-equivalent to  $\sigma_i^*$  is denoted by  $[\sigma_i^*]$ .*

We let  $[s_i^*]$  denote the set of strategies (that is, Dirac distributions) realization-equivalent to  $s_i^*$ .

**Remark 1** *Fix any  $\sigma_i, \sigma_i^* \in \Delta(S_i)$ ;  $\sigma_i \in [\sigma_i^*]$  if and only if  $\sigma_i([s_i^*]) = \sigma_i^*([s_i^*])$  for every  $s_i^* \in S_i$ .*

Let  $H_i(s_i) = \{h \in H_i : s_i \in S_i(h)\}$  denote the subset of information sets of  $DM_i$  that can be reached when  $s_i$  is played. Focusing on pure strategies, we obtain the following observation:

**Remark 2** (*Theorem 1, Kuhn 1953*) *Fix any  $s_i, s_i^* \in S_i$ ;  $s_i \in [s_i^*]$  if and only if  $s_i$  and  $s_i^*$  are behaviorally equivalent, that is, if and only if  $H_i(s_i) = H_i(s_i^*)$  and  $s_{i,h} = s_{i,h}^*$  for each  $h \in H_i(s_i^*)$ .*

Now, suppose that  $DM_i$  is ambiguity neutral:  $\phi_i = \text{Id}_{V_i}$ . Then, by a classical dynamic programming result, unimprovability is (realization) equivalent to “global” (ex ante) subjective EU-maximization.<sup>21</sup>

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<sup>21</sup>All the dynamic programming results of this section can be proved by standard folding-back arguments.

**Proposition 1** For every strategy  $s_i^* \in S_i$  and prior  $\mu_i \in \Delta(\Sigma_{-i})$  the following are equivalent:

- (1)  $[s_i^*]$  contains a  $(\mu_i, \text{Id}_{\mathbb{V}_i})$ -unimprovable strategy,
- (2)  $s_i^* \in \arg \max_{s_i \in S_i} \sum_{s_{-i}} p_{\mu_i}(s_{-i}) u_i(\zeta(s_i, s_{-i}))$ .

We introduce the following strengthening of unimprovability:

**Definition 3** A strategy  $s_i$  is  $(\mu_i, \phi_i)$ -sequentially optimal if

$$\forall h \in H_i(\mu_i), s_i \in \arg \max_{s'_i \in S_i} \phi_i^{-1} \left( \int_{\Sigma_{-i}(h)} \phi_i \left( U_i(s'_i|_h, \sigma_{-i}|_h) \right) \mu_i(d\sigma_{-i}|_h) \right).$$

If  $\text{DM}_i$  has dynamically inconsistent preferences over strategies, a  $(\mu_i, \phi_i)$ -sequentially optimal strategy may not exist, as illustrated in Example 3 below. However, if  $\text{DM}_i$  is ambiguity neutral (hence, his preferences are dynamically consistent), unimprovability coincides with sequential optimality:

**Proposition 2** A strategy  $s_i$  is  $(\mu_i, \text{Id}_{\mathbb{V}_i})$ -unimprovable if and only if it is  $(\mu_i, \text{Id}_{\mathbb{V}_i})$ -sequentially optimal.

**Example 3** Consider the game of Figure 1 and the (marginal) belief of Example 1:

$$\mu_1(\sigma_2) = \begin{cases} \frac{1}{2} & \text{if } \sigma_2 \in \{\delta_L, \delta_R\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$H_1(\mu_1) = H_1 = \{\{\emptyset\}, \{\text{In}, G\}\}.$$

The induced belief on  $S_{-1} \times \Sigma_{-1}$  is

$$\pi_{\mu_1}(s_{-1}, \sigma_{-1}) = \begin{cases} \frac{1}{4} & \text{if } (s_{-1}, \sigma_{-1}) \in \{E, G\} \times \{(L, \bar{\sigma}_0, \delta_L), (R, \bar{\sigma}_0, \delta_R)\}, \\ 0 & \text{otherwise.} \end{cases}$$

For an ambiguity neutral player 1 with belief  $\mu_1$ , the value of  $M$  and  $B$  at  $(\text{In}, G)$  is  $(\frac{1}{2} \cdot 36 + \frac{1}{2} \cdot 1) > 9$ , so it is higher than the value of  $T$ . Therefore, by folding-back, the  $(\mu_1, \text{Id}_{\mathbb{V}_i})$ -unimprovable strategies are  $\text{In}.M$  and  $\text{In}.B$ .

Now suppose instead that the ambiguity attitudes of player 1 are represented by some strictly concave  $\bar{\phi}_1$  with:

$$\begin{aligned} \bar{\phi}_1(u) &= \sqrt{u} \text{ if } 1 \leq u < 36, \\ \bar{\phi}_1\left(\frac{1}{2}\right) &= -1. \end{aligned}$$

At  $(In, G)$ , player 1 still prefers  $M$  (or  $B$ ) over  $T$ , because:

$$\begin{aligned} V_I(M | \{(In, G)\}; s_I, \mu_I; \bar{\phi}_I) &= \bar{\phi}_I^{-1} \left( \frac{1}{2} \cdot \bar{\phi}_I(1) + \frac{1}{2} \cdot \bar{\phi}_I(36) \right) \\ &= (3.5)^2 > 9 = V_I(T | \{(In, G)\}; s_I, \mu_I, \bar{\phi}_I). \end{aligned}$$

Hence, a  $(\mu_I, \bar{\phi}_I)$ -unimprovable strategy must prescribe action  $M$  or  $B$  at  $(In, G)$ . But then, it must also prescribe action  $Out$  at  $\{\emptyset\}$ . Indeed, for every strategy  $s_I$  such that  $s_{I, (In, G)} \in \{M, B\}$  we have:

$$\begin{aligned} V_I(In | \{\emptyset\}; s_I, \mu_I, \bar{\phi}_I) &= \bar{\phi}_I^{-1} \left( \frac{1}{2} \cdot \bar{\phi}_I \left( \frac{1}{2} \cdot 36 \right) + \frac{1}{2} \cdot \bar{\phi}_I \left( \frac{1}{2} \cdot 1 \right) \right) \\ &= \left( \frac{1}{2} \cdot \sqrt{18} - \frac{1}{2} \right)^2 < 4 = V_I(Out | \{\emptyset\}; s_I, \mu_I, \bar{\phi}_I). \end{aligned}$$

So, on the one hand, the only  $(\mu_I, \bar{\phi}_I)$ -unimprovable strategies are  $Out.M$  and  $Out.B$ . On the other hand, from the perspective of the agent at the root of  $\Gamma$ , the value of committing to strategy  $In.T$  is

$$V_I(In | \{\emptyset\}; In.T, \mu_I, \bar{\phi}_I) = \left( \sqrt{\frac{1}{2} \cdot 9} \right)^2 > 4 = V_I(Out | \{\emptyset\}; Out.a_I, \mu_I, \bar{\phi}_I), \quad (7)$$

for all  $a_I \in \{T, M, B\}$ . So, player 1 would commit to  $In.T$  if he only could.  $\blacktriangle$

This example illustrates the well known dynamic inconsistency of preferences of decision makers with non-neutral attitudes towards ambiguity.<sup>22,23</sup> To address this problem, we assume that agents are sufficiently sophisticated to understand the incentives they would face at each information set deemed possible, and plan/predict their contingent behavior by folding back. The resulting  $(\mu_i, \phi_i)$ -unimprovable strategy (or strategies) represents how agents in role  $i$  with belief  $\mu_i$  predict they would choose at future information sets; such strategy and  $\mu_i$  yield a value for each action available at the current information set.

Of course,  $DM_i$  may be indifferent at some information sets. A further refinement can be obtained for ambiguity averse agents by imposing a consistent-planning condition: whenever  $DM_i$  is indifferent at  $h$  then he breaks ties according to the preferences at the immediate predecessor of  $h$  in  $H_i$ . If this does not solve all the indifferences, ties are broken according to the preferences of the twice-removed predecessor, and so on. We omit this refinement for simplicity, and also because we find it arbitrary.

<sup>22</sup>See Siniscalchi (2011), for illustrative examples and an in-depth analysis of this issue.

<sup>23</sup>Note that these dynamic inconsistencies arise as a consequence of the combination of Bayesian updating and non-neutral ambiguity attitudes. Indeed, it is not even obvious from the decision theoretic literature that ambiguity averse players are supposed to update beliefs according to the standard rules of conditional probabilities (see Epstein and Schneider 2007, Hanany and Klibanoff 2009, and Hanany et al. 2017). Instead we take the position that these rules are part of rational cognition, and we stick to them. This position is supported also by works that justify Bayesian updating in an evolutionary perspective, see Blume and Easley (2006) and the references therein.

## 5 Selfconfirming equilibrium

BCMM analyze a notion of smooth self-confirming equilibrium under the assumption that agents play the strategic form of a game with feedback and ambiguity attitudes, as with the strategy method in lab experiments. Specifically, consider a triple  $(\Gamma, f, \phi)$ , where  $\Gamma$  is a standard extensive-form game,  $f = (f_i : Z \rightarrow M)_{i \in I}$  is a profile of feedback functions such that every  $f_i$  describes the message  $m \in M$  that player  $i$  observes ex post as a function of the terminal node, and  $\phi = (\phi_i : \mathbb{V}_i \rightarrow \mathbb{R})_{i \in I}$  is a profile of continuous and strictly increasing functions capturing players attitudes toward ambiguity.

The **structural strategic feedback function** of  $i$  associates each strategy profile  $(s_0, s)$  with a corresponding message, that is,  $F_i = f_i \circ \zeta : S_0 \times S \rightarrow M$ . We let  $\hat{F}_i(s_i, \sigma_{-i}) \in \Delta(M)$  denote the **pushforward** distribution of messages induced by strategy  $s_i$  and the profile of strategy distributions  $\sigma_{-i}$ . Specifically:

$$\forall (s_i, \sigma_{-i}, m) \in S_i \times \Sigma_{-i} \times M, \hat{F}_i(s_i, \sigma_{-i})(m) = \sum_{s_{-i}: F_i(s_i, s_{-i})=m} \sigma_{-i}(s_{-i}).$$

To relate to BCMM it is convenient to define the normal form of a game with feedback  $(\Gamma, f)$ . We first define the strategic form payoff function

$$U_i = u_i \circ \zeta : S_0 \times S \rightarrow \mathbb{R}.$$

With this, the **normal-form** (expected) **payoff function** of player  $i$  is

$$\begin{aligned} \bar{U}_i : S &\rightarrow \mathbb{R}, \\ s &\mapsto \sum_{s_0 \in S_0} \bar{\sigma}_0(s_0) U_i(s_0, s). \end{aligned}$$

Similarly, we define **normal-form feedback function**  $\bar{F}_i : S \rightarrow \bar{M}$  as follows: If strategy profile  $s$  is played in the long run, then  $i$  observes the *distribution* of messages determined by  $s$  and chance probabilities. Therefore, we let  $\bar{M} = \Delta(M)$  and

$$\forall (s, m) \in S \times M, \bar{F}_i(s)(m) = \sum_{s_0: F_i(s_0, s)=m} \bar{\sigma}_0(s_0).$$

With this, the normal form of  $(\Gamma, f)$  is  $\mathcal{N}(\Gamma, f) = (S_i, \bar{U}_i, \bar{F}_i)_{i \in I}$ . The equilibrium concept of BCMM applies to  $(\mathcal{N}(\Gamma, f), \phi) = (S_i, \bar{U}_i, \bar{F}_i, \phi_i)_{i \in I}$  under the assumption that each agent in role  $i$  covertly commits in advance to a strategy  $s_i$ . Here, instead, we analyze an equilibrium concept that is appropriate when agents play  $(\Gamma, f, \phi)$  with the “direct method” making choices as the play unfolds, and we compare it with the strategic-form concept of BCMM.

Given that the information structure of  $\Gamma$  is assumed to satisfy perfect recall, we maintain the assumption that  $(\Gamma, f)$  satisfies “**ex post perfect recall**”:<sup>24</sup>

<sup>24</sup>See Battigalli et al. (2016b). There, the statement of the ex post perfect recall property is slightly different. The two versions are equivalent for extensive-form representations that specify the information of each player  $i$  at each nonterminal node, not only those where  $i$  is active (cf. Battigalli and Bonanno, 1999). Otherwise, one should use the statement of this paper.

**Assumption** (Ex post perfect recall) *For every player  $i \in I$ , the augmented collection of information sets that includes the partition of  $Z$  induced by  $f_i$ ,*

$$\hat{H}_i = H_i \cup \{f_i^{-1}(m) : m \in f_i(Z)\},$$

*satisfies the perfect recall assumption.*

Thus, in particular, for all terminal histories  $z, z' \in Z$ , if there are an information set  $h \in H_i$  and a node  $x \in h$  such that  $x \prec z$  and either  $z'$  has no predecessor in  $h$ , or  $\alpha_i(x, z) \neq \alpha_i(x', z')$  for the predecessor  $x'$  of  $z'$  in  $h$ , then  $f_i(z) \neq f_i(z')$ .

Furthermore, we also consider (but we do not always assume) the following property of feedback:<sup>25</sup>

**Definition 4** *An extensive-form game with feedback  $(\Gamma, f)$  satisfies **observable payoffs** whenever the payoff of every player only depends on his ex post information:*

$$\forall (i, z, z') \in I \times Z^2, f_i(z) = f_i(z') \Rightarrow u_i(z) = u_i(z').$$

In other words, the payoff function is constant on each element of the ex post information partition  $\{f_i^{-1}(m) : m \in f_i(Z)\}$ . We say that  $(\Gamma, f, \phi)$  satisfies observable payoffs if  $(\Gamma, f)$  does.

In some examples, we will assume that agents observe the terminal node they reach, that is,  $f_i = \text{Id}_Z$ . We call this hypothesis **perfect feedback**.

**Definition 5** *A **self-confirming equilibrium** (SCE) of  $(\Gamma, f, \phi)$  is a profile of strategy distributions  $(\bar{\sigma}_i)_{i \in I}$  with the following property: For each  $i \in I$  and  $\bar{s}_i \in \text{Supp} \bar{\sigma}_i$  there is a belief  $\mu_{\bar{s}_i} \in \Delta(\Sigma_{-i})$  such that*

1. (rationality)  $\bar{s}_i$  is  $(\mu_{\bar{s}_i}, \phi_i)$ -unimprovable,
2. (confirmed beliefs)  $\mu_{\bar{s}_i} \left( \left\{ \sigma_{-i} \in \Sigma_{-i} : \hat{F}_i(\bar{s}_i, \sigma_{-i}) = \hat{F}_i(\bar{s}_i, \bar{\sigma}_{-i}) \right\} \right) = 1$ .<sup>26</sup>

*An SCE  $(\bar{\sigma}_i)_{i \in I}$  is a **symmetric SCE** (symSCE) if, for each  $i \in I$ , there is a pure strategy  $\bar{s}_i$  with  $\bar{\sigma}_i(\bar{s}_i) = 1$ , that is, if all agents in the same population  $i$  play the same pure strategy.*

The confirmed beliefs condition requires that the belief  $\mu_{\bar{s}_i}$  justifying  $\bar{s}_i$  exclude all the distributions that are not observationally equivalent to the true one,  $\bar{\sigma}_{-i}$ . When profiles  $(\bar{\sigma}_i)_{i \in I}$  and  $(\mu_{\bar{s}_i})_{i \in I, \bar{s}_i \in \text{Supp} \bar{\sigma}_i}$  satisfy the foregoing SCE conditions, we say that  $(\bar{\sigma}_i)_{i \in I}$  is **justified by confirmed beliefs**  $(\mu_{\bar{s}_i})_{i \in I, \bar{s}_i \in \text{Supp} \bar{\sigma}_i}$ . The set of self-confirming equilibria of

<sup>25</sup>See Battigalli et al. (2016b) for an in-depth analysis of the properties of feedback and how they affect the SCE set.

<sup>26</sup>Recall that  $\bar{\sigma}_{-i}$  includes  $\bar{\sigma}_0$ .

$(\Gamma, f, \phi)$  is denoted by  $SCE(\Gamma, f, \phi)$ , while  $symSCE(\Gamma, f, \phi)$  denotes the set of symmetric SCE.

As in BCMM, the confirmed beliefs condition says that an agent rules out opponents' strategy distributions that are inconsistent with his "empirical distribution" of observations. More specifically, we consider stability conditions for a profile of strategy distributions in a scenario where agents drawn at random from large populations (corresponding to game roles) play the given game recurrently and learn from their personal experience. Suppose each agent keeps playing the same (pure) strategy for a very long time, and consider an agent in role  $i$  who has been playing  $\bar{s}_i$  and accumulated a large dataset of personal observations. With probability 1, and in the limit, this dataset is summarized by the frequency distribution of observations generated by his strategy  $\bar{s}_i$  and by the actual strategy distributions for the opponents' populations, that is,  $\hat{F}_i(\bar{s}_i, \bar{\sigma}_{-i})$ .<sup>27</sup> Every profile of distributions that yields the same distribution of observations is empirically indistinguishable from the true one, and hence it cannot be objectively rejected.

We first observe that every sequential equilibrium of a game  $\Gamma$  corresponds to an SCE of any extension of  $\Gamma$  with feedback and ambiguity attitudes:

**Proposition 3** *For every  $(\Gamma, f, \phi)$ , every sequential equilibrium in behavioral strategies  $(\bar{\beta}_i)_{i \in I}$  of  $\Gamma$  corresponds to a realization-equivalent SCE  $(\bar{\sigma}_i)_{i \in I}$  of  $(\Gamma, f, \phi)$ .*

Intuitively, for every sequential equilibrium  $(\bar{\beta}_i)_{i \in I}$  in behavioral strategies of  $\Gamma$ , we can consider the corresponding realization-equivalent mixed strategy profile  $(\bar{\sigma}_i)_{i \in I}$ . Let  $\mu_{\bar{s}_i} = \delta_{\bar{\sigma}_{-i}}$  for every  $i$  and  $\bar{s}_i \in \text{Supp} \bar{\sigma}_{-i}$ ; since no ambiguity is perceived, agents with these beliefs behave as expected utility maximizers. With this, it can be shown that each  $\bar{s}_i \in \text{Supp} \bar{\sigma}_{-i}$  is  $(\mu_{\bar{s}_i}, \phi_i)$ -unimprovable, because  $(\bar{\sigma}_i)_{i \in I}$  corresponds to a sequential equilibrium of  $\Gamma$ .

Thus, we obtain existence of SCE from well known results of existence for the sequential equilibrium concept:

**Corollary 4** *For every  $(\Gamma, f, \phi)$ , the set  $SCE(\Gamma, f, \phi)$  is not empty. Furthermore, if  $\Gamma$  has perfect information also  $symSCE(\Gamma, f, \phi)$  is not empty.*

The second statement in Corollary 4 follows from Proposition 3 and the well known fact that every finite game with perfect information has a pure strategy subgame perfect equilibrium, which is also a sequential equilibrium.

Observe that our definition does not coincide with the one proposed in BCMM, because the rationality assumption of BCMM is given by the ex ante KMM criterion:

$$\bar{s}_i \in \arg \max_{s_i \in S_i} \int_{\Sigma_{-i}} \phi_i \left( \sum_{s_{-i}} \sigma_{-i}(s_{-i}) U_i(s_i, s_{-i}) \right) \mu_i(d\sigma_{-i}),$$

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<sup>27</sup>See Battigalli et al. (2016c) for a learning foundation of self-confirming equilibrium with non-neutral ambiguity attitudes.

This best reply condition is appropriate only in simultaneous moves games, possibly obtained by having agents play the strategic form of a sequential game (cf. BCMM, pp. 665-667). Here, instead, we require agents to maximize the KMM value over actions at every information set they deem reachable. Therefore, the set of self-confirming equilibria *à la* BCMM of a sequential game  $(\Gamma, f, \phi)$  is  $SCE(\mathcal{N}(\Gamma, f), \phi)$ .

Proposition 1 implies that our definition is (realization) equivalent to the one of BCMM when agents are *ambiguity neutral*: Given  $(\bar{\sigma}_i)_{i \in I} \in SCE(\mathcal{N}(\Gamma, f), (\text{Id}_{V_i})_{i \in I})$  with associated beliefs  $(\mu_{s_i})_{i \in I, s_i \in \text{Supp} \bar{\sigma}_i}$ , replace each  $s_i \in \text{Supp} \bar{\sigma}_i$  with a realization-equivalent  $(\mu_{s_i}, \text{Id}_{V_i})$ -unimprovable strategy  $\hat{s}_i(s_i)$ , where  $\hat{s}_i(\cdot)$  is a suitably defined map (see Proposition 1); then define  $\sigma_i$  as the pushforward of  $\bar{\sigma}_i$  under map  $\hat{s}_i(\cdot)$ , that is,  $\sigma_i = \bar{\sigma}_i \circ \hat{s}_i^{-1}$ ; the resulting profile  $\sigma$  with associated beliefs  $(\mu_{\hat{s}_i(s_i)})_{i \in I, s_i \in \text{Supp} \bar{\sigma}_i}$  satisfies the SCE conditions. To sum up:

**Remark 3** *Suppose that  $\phi_i$  is linear for each  $i \in I$ . Then, for every  $(\bar{\sigma}_i)_{i \in I} \in SCE(\mathcal{N}(\Gamma, f), \phi)$  there is some  $(\sigma_i)_{i \in I} \in SCE(\Gamma, f, \phi)$  such that, for each  $i \in I$ ,  $\bar{\sigma}_i$  and  $\sigma_i$  are realization-equivalent.*

If agents instead are ambiguity averse, the SCE's of a game are not realization-equivalent to SCE's of its strategic-form representation.

**Example 4** *Consider the game of Figure 1. In Example 3 we considered a belief  $\mu_1$  and ambiguity attitudes  $\bar{\phi}_1$  of player 1 such that Out.M and Out.B are  $(\mu_1, \bar{\phi}_1)$ -unimprovable strategies. It follows that (Out.M, L) (for instance) is a symmetric SCE of the game, where Out.M is justified by belief  $\mu_1$  (trivially confirmed for any feedback function) and L is (vacuously) justified by any belief (for any  $\bar{\phi}_2$ ). However, note that inequality (7) implies that (Out.M, L) does not belong to  $SCE(\mathcal{N}(\Gamma, f), \bar{\phi})$ . Specifically, (7) implies that, for every belief  $\mu_1$  and action  $a_1 \in \{T, M, B\}$ , strategy Out. $a_1$  is not ex-ante optimal, and thus it does not satisfy the best reply condition of BCMM.  $\blacktriangle$*

**Comment on knowledge of the game** The definition of SCE relies on very weak interpretive assumptions about agents' knowledge of the game: each agent playing in role  $i$  has to know only his preferences  $(v_i, \phi_i)$ , the extensive-game form (hence, also  $\zeta : S \rightarrow Z$  and  $\gamma : Z \rightarrow C$ ), and his feedback function  $f_i : Z \rightarrow M$ . Therefore, in a SCE an agent in population  $i$  may believe that a positive fraction of agents in population  $j$  are implementing strategies that cannot be justified by any belief, given their true preferences and feedback  $(v_j, \phi_j, f_j)$  (possibly unknown to agents in population  $i$ ). Essentially, SCE is a solution concept for incomplete information games with private values. In Section 8, we analyze a notion of rationalizable SCE that is appropriate when there is common knowledge of  $(\Gamma, f, \phi)$ . This requires a specification of conditional beliefs also at information sets that players deem unreachable and the corresponding strengthening of unimprovability to "full unimprovability" (see Section 7). We note however that this strengthening has bite only

if it is coupled with strategic reasoning based on knowledge of the game (see Proposition 10).

## 6 Monotonicity of selfconfirming equilibrium

In this section we analyze changes in the set of equilibria when feedback and ambiguity attitudes are modified with respect to some baseline  $\bar{f}$  and  $\bar{\phi}$  respectively. Say that the feedback profile  $f$  is **coarser** than  $\bar{f}$  if  $f_i$  is  $\bar{f}_i$ -measurable for each  $i \in I$ .<sup>28</sup> In other words, for each player  $i$ , the partition of  $Z$  induced by  $f_i$  is coarser than the partition of  $Z$  induced by  $\bar{f}_i$ . It is quite straightforward to show that *if  $f$  is coarser than  $\bar{f}$ , then  $SCE(\Gamma, f, \phi) \supseteq SCE(\Gamma, \bar{f}, \phi)$* , because coarser ex post information makes it easier to satisfy the confirmed beliefs condition (see BCMM).

We would also like to prove an extension for sequential games of the following *monotonicity theorem* of BCMM: under observable payoffs, the SCE correspondence is monotone with respect to ambiguity aversion, that is, as ambiguity aversion increases, the set of SCE expands. Formally:

**Definition 6** *We say that  $(\Gamma, f, \phi)$  features **more ambiguity aversion** than  $(\Gamma, f, \bar{\phi})$  if, for each  $i \in I$ ,  $\phi_i = \varphi_i \circ \bar{\phi}_i$  for some concave and strictly increasing function  $\varphi_i$ , and that  $(\Gamma, f, \phi)$  features **ambiguity aversion** if each  $\phi_i$  is concave.*

BCMM proved that if  $(\Gamma, f)$  has observable payoffs and  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$ , then

$$SCE(\mathcal{N}(\Gamma, f), \phi) \supseteq SCE(\mathcal{N}(\Gamma, f), \bar{\phi}).$$

Therefore, if  $(\Gamma, f, \phi)$  features ambiguity aversion, then

$$SCE(\mathcal{N}(\Gamma, f), \phi) \supseteq SCE(\mathcal{N}(\Gamma, f), \text{Id}_{\mathbb{V}}).$$

As already observed, self-confirming equilibria in the strategic and extensive form of the game are not realization equivalent. Hence, the monotonicity result of BCMM cannot be invoked to obtain an equivalent result for SCE in sequential games, not even in terms of induced outcome distributions. Yet, the core of the argument of BCMM can be adapted to sequential games when all the strategies in the support of an SCE with baseline ambiguity attitudes  $\bar{\phi}$  are *sequentially optimal* under the confirmed beliefs that justify them, that is, at every reachable information set  $h$  the prescribed continuation strategy is the one that maximizes the value at  $h$  (see Definition 3).

Recall that  $[\sigma_i]$  (respectively  $[s_i]$ ) is the set of distributions (resp., strategies) realization equivalent to  $\sigma_i$  (resp.  $s_i$ ), and that  $\sigma'_i \in [\sigma_i]$  if and only if  $\sigma'_i([s_i]) = \sigma_i([s_i])$  for every  $s_i$ .<sup>29</sup>

<sup>28</sup>That is,  $f_i = \eta_i \circ \bar{f}_i$  for some  $\eta_i : \bar{f}_i(Z) \rightarrow M$ .

<sup>29</sup>See Definition 2 and Remark 1.



**Lemma 1** *Suppose that  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$  and fix any  $(\bar{\sigma}_i)_{i \in I} \in SCE(\Gamma, f, \bar{\phi})$  justified by confirmed beliefs  $(\mu_{\bar{s}_i})_{i \in I, \bar{s}_i \in \text{Supp}\bar{\sigma}_i}$ . Suppose that for each  $i \in I$ , every  $\bar{s}_i \in \text{Supp}\bar{\sigma}_i$  is  $(\mu_{\bar{s}_i}, \bar{\phi}_i)$ -sequentially optimal. Then, there exists some  $(\sigma_i)_{i \in I} \in SCE(\Gamma, f, \phi)$  such that, for each  $i \in I$ ,  $\sigma_i \in [\bar{\sigma}_i]$  and every  $s_i \in \text{Supp}\sigma_i$  is justified by confirmed belief  $\mu_{\bar{s}_i}$  for some  $\bar{s}_i \in [s_i] \cap \text{Supp}\bar{\sigma}_i$ .*

We provide a sketch of proof of Lemma 1 because it helps to understand how dynamic (in)consistency matters for SCE analysis. Fix  $(\bar{\sigma}_i)_{i \in I} \in SCE(\Gamma, f, \bar{\phi})$  and consider a strategy  $\bar{s}_i \in \text{Supp}\bar{\sigma}_i$  that is sequentially optimal given  $\mu_{\bar{s}_i}$ . Since the confirmed-beliefs condition does not depend on ambiguity attitudes, we only have to argue that some  $s_i \in [\bar{s}_i]$  is  $(\mu_{\bar{s}_i}, \phi_i)$ -unimprovable. Fix any  $h$  consistent with  $\bar{s}_i$ , that is,  $h \in H_i(\bar{s}_i)$ . By ex post perfect recall,  $\mu_{\bar{s}_i}$  assigns probability 1 to the set of distributions  $\sigma_{-i}$  such that  $\sigma_{-i}(S_{-i}(h)) = \bar{\sigma}_{-i}(S_{-i}(h))$ , because  $i$  observes the frequency of  $h$ . There are two cases. (1) If  $\bar{\sigma}_{-i}(S_{-i}(h)) > 0$ , then  $\mu_{\bar{s}_i}(\Sigma_{-i}(h)) > 0$  and conditional belief  $\mu_{\bar{s}_i}(\cdot|h)$  is determined by Bayes rule. Since payoffs are observable, according to conditional belief  $\mu_{\bar{s}_i}(\cdot|h)$  the equilibrium action  $\bar{s}_{i,h}$  is unambiguous (that is, it involves known risks), whereas deviations are untested and can be perceived as ambiguous. Thus, keeping continuation plan and beliefs fixed, an increase in ambiguity aversion from the baseline  $\bar{\phi}_i$  to the more concave  $\phi_i$  decreases the value of deviations without affecting the value of  $\bar{s}_{i,h}$ . Moreover, by sequential optimality, after each deviation the original continuation plan described by  $\bar{s}_i$  is optimal under  $\bar{\phi}_i$  given  $\mu_{\bar{s}_i}(\cdot|h)$ . This implies that any  $(\mu_{\bar{s}_i}, \phi_i)$ -unimprovable continuation plan makes deviations less attractive under  $\bar{\phi}_i$  (if the plan involves further deviations from  $\bar{s}_i$  down the road) and hence less attractive than  $\bar{s}_{i,h}$  under the higher ambiguity aversion represented by  $\phi_i$ . (2) If  $\bar{\sigma}_{-i}(S_{-i}(h)) = 0$ , also  $\mu_{\bar{s}_i}$  assigns probability 0 to  $h$  (that is,  $h \notin H_i(\mu_{\bar{s}_i})$ ) and unimprovability does not impose any optimality requirement on  $\bar{s}_{i,h}$  (see Definition 1). Hence, one can find a  $(\mu_{\bar{s}_i}, \phi_i)$ -unimprovable and realization-equivalent strategy  $s_i \in [\bar{s}_i]$ . The following example illustrates this intuition.

**Example 5** *Let  $\Gamma'$  be the game of Figure 1, but with payoff 5 instead of 4 at outcome Out. The symmetric SCE (Out.M, L) of  $(\Gamma, f, \bar{\phi})$  of Example 4 is also a symmetric SCE of  $(\Gamma', f, \bar{\phi})$  justified by the same confirmed beliefs as in Examples 1 and 4:*

$$\mu_1(\sigma_2) = \begin{cases} \frac{1}{2} & \text{if } \sigma_2 \in \{\delta_L, \delta_R\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Now, Out.M is not only a  $(\mu_1, \bar{\phi}_1)$ -unimprovable strategy, but it is also  $(\mu_1, \bar{\phi}_1)$ -sequentially optimal, because at the root the best alternative strategy In.T (see Example 3) yields an unambiguous expected payoff of  $4.5 < 5$ . Consider now any strictly increasing and concave transformation  $\phi_1 = \varphi_1 \circ \bar{\phi}_1$  such that:*

$$\begin{aligned} \phi_1(u) &= \bar{\phi}_1(u) = \sqrt{u} & \text{if } u \in [0, 9], \\ \bar{\phi}_1(u) &= 6 > \phi_1(36) > \phi_1(9) = 3. \end{aligned}$$

Specifically,  $\phi_1$  (36) is close to 3. In the transformation from  $\bar{\phi}_1$  to  $\phi_1$ , the values of In.M and In.B at the root (given  $\mu_1$ ) decrease, whereas the value of In.T is constant. Then, at the root, In. $a_1^\phi$  is worse than Out, where  $a_1^\phi$  denotes any action that maximizes player 1's  $(\mu_1, \phi_1)$ -value conditional on  $\{(In, G)\}$ . This implies that Out. $a_1^\phi$  is  $(\mu_1, \phi_1)$ -unimprovable and  $(Out.a_1^\phi, L)$  is a symmetric SCE of  $(\Gamma', f, \phi)$  realization equivalent to  $(Out.M, L)$ .  $\blacktriangle$

What can go wrong when the SCE strategies are *not* sequentially optimal under the confirmed beliefs that justify them? Take the viewpoint of an agent in population  $i$  at some information set  $h \in H_i(\mu_i) \cap H(s_i)$ , where  $s_i$  is the agent's strategy in an SCE of  $(\Gamma, \bar{\phi}, f)$  and  $\mu_i$  is the confirmed belief that justifies it. At  $h$ , the agent evaluates a deviation from the equilibrium action  $s_{i,h}$  to an alternative action  $a_i$ , after which he might play once more at an information set  $h' \in H_i(\mu_i)$ . Suppose that, given his belief and some action  $a'_i$  at  $h'$ , the deviation at  $h$  has a higher value than the SCE expected payoff. Yet, since  $s_i$  is not  $(\mu_i, \bar{\phi}_i)$ -sequentially optimal, the agent also realizes that his "future self" at  $h'$  will play action  $s_{i,h'}$  different from  $a'_i$ , and this makes the agent prefer  $s_{i,h}$  to  $a_i$  at  $h$ . But, as his ambiguity aversion increases from  $\bar{\phi}_i$  to  $\phi_i$ , the future self of the agent may switch from  $s_{i,h'}$  to  $a'_i$  at  $h'$  for all the confirmed beliefs that justify  $s_{i,h}$  under  $\bar{\phi}_i$ . Then, although for any fixed belief and action at  $h'$  the value of  $a_i$  compared to  $s_{i,h}$  at  $h$  decreases (because  $a_i$  exposes the agent to ambiguity while  $s_{i,h}$  does not, and the agent has become more ambiguity averse), the value of  $a_i$  at  $h$  under the predicted choice at  $h'$  can increase when moving from  $\bar{\phi}_i$  to  $\phi_i$ . The following example demonstrates this possibility.

**Example 6** Consider the game of Figure 1. Let  $\bar{\phi}_1$  and  $\phi_1$  be the ambiguity attitudes described in Examples 4 and 5. In Appendix 10.2 we show that, if  $\phi$  (36) is sufficiently small, *Out* does not belong to the set of SCE outcomes of  $(\Gamma, \phi, f)$ , although it is an SCE outcome of  $(\Gamma, f, \bar{\phi})$ . The intuition is as follows. For an intermediate level of ambiguity aversion, captured by the baseline second-order utility  $\bar{\phi}_1$ , upon reaching  $(In, G)$  player 1 is tempted by actions *M* and *B* even under the most pessimistic belief  $(\mu_1(\cdot | \{(In, G)\})) = \frac{1}{2}\delta_L + \frac{1}{2}\delta_R$ , cf. Lemma 9). At the root, he anticipates this and, scared by the implied ex-ante objective expected reward of 1/2 under the "bad model" ( $\delta_L$  if he plans *M* and  $\delta_R$  if he plans *B*), he chooses *Out*. For a higher level of ambiguity aversion, captured by  $\phi_1$ , at  $(In, G)$  player 1 is tempted by *M* and *B* only for sufficiently optimistic beliefs. As a consequence, at the root, he is less worried by the "bad model," either because now he plans the unambiguous action *T* for the subgame, or because he plans an ambiguous action and he deems the "bad model" sufficiently unlikely. Therefore, he chooses *In* at the root.  $\blacktriangle$

Thus, the monotonicity result of BCMM does not extend to all sequential games, even if we restrict our attention to distributions of outcomes.<sup>30</sup> Yet, we can use Lemma 1 to show that the monotonicity result holds for classes of games and equilibria of interest.

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<sup>30</sup>This shows that the conjecture informally stated by BCMM (p. 667) is false.

## 6.1 No player moves more than once

We say that **no player moves more than once**<sup>31</sup> in  $\Gamma$  if, for every  $i \in I$  and  $z \in Z$ , there is at most one information set  $h \in H_i$  that contains a predecessor of  $z$ . In this class of games, at any information set  $h \in H_i$ , the agent does not move again after  $h$ ; thus, no information set of  $i$  is prevented by any strategy of  $i$ . In this case the value of an action at  $h \in H_i$  does not depend on  $i$ 's strategy, unimprovability coincides with sequential optimality,<sup>32</sup> strategies (or strategy distributions) are realization-equivalent if and only if they coincide, hence Lemma 1 implies the following result:

**Corollary 5** *Fix two games with observable payoffs where no player moves more than once,  $(\Gamma, f, \phi)$  and  $(\Gamma, f, \bar{\phi})$ , so that  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$ . Then*

$$SCE(\Gamma, f, \phi) \supseteq SCE(\Gamma, f, \bar{\phi}).$$

## 6.2 Symmetric self-confirming equilibria

Consider now a *symmetric* SCE  $\bar{s}$  in a game with observable payoffs and *without chance moves*. We can show that there is a symmetric SCE equilibrium  $s$  with the same outcome.<sup>33</sup>

**Theorem 6** *Fix two games with observable payoffs, ambiguity aversion, and without chance moves,  $(\Gamma, f, \phi)$  and  $(\Gamma, f, \bar{\phi})$ , so that  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$ . Then:*

$$\zeta(\text{symSCE}(\Gamma, f, \phi)) \supseteq \zeta(\text{symSCE}(\Gamma, f, \bar{\phi})).$$

We prove Theorem 6 in the Appendix, here we provide an intuitive argument. The SCE strategies may not be sequentially optimal under the confirmed beliefs that justify them. However, consider alternative beliefs that are supported by Dirac models and give the same predictive beliefs as the original ones. By construction, these beliefs are confirmed by the equilibrium play, and they feature two additional properties. First, they are the most pessimistic beliefs among those that give rise to the same predictive probabilities; thus, by certainty of the equilibrium payoff, they justify a realization equivalent symmetric SCE. This is shown by Lemma 9 in the Appendix and it is based on the following intuition: For any belief  $\mu_i$ , the corresponding belief supported by Dirac models  $\hat{\mu}_i$  with the same predictive as  $\mu_i$  is a “mean-preserving spread” of  $\mu_i$ , and the concavity of  $\phi_i$  implies that player  $i$  feels worse-off with such spread. Second, absent chance moves, beliefs supported by Dirac models cannot entail dynamic inconsistencies of preferences over strategies; thus,

<sup>31</sup>Perfect information games with this property have been called “simple” (see, e.g., Fudenberg and Levine, 2006). We prefer the more explicative phrase because (i) we are not assuming perfect information and (ii) “simple” is vague and had been used earlier with a different meaning (see Battigalli, 2003).

<sup>32</sup>Indeed, Proposition 2 can be extended to any  $\phi_i$  in one-move games; yet, Proposition 1 cannot!

<sup>33</sup>Note that we can identify  $\text{symSCE}(\Gamma, f_i, \bar{\phi})$  with a subset of  $S$ . Hence, it makes sense to write  $\zeta(\text{symSCE}(\Gamma, f_i, \bar{\phi}))$ .

the symmetric SCE's they justify features sequentially optimal strategies. This is shown by Lemma 8 in the Appendix. The main idea behind this result is that dynamic inconsistency is due to the perception of hedging opportunities that may be optimal ex-ante, but would not be implemented ex-post after (partial) resolution of the uncertainty. But, absent chance moves, an agent whose belief is supported by Dirac models does not perceive such hedging opportunities. With this, we can use Lemma 1 to prove the monotonicity result for equilibrium outcomes.

Absence of chance moves and symmetry of the equilibrium are tight conditions. In Example 6, outcome Out is induced by a symSCE of the game with chance moves  $(\Gamma, f, \bar{\phi})$ , but it is not induced by any symSCE of the game  $(\Gamma, f, \phi)$ , which features more ambiguity aversion. As for the role of symmetry (pure equilibrium), consider the following example.

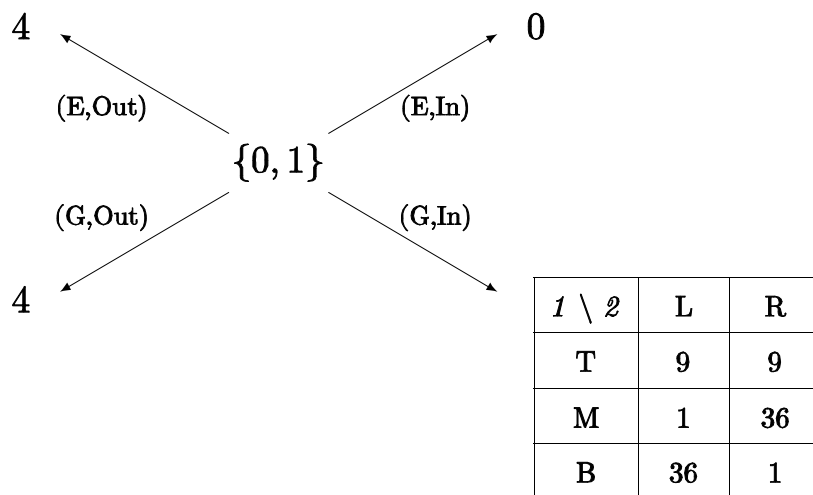


Figure 2: A 3-person common interest game:  $I = \{0, 1, 2\}$

**Example 7** Let  $\Gamma'$  be the following modification of the game of Figure 1: deviating from our standard notation, here 0 is not chance but rather an actual player, choosing between E and G simultaneously with player 1 at the root. Assuming common interests, all players obtain payoff 4 at both outcomes (E, Out) and (G, Out). See Figure 6.2. Assume perfect feedback and consider the same  $\bar{\phi}_1$  and  $\phi_1$  of Example 6. The SCE  $(\sigma_0, \text{Out.M, L})$  of  $(\Gamma', f, \bar{\phi})$ , where  $\sigma_0(\text{E}) = \sigma_0(\text{G}) = \frac{1}{2}$ , yields outcomes (E, Out) and (G, Out) with probability 1/2 (cf. Example 4). By perfect feedback and confirmed beliefs, in any realization-equivalent SCE player 1 is certain of  $\sigma_0$ . Yet, for the same argument as for Example 6, no  $(\mu_1, \phi_1)$ -unimprovable strategy of player 1 prescribes action Out when the marginal of  $\mu_1$  on  $\Sigma_0$  is  $\delta_{\sigma_0}$ . Thus, no SCE of  $(\Gamma', f, \phi)$  yields outcomes (E, Out) and (G, Out) with probability 1/2.  $\blacktriangle$

### 6.3 Ambiguity aversion versus ambiguity neutrality

Fix a sequential game with feedback  $(\Gamma, f)$  and let  $\text{Id}_{\mathbb{V}} = ((\text{Id}_{\mathbb{V}_i})_{i \in I})$  denote the profile of players' identity functions characterizing their neutrality toward ambiguity. We relate the set  $SCE(\Gamma, f, \text{Id}_{\mathbb{V}})$  of SCE's of  $\Gamma$  given feedback functions  $f$  and neutral ambiguity attitudes with the set  $SCE(\Gamma, f, \phi)$  with non-neutral ambiguity attitudes  $\phi$ . We start with a preliminary observation:<sup>34</sup>

**Remark 4** *For every two-person game with feedback  $(\Gamma, f)$  and every profile  $\phi$  of second-order utility functions,  $SCE(\Gamma, f, \text{Id}_{\mathbb{V}}) \subseteq SCE(\Gamma, f, \phi)$ .*

To see this, note that ambiguity neutrality and the convexity of  $\Sigma_{-i} = \Delta(S_{-i})$  in two-person games allow to replace the justified beliefs supporting  $(\bar{\sigma}_i)_{i \in I}$  as an SCE of  $(\Gamma, f, \text{Id}_{\mathbb{V}})$  with the corresponding Dirac beliefs supported by their predictive measure. Since ambiguity attitudes are immaterial for agents with Dirac beliefs,  $(\bar{\sigma}_i)_{i \in I}$  is also an SCE of  $(\Gamma, f, \phi)$ . This argument does not hold with  $n > 2$  players, because in this case  $\Sigma_{-i}$  is not convex, hence, the Dirac measure supported by the predictive of  $\mu_{\bar{\sigma}_i}$  may belong to  $\Delta(\Delta(S_{-i})) \setminus \Delta(\Sigma_{-i})$ . Nonetheless, we can relate  $SCE(\Gamma, f, \text{Id}_{\mathbb{V}})$  and  $SCE(\Gamma, f, \phi)$  for a large class of games.

As a corollary of their main monotonicity result, BCMM show that, under observable payoffs,  $SCE(\mathcal{N}(\Gamma, f), \text{Id}_{\mathbb{V}})$  is contained in  $SCE(\mathcal{N}(\Gamma, f), \phi)$  if each  $\phi_i$  is concave and there are observable payoffs. Even if the main monotonicity result of BCMM does not extend to SCE of sequential games for the entire spectrum of ambiguity attitudes, we are still able to obtain a sequential version of this corollary in terms of induced outcome distributions. By dynamic consistency under ambiguity neutrality (i.e., by Proposition 2), the result about the comparison between ambiguity neutrality and ambiguity aversion is a corollary of Lemma 1.

**Corollary 7** *Suppose that  $(\Gamma, f)$  has observable payoffs and  $(\Gamma, f, \phi)$  features ambiguity aversion. Then, the set of SCE distributions over terminal nodes of  $(\Gamma, f, \phi)$  contains the set of SCE distributions over terminal nodes with ambiguity neutrality.*

Given the large body of empirical evidence supporting the ambiguity aversion hypothesis, we conclude that the standard SCE concept, which implicitly assumes neutral ambiguity attitudes, overestimates the predictability of long-run outcomes of learning dynamics.

Finally, note that Corollary 5 and Corollary 7, via Lemma 1, imply that all beliefs that justify  $SCE(\Gamma, f, \bar{\phi})$  in one-move games or under ambiguity neutrality (linear  $\bar{\phi}$ ) also justify the corresponding equilibria under  $\phi$ . This is not true for symmetric equilibria in games without chance moves (see the proof of Theorem 6).

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<sup>34</sup>Cf. footnote 23 of BCMM.

## 7 Conditional probability systems and full unimprovability

So far we studied how ambiguity aversion and the ensuing possibility of dynamic inconsistency (the incentive to covertly commit, if possible) affect SCE analysis. However, we neglected strategic reasoning based on common knowledge of (some features of) the game. To analyze strategic reasoning in sequential games we have to address an additional and more traditional issue: We need to model how a player thinks that the other agents would react to unexpected moves. Even if players are ambiguity neutral, the analysis of Sections 4-6 is insufficient to address this issue: As a preliminary step, we need to assume that agents have well defined conditional beliefs at all information sets, including the unexpected ones. The conditional beliefs of any given agent at different information sets have to be mutually consistent if we want to preserve/extend the unimprovability principle. Moreover, we want to model players who reason strategically about the game before playing it. For this purpose, we have to consider also their beliefs at the root, even if they are not first movers. Therefore, for every  $i \in I$ , we will consider the expanded collection of information sets  $\bar{H}_i = H_i \cup \{\{\emptyset\}\}$ .

To understand the following definition, consider beliefs  $\mu_i(\cdot|g), \mu_i(\cdot|h) \in \Delta(\Sigma_{-i})$  at two information sets  $g$  and  $h$ , so that  $g$  precedes  $h$  ( $g \prec h$ ), and suppose that  $h$  is possible according to  $\mu_i(\cdot|g)$ , that is,  $p_{\mu_i}(S_{-i}(h)|g) > 0$ . Then, the conditional belief  $\mu_i(\cdot|h)$  can be derived from  $\mu_i(\cdot|g)$  in the way prescribed by (5): for all  $E_{-i} \in \mathcal{B}(\Sigma_{-i})$ ,

$$\mu_i(E_{-i}|h) = \frac{\int_{E_{-i} \cap \Sigma_{-i}(g)} \sigma_{-i}(S_{-i}(h)|g) \mu_i(d\sigma_{-i}|g)}{p_{\mu_i}(S_{-i}(h)|g)}.$$

Next note that we can write the required relation between  $\mu_i(\cdot|g)$  and  $\mu_i(\cdot|h)$  without explicitly stating condition  $p_{\mu_i}(S_{-i}(h)|g) > 0$ : for all  $g, h \in \bar{H}_i$  with  $g \prec h$ , and  $E_{-i} \in \mathcal{B}(\Sigma_{-i})$

$$\mu_i(E_{-i}|h) p_{\mu_i}(S_{-i}(h)|g) = \int_{E_{-i} \cap \Sigma_{-i}(g)} \sigma_{-i}(S_{-i}(h)|g) \mu_i(d\sigma_{-i}|g). \quad (8)$$

**Definition 7** A *conditional probability system (CPS)* on  $(\Sigma_{-i}, \bar{H}_i)$  is an array of probability measures

$$\mu_i(\cdot|\cdot) = (\mu_i(\cdot|h))_{h \in \bar{H}_i} \in [\Delta(\Sigma_{-i})]^{\bar{H}_i}$$

such that

- (1) for all  $h \in \bar{H}_i$ ,  $\mu_i(\Sigma_{-i}(h)|h) = 1$ ,
- (2) for all  $g, h \in \bar{H}_i$  with  $g \prec h$  eq. (8) holds.

**Definition 8** A *CPS* on  $(S_{-i}, \bar{H}_i)$  is an array of probability measures

$$p_i(\cdot|\cdot) = (p_i(\cdot|h))_{h \in \bar{H}_i} \in [\Delta(S_{-i})]^{\bar{H}_i}$$

such that  $(p_i(\cdot|h))_{h \in \bar{H}_i} = (p_{\mu_i(\cdot|h)})_{h \in \bar{H}_i}$  for some CPS  $\mu_i(\cdot|\cdot) = (\mu_i(\cdot|h))_{h \in \bar{H}_i}$  on  $(\Sigma_{-i}, \bar{H}_i)$ .

In Definition 8, we define a CPS on  $(S_{-i}, \bar{H}_i)$  indirectly as the “predictive” of some CPS on  $(\Sigma_{-i}, \bar{H}_i)$ . We could have given a direct definition: Indeed,  $p_i \in [\Delta(S_{-i})]^{\bar{H}_i}$  is a CPS on  $(S_{-i}, \bar{H}_i)$  if and only if (1) for every  $h \in \bar{H}_i$ ,  $p_i(S_{-i}(h)|h) = 1$  and (2) for all  $g, h \in \bar{H}_i$  with  $g \prec h$  and all  $s_{-i} \in S_{-i}(h)$ ,

$$p_i(s_{-i}|h)p_i(S_{-i}(h)|g) = p_i(s_{-i}|g). \quad (9)$$

**Notation 8** The set of CPS’s on  $(\Sigma_{-i}, \bar{H}_i)$   $[(S_{-i}, \bar{H}_i)]$  is denoted by  $\Delta^{\bar{H}_i}(\Sigma_{-i})$   $[\Delta^{\bar{H}_i}(S_{-i})]$ .

With this, we can give a stronger definition of unimprovability:

**Definition 9** A strategy  $s_i$  is **fully**  $(\mu_i(\cdot|\cdot), \phi_i)$ -**unimprovable** (where  $\mu_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$ ) if

$$\forall h \in H_i, s_{i,h} \in \arg \max_{a_i \in A_i(h)} V_i(a_i|h; s_i, \mu_i, \phi_i).$$

It is well known that, for ambiguity neutral agents, we have a refined dynamic programming result:<sup>35</sup>

**Proposition 9** For every  $s_i^* \in S_i$  and  $\mu_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$ , the following are true:  
 (1\*)  $[s_i^*]$  contains a fully  $(\mu_i(\cdot|\cdot), \text{Id}_{V_i})$ -unimprovable strategy if and only if

$$\forall h \in \bar{H}_i(s_i^*), s_i^* \in \arg \max_{s_i \in S_i(h)} \sum_{s_{-i}} U_i(s_i, s_{-i}) \cdot p_{\mu_i}(s_{-i}|h);$$

(2\*)  $s_i^*$  is a fully  $(\mu_i(\cdot|\cdot), \text{Id}_{V_i})$ -unimprovable strategy if and only if

$$\forall h \in \bar{H}_i, (s_{i|h}^*, s_{i,h}^*) \in \arg \max_{s_i \in S_i(h)} \sum_{s_{-i}} U_i(s_i, s_{-i}) \cdot p_{\mu_i}(s_{-i}|h).$$

Full-improvability solves the difficulties described above. Intuitively, we can interpret a  $(\mu_i(\cdot|\cdot), \phi_i)$ -fully unimprovable strategy  $s_i^*$  as the **plan** of an agent in role  $i$ , which can be obtained with a “folding back” dynamic programming procedure on the subjective decision tree implied by CPS  $\mu_i(\cdot|\cdot)$ . In our perspective, the fact that the beliefs of an agent with perfect recall are given by a CPS reflects the **epistemic unity of the agent’s self**: the agent always incorporates new information into his system of knowledge and beliefs in a way that is consistent with his previous beliefs and with the rules of conditional probabilities, even when new information follows previously unexpected information.

Before we move on to model strategic reasoning, we verify that even if SCE is strengthened by requiring full unimprovability, the set of possible outcomes does not change.

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<sup>35</sup> Definition 9 requires maximization over actions; hence, it considers only the information sets where  $i$  is active. The following propositions relate to maximization over strategies; hence, they also include (ex ante) maximization at  $\{\emptyset\}$  even if  $i$  is not a first mover. Also, recall that, according to our notation,  $(s_{i|h}^*, s_{i,h}^*)$  is the minimal modification of  $s_i^*$  that makes  $h$  reachable and plays like  $s_i^*$  at all information sets  $h'$  that do not strictly precede  $h$ .

**Definition 10** A *fully unimprovable self-confirming equilibrium* of  $(\Gamma, f, \phi)$  is a profile of strategy distributions  $(\sigma_i^*)_{i \in I}$  with following property: For every  $i \in I$  and  $s_i^* \in \text{Supp}\sigma_i^*$  there is a CPS  $\mu_{s_i^*}(\cdot|\cdot) \in \Delta^{\hat{H}_i}(\Sigma_{-i})$  such that  
 (rationality)  $s_i^*$  is fully  $(\mu_{s_i^*}(\cdot|\cdot), \phi_i)$ -unimprovable,  
 (confirmed beliefs)  $\mu_{s_i^*} \left( \left\{ \sigma_{-i} \in \Sigma_{-i} : \hat{F}_i(s_i^*, \sigma_{-i}) = \hat{F}_i(s_i^*, \sigma_{-i}^*) \right\} \mid \{\emptyset\} \right) = 1$ .

Note that the confirmed beliefs condition refers only to the initial beliefs  $\mu_{s_i^*}(\cdot|\{\emptyset\})$  because it implies that conditional beliefs are confirmed by observed conditional frequencies at every history that is reached with positive probability in equilibrium (see the proof of Lemma 4). By inspection of Definitions 5 and 10 it is also clear that every fully unimprovable SCE is also an SCE. The following proposition implies that SCE and fully unimprovable SCE are realization equivalent.

**Proposition 10** For every SCE there is a corresponding fully unimprovable SCE that yields the same probability distribution on terminal nodes.

Intuitively, since SCE does not rely on complete information and does not model strategic thinking, requiring rational reactions to unexpected moves adds little to the analysis: Agents are not assumed to know the preferences of others, hence they are not assumed to rule out irrational reactions to deviations.

## 8 Knowledge of the game and strategic reasoning

What if  $(\Gamma, f, \phi)$  (or a part of it) is common knowledge? Then it makes sense to explore a notion of “rationalizable SCE” according to which agents reason strategically about their opponents taking into account their preferences and feedback functions (cf. Rubinstein and Wolinsky 1994, Battigalli 1999, Dekel *et al.* 1999, Esponda 2013, Fudenberg and Kamada 2015). We illustrate this with two simple examples.



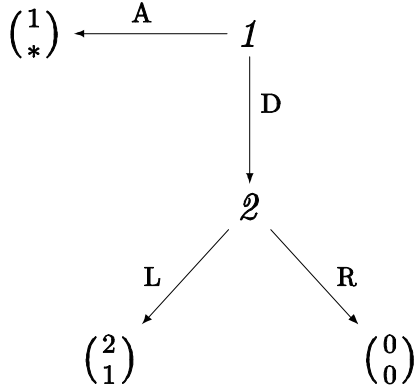


Figure 3: Entry game

**Example 8** (Cf. Fudenberg and Levine, 1993). Figure 8 depicts the so called “Entry game” often used to illustrate the shortcomings of the Nash equilibrium concept. To complete the specification of  $(\Gamma, f, \phi)$  assume observable payoffs (hence, in this case, perfect feedback) and let  $\phi$  be an arbitrary pair of strictly increasing functions.<sup>36</sup> It can be checked that the set of SCE’s of  $(\Gamma, f, \phi)$  is

$$\{(\sigma_1^*, \sigma_2^*) \in \Delta(S_1) \times \Delta(S_2) : \sigma_1^*(A) = 1, \text{ or } (\sigma_1^*(A) < 1, \sigma_2^*(L) = 1)\}.$$

First-movers who choose A (go Across) get no feedback and can thus hold trivially confirmed beliefs that make them play A. As for second-movers, their plan becomes relevant only if a positive fraction of first-movers choose D (go Down). In that case, the only rational choice is L (Left). In equilibrium, first-movers going Down correctly predict Left, which makes Down a strict best response. How can a first mover expect that the second mover goes Right with high probability (i.e., that a large fraction of second movers play Right)? Informally, this is possible if either the first mover gives a high probability to the second mover being irrational, or—more reasonably—if the first mover does not know  $v_2$ . If instead the first mover knows  $v_2$  and believes in the rationality of the second mover, then he predicts that Down would be followed by Left, and he would go Down. Intuitively, only one SCE is consistent with belief in rationality and knowledge of the game: (D,L). However, this is not formally captured by the confirmed beliefs condition of Definition 10. According to such definition, if an agent in population 1 does not play Down, he can keep the belief that the second mover would play Right after Down: If  $v_2$  is unknown to first movers, they have no way to understand that only Left is rational. Relatedly, the set of equilibrium distributions on terminal nodes is unchanged if we adopt the stronger definition of unimprovability: The latter implies  $\sigma_2^*(L) = 1$ , but the first movers who go Across because they have wrong beliefs

<sup>36</sup>One can show that ambiguity aversion does not matter when players have only two strategies.

have no way to find out they are wrong; hence, the equilibrium distributions on terminal nodes are the same.  $\blacktriangle$

**Example 9** (Rubinstein and Wolinsky 1994, Battigalli 1999). Two players must simultaneously choose a location among the equally spaced points 0, 1, 2, 3 on the real line. Their payoff is the negative of the distance between their chosen locations and each player, that is,  $U_i(s_1, s_2) = -|s_1 - s_2|$ ; furthermore, each player observes this distance and, of course, remembers his action:  $F_i(s_1, s_2) = (s_i, |s_1 - s_2|)$ . Again, let us fix an arbitrary concave  $\phi$  and—for simplicity—let us focus on the symmetric equilibria. The set of symmetric SCE's is

$$\text{symSCE} = \{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}.$$

Of course, (1, 2) and (2, 1) are not Nash equilibria. They are symmetric SCE's supported by confirmed beliefs according to which each agent assigns the same probability to the co-player being either on his right, or on his left. However, if the game is common knowledge, an agent with such beliefs infers that with probability .5 his co-player is “cornered” at an extreme point and cannot be best responding to confirmed beliefs. This is unlikely to be a stable situation under complete information if players reason strategically, because no player has reasons to believe that the co-player would keep playing in the same way.  $\blacktriangle$

**Symmetrically rationalizable SCE** We capture with an inductive definition the behavioral consequences of the following assumptions on rationality and interactive beliefs: (1) agents are rational (in the sense of full unimprovability) and have confirmed beliefs, and (2) there is common belief at the beginning of the game that (1) holds. This is easier to do in a particular case, i.e., when all agents in the same population follow the same plan (symmetric SCE) and there is common full belief of this.<sup>37</sup> To simplify the analysis we focus on games *without chance moves*. Then, the CPS  $\mu_i(\cdot|\cdot)$  on  $(\Sigma_{-i}, \bar{H}_i)$  is supported by Dirac models, and therefore it is isomorphic to a CPS  $p_{\mu_i}$  on  $(S_{-i}, \bar{H}_i)$ . A variation of the algorithm defined by Battigalli (1999) (see also Esponda 2013) precisely captures the foregoing epistemic assumptions.

Given a CPS  $\mu_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$ , we let

$$r_i(\mu_i(\cdot|\cdot), \phi_i) = \left\{ s_i \in S_i : \forall h \in H_i, s_{i,h} \in \arg \max_{a_i \in A_i(h)} V_i(a_i|h; s_i, \mu_i(\cdot|\cdot), \phi_i) \right\}$$

denote the set of fully  $(\mu_i, \phi_i)$ -unimprovable strategies of  $i$ . When  $\mu_i(\cdot|\cdot)$  is isomorphic to a predictive CPS  $p_i(\cdot|\cdot)$ , as in the case we are considering right now, it makes sense to write  $r_i(p_i(\cdot|\cdot), \phi_i)$ .

<sup>37</sup>Note, we did not say “belief at the beginning of the game,” because now we mean “probability 1 belief conditional on every information set.” This is called “full belief” in epistemic game theory (e.g., Battigalli et al. 2017).

**Definition 11** For each  $i \in I$ , let  $B_i^0 = S_i \times M$ , and

$$B_i^{k+1} = \left\{ (\bar{s}_i, \bar{m}_i) \in B_i^k : \begin{array}{l} \exists p_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(S_{-i}), \bar{s}_i \in r_i(p_i(\cdot|\cdot), \phi_i), \\ p_i(F_{\bar{s}_i}^{-1}(\bar{m}_i) \cap \{s_{-i} : (s_j, F_j(\bar{s}_i, s_{-i}))_{j \neq i} \in B_{-i}^k\} | \{\emptyset\}) = 1 \end{array} \right\}$$

for each  $k \in \mathbb{N}_0$ . A strategy profile  $\bar{s}$  is a **symmetrically rationalizable SCE** for game  $(\Gamma, f, \phi)$  (without chance moves) if  $(\bar{s}_i, F_i(\bar{s}))_{i \in I} \in \times_{i \in I} \bigcap_{k \in \mathbb{N}} B_i^k$ . The set of symmetrically rationalizable self-confirming equilibria of  $(\Gamma, f, \phi)$  is denoted by  $\text{symRSCE}(\Gamma, f, \phi)$ .

Intuitively,  $(\bar{s}_i, \bar{m}_i) \in B_i^1$  if  $\bar{s}_i$  is justified by some CPS such that  $i$  is initially certain to get message  $\bar{m}_i$  if he plays  $\bar{s}_i$ . Thus,  $(\bar{s}_i, F_i(\bar{s}))_{i \in I} \in \times_{i \in I} \bigcap_{k \in \mathbb{N}} B_i^1$  if  $\bar{s}$  is an unimprovable SCE (with beliefs supported by Dirac models). Then,  $(\bar{s}_i, \bar{m}_i) \in B_i^2$  if  $\bar{s}_i$  is justified by some CPS such that  $i$  is initially certain to get message  $\bar{m}_i$  if he plays  $\bar{s}_i$ , and furthermore he is initially certain that everybody else is playing strategies justified by confirmed beliefs. The iterations capture higher levels of (initial) mutual belief in rationality and in confirmation of beliefs.

**Remark 5** A strategy profile  $\bar{s}$  is a **symmetrically rationalizable SCE** for  $(\Gamma, f, \phi)$  if and only if there is a profile of finite subsets  $(\bar{B}_i)_{i \in I} \in \times_{i \in I} 2^{S_i \times M}$  such that, for every  $i \in I$ , there is  $\bar{m}_i \in M$  with  $(\bar{s}_i, \bar{m}_i) \in \bar{B}_i$ , and for every  $(\hat{s}_i, \hat{m}_i) \in \bar{B}_i$ , there is  $p_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(S_{-i})$  with  $\hat{s}_i \in r_i(p_i(\cdot|\cdot), \phi_i)$  and

$$p_i(F_{\hat{s}_i}^{-1}(\hat{m}_i) \cap \{s_{-i} : (s_j, F_j(\hat{s}_i, s_{-i}))_{j \neq i} \in \bar{B}_{-i}\} | \{\emptyset\}) = 1. \quad (10)$$

The Remark above shows that our inductive definition is an extensive-form version with ambiguity attitudes of Rubinstein's and Wolinsky's (1994) rationalizable conjectural equilibrium. Next we offer a characterization with sets of strategy profiles. Let  $\text{symSCE}^0(\Gamma, f, \phi) = S$  and

$$\text{symSCE}^{k+1}(\Gamma, f, \phi) = \left\{ \bar{s} \in \text{symSCE}^k(\Gamma, f, \phi) : \begin{array}{l} \forall i \in I, \exists p_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(S_{-i}), \bar{s}_i \in r_i(p_i(\cdot|\cdot), \phi_i), \\ p_i(F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i} | \{\emptyset\}) = 1 \end{array} \right\},$$

where, for any subset  $X \subseteq S$  and strategy  $\bar{s}_i \in S_i$ ,

$$X_{\bar{s}_i} = \{s_{-i} \in S_{-i} : (\bar{s}_i, s_{-i}) \in X\}$$

is the section of  $X$  at  $\bar{s}_i$  (thus,  $\text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i}$  is the section of  $\text{symSCE}^k(\Gamma, f, \phi)$  at  $\bar{s}_i$ ). Note that  $\text{symSCE}^1(\Gamma, f, \phi)$  coincides with the set of fully unimprovable symmetric SCE's of  $(\Gamma, f, \phi)$  justified by confirmed beliefs supported by Dirac models. Thus,  $\bar{s} \in \text{symSCE}^2(\Gamma, f, \phi)$  if each  $\bar{s}_i$  is a best reply to a confirmed belief that assigns probability

1 to other players choosing best replies to confirmed beliefs supported by Dirac models, given  $\bar{s}_i$  and message  $m_i = F_i(\bar{s})$ .

The following result shows that symmetric RSCE is characterized by the iterated deletion of strategy profiles  $(\text{symSCE}^k(\Gamma, f, \phi))_{k \in \mathbb{N}}$ :

**Lemma 2** *For every  $k \in \mathbb{N}$ , and  $\bar{s} \in S$ ,  $\bar{s} \in \text{symSCE}^k(\Gamma, f, \phi)$  if and only if  $(\bar{s}_i, F_i(\bar{s}))_{i \in I} \in B^k$ . Therefore,*

$$\bigcap_k \text{symSCE}^k(\Gamma, f, \phi) = \text{symRSCE}(\Gamma, f, \phi).$$

**Example 10** *One can easily check that the only rationalizable SCE for the game of Figure 3 is (D,L), independently of  $\phi$ : Only L is fully unimprovable for player 2, thus  $\text{symSCE}^1(\Gamma, f, \phi) = \{(A, L), (D, L)\}$  and  $\text{symSCE}^1(\Gamma, f, \phi)_A = \{L\}$ . Therefore,*

$$\text{symSCE}^2(\Gamma, f, \phi) = \{(D, L)\}.$$

▲

**Example 11** *Now consider the rationalizable SCE's set for the distance game of Example 9:*

$$\text{symSCE}^1(\Gamma, f, \phi) = \text{symSCE}(\Gamma, f, \phi) = \{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}.$$

Let  $\bar{s} = (1, 2)$ . We show that  $\bar{s} \notin \text{symSCE}^2(\Gamma, f, \phi)$ . Consider player 1. If we had  $\bar{s} \in \text{symSCE}^2(\Gamma, f, \phi)$ , then  $\bar{s}_1 = 1$  would be a best reply to a belief  $p_1$  such that  $p_1(\{0, 2\} \cap \text{symSCE}^1(\Gamma, f, \phi)_{\bar{s}_2} | \{\emptyset\}) = 1$ . But  $\text{symSCE}^1(\Gamma, f, \phi)_{\bar{s}_2} = \{1, 2\}$ . Thus

$$\{0, 2\} \cap \text{symSCE}^1(\Gamma, f, \phi)_{\bar{s}_2} = \{2\},$$

and the best reply to  $s_2 = 2$  is  $s_1 = 2$ . A similar argument shows that  $(2, 1) \notin \text{symSCE}^1(\Gamma, f, \phi)$ . Hence only the pure Nash equilibria are  $\text{symRSCE}$ . ▲

We can prove for the  $\text{symRSCE}$  correspondence two monotonicity results analogous to those obtained for the  $\text{symSCE}$  correspondence (cf. Corollary 5 adapted to  $\text{symSCE}$  and Theorem 6):

**Theorem 11** *Fix two games with observable payoffs and no chance moves where no player moves more than once,  $(\Gamma, f, \phi)$  and  $(\Gamma, f, \bar{\phi})$ , so that  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$ . Then,*

$$\text{symRSCE}(\Gamma, f, \bar{\phi}) \subseteq \text{symRSCE}(\Gamma, f, \phi).$$

**Theorem 12** Fix two games with ambiguity aversion, observable payoffs, and no chance moves,  $(\Gamma, f, \phi)$  and  $(\Gamma, f, \bar{\phi})$ , so that  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$ . Then, for every  $\bar{s} \in \text{symRSCE}(\Gamma, f, \bar{\phi})$  there is some  $s \in \text{symRSCE}(\Gamma, f, \phi)$  such that, for each player  $i$ ,  $s_i$  is realization equivalent to  $\bar{s}_i$ ; therefore,

$$\zeta(\text{symRSCE}(\Gamma, f, \bar{\phi})) \subseteq \zeta(\text{symRSCE}(\Gamma, f, \phi)).$$

Intuitively, these results rely on the following intermediate step:<sup>38</sup>

**Monotonicity of the justifiability correspondence:** For every information set  $h \in H_i$  and compact subset  $\Sigma_{i|h} \subseteq \Delta(S_{-i}(h))$ , if an action  $a_i$  is justified as a best reply to some conditional belief  $\bar{p}_{i,h} \in \Delta(\Sigma_{i|h})$  given baseline ambiguity attitudes  $\bar{\phi}_i$ , then there is some  $p_{i,h} \in \Delta(\Sigma_{i|h})$  that justifies  $a_i$  as a best reply given the more ambiguity averse attitudes  $\phi_i$ . This holds, in particular, when  $\Sigma_{i|h} = \{\delta_{s_{-i}} : s_{-i} \in S_{-i}(h)\}$  is the set of Dirac distributions in  $\Delta(S_{-i}(h))$ .

With this, we can show that if a strategy is fully  $(\bar{p}_i, \bar{\phi}_i)$ -unimprovable, then it is also fully  $(p_i, \phi_i)$ -unimprovable for some suitably chosen belief system  $p_i$  that coincides with  $\bar{p}_i$  on the equilibrium path (by the usual argument that equilibrium actions are unambiguous, hence deviations become less appealing), and can be chosen off the equilibrium path invoking monotonicity of the justifiability correspondence.

**Rationalizable SCE** We now define rationalizable SCE for population games, that is, we consider the general non-symmetric version of rationalizable SCE. In doing so, we reintroduce chance moves. By analogy with Lemma 2, we perform an iterated deletion of distributions of strategy profiles.

To ease notation, let

$$\hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) = \left\{ \sigma_{-i} \in \Sigma_{-i} : \hat{F}_i(s_i, \sigma_{-i}) = \hat{F}_i(s_i, \bar{\sigma}_{-i}) \right\}$$

denote the partially identified set of co-players strategy distributions observationally equivalent for  $i$  to  $\bar{\sigma}_{-i}$  given  $s_i$ . Moreover, let

$$\begin{aligned} \psi_{-i} : \quad & \times_{j \in I_0 \setminus \{i\}} \Delta(S_j) & \rightarrow & \cong(S_{-i}), \\ & (\sigma_j)_{j \in I_0 \setminus \{i\}} & \mapsto & \times_{j \in I_0 \setminus \{i\}} \sigma_j. \end{aligned}$$

denote the homeomorphism that associates each profile of opponents' distributions (including  $\bar{\sigma}_0$ ) with the corresponding product measure.

With this, let  $\text{SCE}^0(\Gamma, f, \phi) = \times_{i \in I} \Delta(S_i)$ , and for every  $k \in \mathbb{N}_0$ ,<sup>39</sup>

$$\text{SCE}^{k+1}(\Gamma, f, \phi) =$$

<sup>38</sup>See Lemma 10 and its corollary in the Appendix; cf. Battigalli et al. (2016a) and Weinstein (2016).

<sup>39</sup>Note the following slight abuse of notation. If for some  $k \in \mathbb{N}_0$ ,  $s_i \in S_i$ , and  $\bar{\sigma}_{-i} \in \Sigma_{-i}$  the set  $\Upsilon = \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) \cap \psi_{-i}(\text{proj}_{\Sigma_{-i}}(\{\bar{\sigma}_0\} \times \text{SCE}^k(\Gamma, f, \phi)))$  is not measurable, then the requirement  $\mu_{s_i}(\Upsilon|\{\emptyset\}) = 1$  means that there is a measurable set  $\Upsilon' \subset \Upsilon$  with  $\mu_{s_i}(\Upsilon'|\{\emptyset\}) = 1$ .

$$\left\{ (\bar{\sigma}_i)_{i \in I} \in SCE^k(\Gamma, f, \phi) : \begin{array}{l} \forall i \in I, \forall s_i \in \text{Supp} \bar{\sigma}_i, \exists \mu_{s_i}(\cdot | \cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i}), s_i \in r_i(\mu_{s_i}(\cdot | \cdot), \phi_i), \\ \mu_{s_i}(\hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) \cap \psi_{-i}(\text{proj}_{\Sigma_{-i}}(\{\bar{\sigma}_0\} \times SCE^k(\Gamma, f, \phi))) | \{\emptyset\}) = 1 \end{array} \right\}.$$

Note that  $SCE^1(\Gamma, f, \phi)$  is the set of fully unimprovable SCE's of  $(\Gamma, f, \phi)$ .

**Definition 12** A profile of strategy distributions  $(\bar{\sigma}_i)_{i \in I}$  is a rationalizable self-confirming equilibrium for  $(\Gamma, f, \phi)$  if

$$(\bar{\sigma}_i)_{i \in I} \in \bigcap_{k \in \mathbb{N}} SCE^k(\Gamma, f, \phi).$$

The set of rationalizable self-confirming equilibria of  $(\Gamma, f, \phi)$  is denoted by  $RSCE(\Gamma, f, \phi)$ .

The most important difference with respect to the definition of rationalizable symmetric self-confirming equilibrium is best understood by looking at the second step, where agents check if it is possible that others are best responding to confirmed beliefs. If there is common belief of symmetry, an agent playing strategy  $s_i$  is certain that all the other agents in population  $i$  also use  $s_i$ , and this is taken into account when he checks whether the agents in co-players' populations are best responding to confirmed beliefs. If instead it is understood that different agents in population  $i$  may use different strategies, an agent playing  $s_i$  may think that only a negligible fraction of other agents in  $i$  is playing  $s_i$ , therefore, the fact that he is playing  $s_i$  does not enter this calculation. Essentially, we are assuming that each agent has a belief about the whole profile of distributions, that is, a belief over  $\Sigma = \times_{j \in I} \Delta(S_j)$ . For the purpose of computing best replies, only the marginal over  $\Sigma_{-i}$  matters. But in order to check whether everybody is best replying to confirmed beliefs, the distribution of strategies in population  $i$  is crucial (see Fudenberg and Kamada, 2015, 2017).

Say that  $\Gamma$  (hence also  $(\Gamma, f, \phi)$ ) has **observable deviators** if the factorization

$$S_{I_0}(h) = \times_{j \in I_0} S_j(h)$$

holds for each information set  $h$  of each player. Intuitively, this means that if an information set is reached unexpectedly the active player is able to understand who deviated from the expected path (cf. Fudenberg and Levine 1993, and Battigalli and Guaitoli 1988). We can prove for the RSCE correspondence a monotonicity result analogous to the one obtained for the SCE correspondence (cf. Corollary 5):

**Theorem 13** Fix two games with observable payoffs and observable deviators where no player moves more than once,  $(\Gamma, f, \phi)$  and  $(\Gamma, f, \bar{\phi})$ , so that  $(\Gamma, f, \bar{\phi})$  features more ambiguity aversion than  $(\Gamma, f, \phi)$ . Then  $RSCE(\Gamma, f, \bar{\phi}) \subseteq RSCE(\Gamma, f, \phi)$ .

We assume observable deviators because in the proof we use a version of the previous result about monotonicity of the justifiability correspondence where  $S_{-i}(h)$  is replaced by the (compact) space of product distributions  $\{\bar{\sigma}_0\} \times (\times_{j \in I \setminus \{i\}} \Delta(S_j(h)))$ , which makes sense if  $S_{-i}(h)$  is a product set.

## 9 Discussion

We discuss the relevance of some assumptions, some extensions, and possible modifications of solution concepts.

### 9.1 Pure equilibrium and absence of chance moves

We extended the monotonicity result of BCMM and proved monotonicity of the (rationalizable) SCE correspondence with respect to ambiguity aversion in games where players move at most once on any path, and in games without chance moves if attention is restricted to symmetric, i.e. “pure,” equilibria. Here we discuss the relevance of symmetric (rationalizable) SCE in games without chance moves.

We justified our assumption that players do not randomize arguing that if randomizations were feasible they could be represented in the extensive form by means of chance moves. Thus, when we exclude chance moves we implicitly assume that players cannot randomize at all. We point out that, in our analysis of symmetric rationalizable SCE, this assumption is made for simplicity and that it is crucial for Theorem 12 only.

A methodological reason to be interested in pure equilibria is that we may want to analyze explicitly the population game as a grand game with many players who are randomly matched.<sup>40</sup> Assuming that individual agents do not randomize (see the previous comment), we should consider the pure equilibria of such grand game. While this argument justifies the focus on pure equilibria, it also implies that we should allow for chance moves.

We focus on pure equilibria of games without chance moves because this improves our understanding of the interaction between non-monotonicity of the equilibrium correspondence and dynamic inconsistency of preferences. We do not claim that they are plausible, or interesting, in all applications.

### 9.2 Heterogeneous populations

Our analysis can be easily extended to the case of populations with heterogeneous personal traits affecting preferences, or feedback. Let  $\Theta_i$  be a parameter space of possible personal traits of agents in population  $i$ . Parameter  $\theta_i \in \Theta_i$  may affect tastes and risk attitudes ( $v_{\theta_i}$ ), ambiguity attitudes ( $\phi_{\theta_i}$ ), and feedback ( $f_{\theta_i}$ ). Each agent’s personal traits are privately known to him. Assume for simplicity that  $\Theta_i$  is finite and let  $\tau_i \in \Delta(\Theta_i)$  denote the exogenous distribution of personal traits in population  $i$ . For any  $j \in I$  and  $\theta_j \in \Theta_j$  we let  $\sigma_{\theta_j}$  denote the strategy distribution in the subpopulation of agents with trait  $\theta_j$ . For any profile of distributions  $(\sigma_{\theta_j})_{\theta_j \in \Theta_j} \in \Delta(S_j)^{\Theta_j}$  and any  $s_j \in S_j$  the aggregate fraction of agents playing  $s_j$  is  $\sum_{\theta_j \in \Theta_j} \tau_j(\theta_j) \sigma_{\theta_j}(s_j)$ . With this, we can extend SCE and the refinements analyzed in this paper by taking into account that unimprovability and belief confirmation depend on personal traits (cf. Ch. 7 in Battigalli 2017, and Dekel et al. 2004):

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<sup>40</sup>That is, if each population  $i$  has  $n$  agents, the grand game has  $n \times |I|$  players.

A profile of distributions  $\left((\sigma_{\theta_i})_{\theta_i \in \Theta_i}\right)_{i \in I}$  is a selfconfirming equilibrium if there is a profile of beliefs  $\left((\mu_{\theta_i, s_i})_{\theta_i \in \Theta_i, s_i \in \text{Supp}\sigma_{\theta_i}}\right)_{i \in I}$  such that, for all  $i \in I$ ,  $\theta_i \in \Theta_i$ , and  $s_i \in \text{Supp}\sigma_{\theta_i}$

1.  $s_i$  is unimprovable given  $\mu_{\theta_i, s_i}$ ,  $v_{\theta_i}$ , and  $\phi_{\theta_i}$ ,
2.  $\mu_{\theta_i, s_i} \left( \left\{ \sigma'_{-i} \in \Sigma_{-i} : \hat{F}_{\theta_i}(s_i, \sigma'_{-i}) = \hat{F}_{\theta_i} \left( s_i, \times_{j \neq i} \left( \sum_{\theta_j \in \Theta_j} \tau_j(\theta_j) \sigma_{\theta_j} \right) \right) \right\} \right) = 1$ .

The definition of rationalizable SCE can be extended in a similar way, taking into account what is assumed to be commonly known about the exogenous distributions of traits. Our results about nonsymmetric SCE can be seamlessly extended to such notion of (rationalizable) SCE. As for the results about symmetric (rationalizable) SCE in games without chance moves, they only apply to the case in which *all* agents of the same population use the same strategy despite differences in personal traits. Therefore, these results are less relevant.

### 9.3 Incomplete information

A related extension concerns the case where some features of game are not known or commonly known. It is straightforward to relax the assumption that the probabilities of chance moves are (commonly) known: Chance can be analyzed as an indifferent player;<sup>41</sup> then, one just applies the definitions we already have for games without chance moves, letting the chance player distribution be the one obtained from the objective probabilities of chance moves. Of course, there cannot be symmetric equilibria of such game, because the strategy distribution of the chance player is strictly positive. More generally, we can parametrize all the features of the game that are not commonly known with a parameter profile  $\theta = (\theta_j)_{j \in I_0} \in \Theta$ , where  $\theta_0$  parameterizes chance probabilities and other non personal aspects of the game such as some aspects of the consequence function, and each  $\theta_i$  ( $i \in I$ ) parameterizes privately known personal features. Then we can give definitions of SCE at a given  $\theta$ , and of rationalizable SCE given that it is common knowledge that  $\theta \in \Theta$  (cf. Ch. 7 in Battigalli 2017, Battigalli and Guaitoli 1988, and Esponda 2013).

### 9.4 Rationalizable selfconfirming equilibrium

We put forward a relatively simple definition of rationalizable SCE that can be formally derived from the epistemic assumptions of (a) subjective rationality, (b) belief confirmation, and (c) initial common belief of (a) and (b). These epistemic assumptions can be formally stated by means of extensive-form type structures (see, e.g., Battigalli et al. 2017). The derivation of symmetric rationalizable SCE from assumptions (a)-(c) is quite straightforward for the symmetric case, which is relevant for repeated games played by impatient

<sup>41</sup>More precisely, chance can be analyzed as an indifferent player with no feedback. Ex post perfect recall applies only to real players.



players. The derivation from (a)-(c) of nonsymmetric rationalizable SCE in population games is more complex. We can think of one that goes through approximations *via* large finite populations. We provide below a critical discussion of rationalizable SCE based on our knowledge of dynamic epistemic game theory and our intuitions about learning dynamics. We emphasize that this discussion is not specifically related to the issue of ambiguity aversion.

While plain SCE can be justified as the set of rest points of learning dynamics (e.g., Battigalli et al. 1992, Battigalli et al. 2016c and references therein), we are not aware of any learning foundation for rationalizable SCE. As we think about one, the very definition of rationalizable SCE appears questionable. Let us consider the simplest possible scenario: A fixed set  $I$  of completely *impatient* players plays  $\Gamma$  infinitely many times, with imperfect monitoring about past periods outcomes (terminal nodes) given by the profile of feedback functions  $f = (f_i)_{i \in I}$ ; if players have nonneutral attitudes toward ambiguity, they are represented by the profile of second-order utility functions  $\phi = (\phi_i)_{i \in I}$ ;  $(\Gamma, f, \phi)$  is common knowledge. Suppose that players are rational and there is *common belief in rationality at the beginning of the first period*. Also suppose for simplicity that players' initial beliefs about first-period strategies assign strictly positive probability to opponents' first-period strategies consistent with rationality and initial common belief in rationality, called “**initially rationalizable**” strategies (Battigalli et al. 2017).<sup>42</sup> Then players never observe zero-probability events and have well defined updated beliefs that always satisfy one-period rationality and common initial belief in rationality; hence, they play initially rationalizable strategies of  $(\Gamma, f, \phi)$  in every period. If one-period strategies and updated beliefs converge, then the limit is a pure SCE in initially rationalizable strategies of  $(\Gamma, f, \phi)$ . This is a weaker equilibrium concept than rationalizable SCE, because it requires common initial belief in rationality, but it does not require common initial belief in confirmation. This (somewhat informal) argument shows that the conjunction of SCE and initial rationalizability is a necessary condition of steady states of learning dynamics that satisfy the foregoing epistemic assumptions. The arguments offered by Rubinstein and Wolinsky (1994) suggest, however, that this condition may not be sufficient for stability (cf. Examples 9 and 11). We conjecture that rationalizable SCE is a sufficient condition for stability, but it is not clear to us why it should also be necessary.

Be as it may, why should we assume players hold strictly positive beliefs over the set of initially rationalizable strategies? After all, the epistemic assumptions that justify initial rationalizability do not require any kind of strict positivity of beliefs. If we drop this assumption about the support of initial beliefs, we open the door to surprises as the learning dynamic unfolds and we have no reason to suppose that surprised players keep believing in the strategic sophistication of the opponents, or even their subjective rationality, unless we strengthen the assumptions about players' belief in the strategic sophistication

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<sup>42</sup>Initially rationalizable strategies can be obtained with an iterated elimination algorithm similar to the one used to define symSCE, but without the confirmed-belief requirement.

of opponents. Indeed, the very definition of initial rationalizability allows surprised players to assign positive updated probabilities to never-best-replies of the opponents. Thus, we suggest that a more interesting approach is to assume “common strong belief” in rationality: players always assign probability 1 to the “highest level of strategic sophistication” of the opponents consistent with what they observe. Within a one-period play, these epistemic assumptions yield a solution concept called “**strong rationalizability**” (Battigalli et al. 2017). Suppose that these epistemic assumptions hold and the learning dynamic converges; then the limit must be a pure SCE in strongly rationalizable strategies, which is—essentially—the concept first put forward by Battigalli (1987) and Battigalli and Guaitoli (1988). Thus, SCE in strongly rationalizable strategies is a necessary condition for stability under the foregoing epistemic assumptions. Again, arguments *à la* Rubinstein and Wolinsky (1994) suggest that it may not be sufficient. We can give a definition of “**strongly rationalizable SCE**” in which “initial belief” is replaced by “strong belief.” We conjecture that strongly rationalizable SCE provides sufficient conditions for stability, but it is not clear to us that these conditions are also necessary.

Of course, the assumption that players are impatient is just a simplification. If a player is somewhat patient and plays a repeated game, he may sacrifice short-run expected payoffs to experiment, or to teach opponents to play in future periods in ways he finds advantageous to him. This is why the literature on learning and convergence to equilibrium in games often focuses on population games: if populations are large, the incentive to teach is shut down, and the incentive to experiment disappears in the long run if updated beliefs converge. The interaction of epistemic assumptions and learning dynamics in recurrent play within a population game scenario is hard to analyze, and learning foundations for (nonsymmetric) rationalizable SCE are therefore harder to provide. An alternative route is to put forward a definition of rationalizable SCE for repeated games, similar to the definition of “subjective equilibrium” of Kalai and Lehrer (1993a,b, 1995), but with the additional requirement of common belief in rationality. Arguments similar to those offered above suggest that an equilibrium concept based on common strong belief in rationality and—possibly—in belief confirmation could capture interesting necessary and/or sufficient conditions for the stability of beliefs (about repeated games strategies) under learning.

To sum up, it is not surprising that more than two decades after Rubinstein and Wolinsky (1994) put forward the rationalizable SCE concept a learning foundation is still lacking. We suggest that, as one seriously tries to provide such a foundation, different versions of the rationalizable SCE will concepts turn out to be relevant.

## 10 Appendix

To ease notation, throughout the appendix we let  $\sigma$  denote a profile of distributions  $(\sigma_i)_{i \in I}$ , whereas  $\sigma_{-i}$  will still denote a product distribution in  $\Sigma_{-i}$  (thus including  $\bar{\sigma}_0$ ) like in the main text.

## 10.1 Proof of Proposition 3

Fix any sequential equilibrium in behavioral strategies  $\bar{\beta} = (\bar{\beta}_i)_{i \in I_0}$  of game  $\Gamma$  (chance has been included for notational convenience). For each  $i \in I_0$ , let  $\bar{\sigma}_i$  denote the mixed strategy associated with  $\bar{\beta}_i$  according to Kuhn's (1953) transformation:

$$\forall s_i \in S_i, \bar{\sigma}_i(s_i) = \prod_{h \in H_i} \bar{\beta}_i(s_{i,h}|h).$$

Let  $\mu_{\bar{s}_i} = \delta_{\bar{\sigma}_i}$  for every  $i$ . By construction, these beliefs are correct, hence confirmed. To show that  $\bar{\sigma} = (\bar{\sigma}_j)_{j \in I}$  is an SCE justified by these confirmed beliefs, we must prove that, for each  $i \in I$ , each  $\bar{s}_i \in \text{Supp} \bar{\sigma}_i$  is  $(\mu_{\bar{s}_i}, \phi_i)$ -unimprovable.

It is well known that every pure strategy in the support of a sequential equilibrium is a sequential best reply to the equilibrium beliefs. Therefore, for every  $i \in I$ ,  $h \in H_i$  and  $\bar{s}_i$  such that  $\prod_{h' \in H_i} \bar{\beta}_i(\bar{s}_{i,h'}|h') > 0$ , that is, for every  $\bar{s}_i \in \text{Supp} \bar{\sigma}_i$ ,

$$\bar{s}_{i,h} \in \arg \max_{a_i \in A_i(h)} \sum_{x \in h} \mathbb{P}_{(\bar{s}_{i,h}, a_i), \bar{\beta}_{-i}}(x|h) \sum_{z \in Z} \mathbb{P}_{(\bar{s}_{i,h}, a_i), \bar{\beta}_{-i}}(z|x) u_i(z),$$

where  $\mathbb{P}_{s_i, \bar{\beta}_{-i}}(\cdot|\cdot)$  denotes the probability of a node conditional on an information set, or an earlier node, determined by the behavioral strategy profile  $(s_i, \bar{\beta}_{-i})$ . Since  $\bar{\sigma}_{-i}$  is by construction realization-equivalent to  $\bar{\beta}_{-i}$ ,  $\mathbb{P}_{s_i, \bar{\beta}_{-i}}(\cdot|\cdot) = \mathbb{P}_{s_i, \bar{\sigma}_{-i}}(\cdot|\cdot)$ ; hence,

$$\forall s_i \in S_i(h), U_i(s_i, \bar{\sigma}_{-i}|h) = \sum_{x \in h} \mathbb{P}_{s_i, \bar{\beta}_{-i}}(x|h) \sum_{z \in Z} \mathbb{P}_{s_i, \bar{\beta}_{-i}}(z|x) u_i(z).$$

Since  $\mu_{\bar{s}_i} = \delta_{\bar{\sigma}_{-i}}$ ,

$$\forall a_i \in A_i(h), V_i(a_i|h; \bar{s}_i, \mu_{\bar{s}_i}, \phi_i) = \phi_i^{-1}(\phi_i(U_i((\bar{s}_{i,h}, a_i), \bar{\sigma}_{-i}|h))) = U_i((\bar{s}_{i,h}, a_i), \bar{\sigma}_{-i}|h).$$

Therefore  $\bar{s}_i$  is  $(\mu_{\bar{s}_i}, \phi_i)$ -unimprovable. ■

## 10.2 Example 6

We prove that in Example 6, *there exists a concave and strictly increasing transformation  $\phi_1 = \varphi_1 \circ \bar{\phi}_1$  such that for every belief  $\mu_1$ , Out is not prescribed by any  $(\phi_1, \mu_1)$ -unimprovable strategy of player 1.* As anticipated in Example 5, we look for a  $\phi_1$  such that

$$\begin{aligned} \phi_1(u) &= \bar{\phi}_1(u) = \sqrt{u} \quad \text{if } u \in [0, 9], \\ \bar{\phi}_1(36) &= 6 > k = \phi_1(36) > \phi_1(9) = 3, \end{aligned}$$

with  $k = \phi_1(36)$  close to 3.

For every  $\sigma_2$  in  $\Delta(A_2(\{\text{In}, \text{G}\})) = \Delta(\{\text{L}, \text{R}\})$ , let

$$e_M(\sigma_2) = 1 \cdot \sigma_2(\text{L}) + 36 \cdot \sigma_2(\text{R})$$

denote the objective expected payoff of M in the subgame. For every probability measure  $\nu$  on  $\Delta(\{L, R\})$ , define  $G_\nu : \Delta(\{L, R\}) \rightarrow [0, 1]$  as

$$G_\nu(\sigma_2) = \nu(\{\sigma'_2 \in \Delta(\{L, R\}) : e_M(\sigma'_2) \leq e_M(\sigma_2)\}).$$

Player 1 prefers M to T only if

$$V_1(M | \{(In, G)\}; s_1, \mu_1, \phi_1) \geq 9 = V_1(T | \{(In, G)\}; s_1, \mu_1, \phi_1).$$

Let  $\nu = \mu_1(\cdot | \{(In, G)\})$ . Then, for every  $\sigma_2 \in \Delta(\{L, R\})$ ,<sup>43</sup>

$$\phi_1^{-1}(G_\nu(\sigma_2) \cdot \phi_1(e_M(\sigma_2)) + (1 - G_\nu(\sigma_2)) \cdot \phi_1(36)) \geq V_1(M | \{(In, G)\}; s_1, \mu_1, \phi_1),$$

that is,

$$G_\nu(\sigma_2) \cdot \phi_1(e_M(\sigma_2)) + (1 - G_\nu(\sigma_2)) \cdot \phi_1(36) \geq \phi_1(V_1(M | \{(In, G)\}; s_1, \mu_1, \phi_1)).$$

Therefore, a necessary condition for 1 to choose M over T is that, for every  $\sigma_2 \in \Delta(\{L, R\})$ ,

$$G_\nu(\sigma_2) \cdot \phi_1(e_M(\sigma_2)) + (1 - G_\nu(\sigma_2)) \cdot k \geq 3.$$

Solving for  $\hat{\sigma}_2$  such that  $e_M(\hat{\sigma}_2) = 17/2$ , we get

$$G_\nu(\hat{\sigma}_2) \leq \frac{k - 3}{k - \sqrt{17/2}}. \quad (11)$$

The interpretation is that the probability assigned to the models under which action M has an (objective) value lower or equal than  $17/2$  must be sufficiently small if player 1 is to choose M at (In, G).

Now, we return to the choice of player 1 at the root of the game. Note that

$$\mu_1 = \mu_1(\cdot | \{\emptyset\}) = \mu_1(\cdot | \{(In, G)\})$$

because the belief about the co-player cannot change upon observing one's own move, or a chance move. The DM is sophisticated, therefore, before choosing between In and Out, he predicts his behavior at (In, G). There are three cases:

1. Player 1 understands that his subjective belief  $\mu_1$  makes him play T at (In, G). Then, since In.T is preferred to Out for every belief, he will play In at  $\{\emptyset\}$ .

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<sup>43</sup>Given the belief  $\nu$  and a model  $\sigma_2$ ,  $G_\nu(\sigma_2)$  is the probability assigned by  $\nu$  to the models that, paired with action M, yield an (objective) expected utility lower or equal to the one obtained under model  $\sigma_2$ . Therefore, the value of action M under  $\nu$  cannot exceed the value of M under the following belief: the models that, paired with action M, yield an (objective) expected utility equal to the one obtained under model  $\sigma_2$  have probability  $G_\nu(\sigma_2)$ , whereas  $\delta_R$  has probability  $(1 - G_\nu(\sigma_2))$ .

2. Player 1 understands that his subjective belief  $\mu_1$  makes him play M at (In, G). But then, it must be the case that (11) holds. Therefore, the probability assigned to the models that (given In.G) yield an objective expected utility of choosing action M larger than or equal to  $17/2$  is at least  $1 - \frac{k-3}{k-\sqrt{17/2}}$ . In turn, this implies that the probability assigned to the models that, ex-ante, yields an objective expected utility of strategy In.M larger than or equal to  $17/4$  is at least  $1 - \frac{k-3}{k-\sqrt{17/2}}$ . But then, the evaluation of strategy In.M under belief  $\mu_1$  satisfies the following conditions:<sup>44</sup>

$$\begin{aligned} \phi_1(V_1(\text{In}|\{\emptyset\}; \text{In.M}, \mu_1, \phi_1)) &\geq \phi_1\left(\frac{1}{2}\right) \cdot \frac{k-3}{k-\sqrt{17/2}} + \phi_1\left(\frac{17}{4}\right) \cdot \left(1 - \frac{k-3}{k-\sqrt{17/2}}\right) \\ &= \frac{\sqrt{17}}{2} - \frac{k-3}{k-\sqrt{17/2}} \cdot \left(\frac{\sqrt{17}}{2} + 1\right). \end{aligned}$$

For  $k = \phi_1$  (36) sufficiently close to 3,  $\phi_1(V_1(\text{In}|\{\emptyset\}; \text{In.M}, \mu_1, \phi_1))$  is higher than 2, thus player 1 chooses In over Out.

3. Player 1 understands that his subjective belief  $\mu_1$  makes him play B at (In, G). A similar argument as for the previous case shows that for  $k$  sufficiently close to 3 also in this case player 1 chooses In over Out.

Summing up, there is a concave and strictly increasing transformation  $\phi_1 = \varphi_1 \circ \bar{\phi}_1$  such that Out is not prescribed by any  $(\mu_1, \phi_1)$ -unimprovable strategy for all beliefs  $\mu_1$ .

### 10.3 Monotonicity of the SCE correspondence

Define the set of messages for  $i$  consistent with an information set  $h \in H_i$  as follows:

$$M_i(h) = \{m : \exists (x, z) \in h \times Z, (x \prec z) \wedge (m = f_i(z))\}.$$

**Lemma 3** *Ex post perfect recall implies that*

$$S_{-i}(h) = \bigcup_{m \in M_i(h)} F_{i, s_i}^{-1}(m)$$

for all  $i \in I$ ,  $h \in H_i$ , and  $s_i \in S_i(h)$ .

<sup>44</sup>The first inequality is due to the following fact. The probability assigned by  $\mu_1$  to the models that, ex-ante, given strategy In.M, yield an (objective) expected utility larger or equal than  $\frac{17}{4}$  must be larger or equal than  $1 - \frac{k-3}{k-\sqrt{17/2}}$ . Therefore, the value of action In given strategy In.M under  $\mu_1$  cannot be lower than its value under the following belief: the set of models that, given strategy In.M, yield an (objective) expected utility equal to  $\frac{17}{4}$  has subjective probability  $1 - \frac{k-3}{k-\sqrt{17/2}}$ , whereas  $\delta_L$  has subjective probability  $\frac{k-3}{k-\sqrt{17/2}}$ .

In words,  $S_{-i}(h)$  is the union of the sets of preimages of messages consistent with  $h$ , because these messages “record” that  $h$  has been reached.

**Proof.** Fix  $i \in I$ ,  $h \in H_i$ , and  $s_i \in S_i(h)$  arbitrarily. First recall that perfect recall implies

$$S_{I_0}(h) = S_i(h) \times S_{-i}(h).$$

We first prove that

$$S_{-i}(h) \subseteq \bigcup_{m \in M_i(h)} F_{i,s_i}^{-1}(m).$$

Fix any  $s_{-i} \in S_{-i}(h)$ ; since  $s_i \in S_i(h)$  and  $S_{I_0}(h) = S_i(h) \times S_{-i}(h)$ , then  $(s_i, s_{-i}) \in S_{I_0}(h)$ , that is,  $x \prec \zeta(s_i, s_{-i})$  for some  $x \in h$ . Thus, by definition of  $M_i(h)$ ,  $f_i(\zeta(s_i, s_{-i})) \in M_i(h)$ . Hence,  $s_{-i} \in F_{i,s_i}^{-1}(m)$  for some  $m \in M_i(h)$ .

Next we prove by contraposition that the converse

$$S_{-i}(h) \supseteq \bigcup_{m \in M_i(h)} F_{i,s_i}^{-1}(m)$$

is implied by *ex post* perfect recall. Suppose that we can find some  $m \in M_i(h)$  and  $s'_{-i} \in F_{i,s_i}^{-1}(m) \setminus S_{-i}(h)$ . We show that this implies a violation of *ex post* perfect recall. Since  $m \in M_i(h)$ , there is a pair  $(x, z) \in h \times Z$  such that  $x \prec z$  and  $f_i(z) = m$ . Fix any  $s_{-i} \in \text{proj}_{S_{-i}} \zeta^{-1}(z)$ , so that  $(s_i, s_{-i}) \in S_i(h) \times S_{-i}(h) = S_{I_0}(h)$  for some  $s_i \in S_i(h)$ . Let  $z = \zeta(s_i, s_{-i})$  and  $z' = \zeta(s_i, s'_{-i})$ . Then, by choice of  $s_{-i}$  and  $s'_{-i}$ ,  $f_i(z) = m = f_i(z')$ ,  $z$  is preceded by a node of  $h$ , and  $z'$  is not preceded by any node of  $h$ . Hence, there are  $z, z' \in Z$  such that  $z$  has a predecessor in  $h$ ,  $z'$  has no predecessor in  $h$ , and yet  $f_i(z) = m = f_i(z')$ , thus violating *ex post* perfect recall.  $\blacksquare$

The following result says that the value of equilibrium actions is unambiguous, hence independent of ambiguity attitudes:

**Lemma 4** *Let  $\bar{\sigma}$  be an SCE of the game with observable payoffs  $(\Gamma, f, \phi)$  justified by the confirmed beliefs  $(\mu_{s_i})_{i \in I, s_i \in \text{Supp} \bar{\sigma}_i}$ . For every  $i \in I$  and  $s_i \in \text{Supp} \bar{\sigma}_i$  and  $h \in H_i(\mu_{s_i}) \cap H_i(s_i)$ , action  $s_{i,h}$  is  $\mu_i(\cdot|h)$ -unambiguous, and its value is the conditional objective expected payoff, that is,*

$$V_i(s_{i,h}|h; s_i, \mu_{s_i}, \phi_i) = U_i(s_i, \bar{\sigma}_{-i}|h).$$

**Proof.** By *ex post* perfect recall and Lemma 3,

$$S_{-i}(h) = \bigcup_{m \in M_i(h)} F_{i,s_i}^{-1}(m) \tag{12}$$

for each  $h \in H_i$ . Also, recall that

$$\hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) = \left\{ \sigma_{-i} \in \Sigma_{-i} : \forall m, \sigma_{-i}(F_{i,s_i}^{-1}(m)) = \bar{\sigma}_{-i}(F_{i,s_i}^{-1}(m)) \right\}$$

is the partially identified set of co-players distributions of strategies observationally equivalent (obs.eq) for  $i$  to  $\bar{\sigma}_{-i}$  given  $s_i$ .

Fix any  $h \in H_i(\mu_{s_i}) \cap H_i(s_i)$ . Then,

$$\begin{aligned} \forall \sigma_{-i} \in \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}), \\ \sigma_{-i}(S_{-i}(h)) \stackrel{(12, \text{obs.eq})}{=} \bar{\sigma}_{-i}(S_{-i}(h)) \stackrel{(12, \text{conf})}{=} p_{\mu_{s_i}}(S_{-i}(h)) > 0, \end{aligned} \quad (13)$$

where the first equality follows from eq. (12) and the fact that  $\sigma_{-i}$  is observationally equivalent to  $\bar{\sigma}_{-i}$ , the second equality follows from eq. (12) and belief confirmation (conf), that is  $\mu_{s_i}(\hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i})) = 1$ , and the inequality follows from  $h \in H_i(\mu_{s_i})$ .

Fix any  $m \in M_i(h)$ ; by observable payoffs (obs.p), there is  $u^m \in \mathbb{R}$  such that  $u_i(\zeta(s_i, s_{-i})) = u^m$  for all  $s_{-i} \in S_{-i}$  with  $f_i(\zeta(s_i, s_{-i})) = m$ . Then, observable payoffs and eq.s (12)-(13) imply that

$$\forall \sigma_{-i} \in \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}), U_i(s_i, \sigma_{-i}|h) = U_i(s_i, \bar{\sigma}_{-i}|h). \quad (14)$$

Indeed, we have

$$\begin{aligned} U_i(s_i, \sigma_{-i}|h) &\stackrel{(\text{def})}{=} \sum_{s_{-i} \in S_{-i}(h)} \frac{\sigma_{-i}(s_{-i})}{\sigma_{-i}(S_{-i}(h))} u_i(\zeta(s_i, s_{-i})) \\ &\stackrel{(12, \text{obs.p})}{=} \sum_{m \in M_i(h)} \frac{\sigma_{-i}(F_{i, s_i}^{-1}(m)) v^m}{\sigma_{-i}(S_{-i}(h))} \stackrel{(\text{obs.eq})}{=} \sum_{m \in M_i(h)} \frac{\bar{\sigma}_{-i}(F_{i, s_i}^{-1}(m)) v^m}{\bar{\sigma}_{-i}(S_{-i}(h))} \\ &\stackrel{(12, \text{obs.p})}{=} \sum_{s_{-i} \in S_{-i}(h)} \frac{\bar{\sigma}_{-i}(s_{-i})}{\bar{\sigma}_{-i}(S_{-i}(h))} u_i(\zeta(s_i, s_{-i})) \stackrel{(\text{def})}{=} U_i(s_i, \bar{\sigma}_{-i}|h). \end{aligned}$$

Confirmed beliefs ( $\mu_{s_i}(\hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i})) = 1$ ), the updating formula (5), and eq.s (13)-(14) yield

$$\begin{aligned} V_i(s_{i,h}|h; s_i, \mu_{s_i}, \phi_i) &= \phi_i^{-1} \left( \int_{\Sigma_{-i}(h)} \phi_i(U_i(s_i, \sigma_{-i}|h)) \mu_{s_i}(d\sigma_{-i}|h) \right) \\ &\stackrel{(5)}{=} \phi_i^{-1} \left( \int \phi_i(U_i(s_i, \sigma_{-i}|h)) \frac{\sigma_{-i}(S_{-i}(h))}{p_{\mu_{s_i}}(S_{-i}(h))} \mu_{s_i}(d\sigma_{-i}) \right) \\ &\stackrel{(14, \text{conf.})}{=} \phi_i^{-1} \left( \phi_i(U_i(s_i, \sigma_{-i}|h)) \int \frac{\sigma_{-i}(S_{-i}(h))}{p_{\mu_{s_i}}(S_{-i}(h))} \mu_{s_i}(d\sigma_{-i}) \right) \\ &\stackrel{(13)}{=} \phi_i^{-1}(\phi_i(U_i(s_i, \bar{\sigma}_{-i}|h))) = U_i(s_i, \bar{\sigma}_{-i}|h). \end{aligned}$$

■

An increase in ambiguity aversion (weakly) decreases the values of actions:

**Lemma 5** *If  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$ . Then,*

$$V_i(a_i|h; s_i, \mu_i, \bar{\phi}_i) \geq V_i(a_i|h; s_i, \mu_i, \phi_i)$$

for all  $i \in I$ ,  $\mu_i \in \Delta(\Sigma_{-i})$ ,  $s_i \in S_i$ ,  $h \in H_i(\mu_i)$ , and  $a_i \in A_i(h)$ .

**Proof.** Fix  $i$  and  $s_i$  arbitrarily. For every  $h \in H_i$  and  $a_i \in A_i(h)$ , define the following auxiliary function:

$$\begin{aligned} U_{s_i, h, a_i} : \Sigma_{-i}(h) &\rightarrow \mathbb{R}, \\ \sigma_{-i} &\mapsto U_i((s_i|h, a_i), \sigma_{-i}|h). \end{aligned}$$

With this, for every  $\mu_i \in \Delta(\Sigma_{-i})$  and  $h \in H_i(\mu_i)$ , the conditional belief  $\mu_i(\cdot|h)$  is well defined and

$$\begin{aligned} V_i(a_i|h; s_i, \mu_i, \bar{\phi}_i) &= \bar{\phi}_i^{-1} \left( \mathbb{E}_{\mu_i(\cdot|h)} [\bar{\phi}_i \circ U_{s_i, h, a_i}] \right), \\ V_i(a_i|h; s_i, \mu_i, \phi_i) &= \phi_i^{-1} \left( \mathbb{E}_{\mu_i(\cdot|h)} [\phi_i \circ U_{s_i, h, a_i}] \right). \end{aligned} \quad (15)$$

Since  $\phi_i = \varphi \circ \bar{\phi}_i$  for some concave and strictly increasing  $\varphi : \bar{\phi}_i(\mathbb{V}_i) \rightarrow \mathbb{R}$ , Jensen's inequality implies

$$\varphi \left( \mathbb{E}_{\mu_i(\cdot|h)} [\bar{\phi}_i \circ U_{s_i, h, a_i}] \right) \geq \mathbb{E}_{\mu_i(\cdot|h)} [\varphi \circ \bar{\phi}_i \circ U_{s_i, h, a_i}] = \mathbb{E}_{\mu_i(\cdot|h)} [\phi_i \circ U_{s_i, h, a_i}].$$

By monotonicity of  $\varphi$  and  $\bar{\phi}_i$ , and recalling that  $\phi_i^{-1} = \bar{\phi}_i^{-1} \circ \varphi^{-1}$ , we obtain

$$\mathbb{E}_{\mu_i(\cdot|h)} [\bar{\phi}_i \circ U_{s_i, h, a_i}] \geq \varphi^{-1} \left( \mathbb{E}_{\mu_i(\cdot|h)} [\phi_i \circ U_{s_i, h, a_i}] \right),$$

$$\bar{\phi}_i^{-1} \left( \mathbb{E}_{\mu_i(\cdot|h)} [\bar{\phi}_i \circ U_{s_i, h, a_i}] \right) \geq \left( \bar{\phi}_i^{-1} \circ \varphi^{-1} \right) \left( \mathbb{E}_{\mu_i(\cdot|h)} [\phi_i \circ U_{s_i, h, a_i}] \right) = \phi_i^{-1} \left( \mathbb{E}_{\mu_i(\cdot|h)} [\phi_i \circ U_{s_i, h, a_i}] \right).$$

By eq. (15),  $V_i(a_i|h; s_i, \mu_i, \bar{\phi}_i) \geq V_i(a_i|h; s_i, \mu_i, \phi_i)$ . ■

**Lemma 6** *Fix two games with observable payoffs,  $(\Gamma, f, \phi)$  and  $(\Gamma, f, \bar{\phi})$ , such that  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$ . Fix an SCE  $\bar{\sigma}$  of  $(\Gamma, f, \bar{\phi})$  and justifying beliefs  $(\mu_{s_i})_{i \in I, s_i \in \text{Supp} \bar{\sigma}_i}$ . Suppose that for each  $i \in I$ , every  $s_i \in \text{Supp} \bar{\sigma}_i$  is  $(\mu_{s_i}, \phi_i)$ -sequentially optimal. Then, there exist maps  $\bar{s}_i : \text{Supp} \bar{\sigma}_i \rightarrow S_i$ ,  $i \in I$ , such that*

- (i) for every  $i \in I$  and  $h \in H_i(s_i)$ ,  $\bar{s}_i(s_i)_h = s_{i, h}$ , and
- (ii)  $\sigma = (\bar{\sigma}_i \circ \bar{s}_i^{-1})_{i \in I}$  is an SCE of  $(\Gamma, f, \phi)$  where for every  $i \in I$  and  $s_i \in \text{Supp} \bar{\sigma}_i$ ,  $\mu_{s_i}$  justifies  $\bar{s}_i(s_i)$ .

(So,  $\sigma$  and  $\bar{\sigma}$  are realization equivalent and are justified by the same beliefs.)



**Proof** For every  $i \in I$  and  $s_i \in \text{Supp} \bar{\sigma}_i$ , we construct a  $(\mu_{s_i}, \phi_i)$ -unimprovable strategy  $\bar{s}_i(s_i)$ . Map  $\bar{s}_i : \text{Supp} \bar{\sigma}_i \rightarrow S_i$  is such that (1)  $\bar{s}_i(s_i)_h = s_{i,h}$  for all  $h \in H_i(s_i)$ , and (2)  $\bar{s}_i(s_i)$  is derived by folding back on  $H_i(\mu_{s_i}) \setminus H_i(s_i)$  given  $\mu_{s_i}$  and  $\phi_i$ .<sup>45</sup> Therefore, by construction, for every  $h \in H_i(\mu_{s_i}) \setminus H_i(s_i)$ ,

$$\bar{s}_i(s_i)_h \in \arg \max_{a_i \in A_i(h)} V_i(a_i|h; \bar{s}_i(s_i), \mu_{s_i}, \phi_i).$$

Now, let  $h \in H_i(\mu_{s_i}) \cap H_i(s_i)$ . For every  $a_i \in A_i(h)$ , we have

$$\begin{aligned} V_i(\bar{s}_i(s_i)_h|h; \bar{s}_i(s_i), \mu_{s_i}, \phi_i) &= V_i(s_{i,h}|h; s_i, \mu_{s_i}, \phi_i) \\ &\stackrel{\text{(L4)}}{=} U_i(s_i, \bar{\sigma}_{-i}|h) \\ &\stackrel{\text{(L4)}}{=} V_i(s_{i,h}|h; s_i, \mu_{s_i}, \bar{\phi}_i) \\ &\stackrel{\text{(s.opt)}}{\geq} V_i(a_i|h; \bar{s}_i(s_i), \mu_{s_i}, \bar{\phi}_i) \\ &\stackrel{\text{(L5)}}{\geq} V_i(a_i|h; \bar{s}_i(s_i), \mu_{s_i}, \phi_i), \end{aligned}$$

where the first equality follows from construction of  $\bar{s}_i(s_i)$ , the second and the third equalities from Lemma 4, the first inequality from sequential optimality (s.opt), and the second inequality from Lemma 5. This shows that  $\bar{s}_i(s_i)$  is  $(\mu_{s_i}, \phi_i)$ -unimprovable.

To conclude, note that the profile  $(\bar{\sigma}_i \circ \bar{s}_i^{-1}, (\mu_{\hat{s}_i})_{\hat{s}_i \in \text{Supp}(\bar{\sigma}_i \circ \bar{s}_i^{-1})})_{i \in I}$ , where  $\mu_{\hat{s}_i}(\cdot) = \mu_{s_i}(\cdot)$  for some  $s_i$  with  $\hat{s}_i = \bar{s}_i(s_i)$ , also satisfies the confirmed beliefs condition (because  $(\bar{s}_i(s_i), (\bar{\sigma}_0, (\bar{\sigma}_j \circ \bar{s}_j^{-1})_{j \in I \setminus \{i\}}))$  and  $(s_i, \bar{\sigma}_{-i})$ , induce the same distribution over terminal nodes). Therefore  $(\bar{\sigma}_i \circ \bar{s}_i^{-1})_{i \in I}$  is an SCE of  $(\Gamma, f, \phi)$ .  $\blacksquare$

## 10.4 Symmetric equilibria

In this subsection and the following ones, we consider games *without chance moves*. Therefore, the outcome function and the strategic-form feedback and payoff functions of player  $i$  are, respectively,

$$\zeta : S \rightarrow Z,$$

$$F_i = f_i \circ \zeta : S \rightarrow M,$$

and

$$U_i = u_i \circ \zeta : S \rightarrow \mathbb{R}.$$

The proof of Theorem 6 requires some preliminary results. For every  $i \in I$ , consider the augmented collection

$$\hat{H}_i = H_i \cup \{f_i^{-1}(m) : m \in f_i(Z)\}$$

<sup>45</sup>Note that if  $h \notin H_i(s_i)$ , and  $h' \succ h$ ,  $h' \notin H_i(s_i)$ , therefore the folding back construction is well defined. See Section 4 for the description of folding back optimality.

that includes also  $i$ 's terminal information sets. By ex post perfect recall of  $\hat{H}_i$ , we can derive from the game tree a transitive and antisymmetric precedence relation  $\preceq$  on  $\hat{H}_i$  that makes it a directed forest (collection of directed trees). Furthermore, we can extend to  $\hat{H}_i$  the definition of the sets  $S(h)$ ,  $S_i(h)$  and  $S_{-i}(h)$  so that  $S(h) = S_i(h) \times S_{-i}(h)$ ; in particular, for each  $m \in f_i(Z)$ ,

$$\begin{aligned} S(f_i^{-1}(m)) &= F_i^{-1}(m) = \text{proj}_{S_i} F_i^{-1}(m) \times \text{proj}_{S_{-i}} F_i^{-1}(m) \\ &= S_i(f_i^{-1}(m)) \times S_{-i}(f_i^{-1}(m)). \end{aligned}$$

Note that, for each  $\mu_i$ , we can define the collection of information sets in  $\hat{H}_i$  that are possible under  $\mu_i$ :

$$\hat{H}_i(\mu_i) = \left\{ h \in \hat{H}_i : p_{\mu_i}(S_{-i}(h)) > 0 \right\}.$$

Note that,  $H_i(\mu_i) \subset \hat{H}_i(\mu_i)$ . Also, for each  $h \in H_i$  and  $a_i \in A_i(h)$ , we define the collection of information sets in  $\hat{H}_i$  that “immediately” follow  $h$  and action  $a_i$ :

$$\hat{H}_i(h, a_i) = \left\{ h' \in \hat{H}_i : (h \prec h') \wedge (\alpha_i(h, h') = a_i) \wedge (\nexists h'' \in H_i, h \prec h'' \prec h') \right\}.$$

Finally, it is convenient to extend the formula for the values of actions (given strategy and belief) from  $H_i(\mu_i)$  to  $\hat{H}_i(\mu_i)$ . Thus, we stipulate that if  $h = f_i^{-1}(m)$  then  $A_i(h) = \{a_i^m\}$  is the singleton that contains only the pseudo-action “ $i$  observes  $m$ .” By observability of payoff, if  $h = f_i^{-1}(m)$ , then  $V_i(a_i^m | h; s_i, \mu_i, \phi_i) = u_i(z)$  for every  $z \in h$ , independently of  $(s_i, \mu_i, \phi_i)$ .

**Remark 6** *By ex post perfect recall, for every  $i \in I$ ,  $h \in H_i$  and  $a_i \in A_i(h)$ ,  $\{S_{-i}(h')\}_{h' \in \hat{H}_i(h, a_i)}$  is a partition of  $S_{-i}(h)$ .*

**Proof** We first prove that distinct elements of the collection  $\{S_{-i}(h')\}_{h' \in \hat{H}_i(h, a_i)}$  are disjoint. By ex post perfect recall  $S(h') = S_i(h') \times S_{-i}(h')$  for each  $h' \in \hat{H}_i$ . Let

$$S_i(h, a_i) = \{s_i \in S_i(h) : s_{i,h} = a_i\}$$

denote the strategies of  $i$  allowing for  $h$  and choosing  $a_i$  at  $h$ . Then, by definition of  $\hat{H}_i(h, a_i)$ ,  $S_i(h') = S_i(h, a_i)$  for each  $h' \in \hat{H}_i(h, a_i)$ , because  $i$  does not move again after choosing  $a_i$  and before any such  $h'$ . Therefore,

$$\forall h' \in \hat{H}_i(h, a_i), S(h') = S_i(h, a_i) \times S_{-i}(h').$$

Take any  $h', h'' \in \hat{H}_i(h, a_i)$  with  $h' \neq h''$  (if such distinct information sets exist). By definition of  $\hat{H}_i(h, a_i)$ ,  $h' \not\prec h''$  and  $h'' \not\prec h'$ . By perfect recall, this implies that

$$\{z' \in Z : \exists x' \in h', x' \preceq z'\} \cap \{z'' \in Z : \exists x'' \in h'', x'' \preceq z''\} = \emptyset,$$

which in turn implies that  $S(h') \cap S(h'') = \emptyset$ . Since  $S_i(h') = S_i(h, a_i) = S_i(h'')$ , then  $S_{-i}(h') \cap S_{-i}(h'') = \emptyset$ .

Note that

$$\bigcup_{h' \in \hat{H}_i(h, a_i)} S_{-i}(h') \subseteq S_{-i}(h)$$

because  $h \prec h'$ , hence  $S_{-i}(h') \subseteq S_{-i}(h)$ , for each  $h' \in \hat{H}_i(h, a_i)$ .

To prove the converse, pick any  $(s_i, s_{-i}) \in S_i(h, a_i) \times S_{-i}(h)$  and let  $h'$  be the first information set in  $\hat{H}_i$  after  $h$  (possibly a terminal information set) reached by the path with terminal node  $\zeta(s_i, s_{-i})$ . Then  $h' \in \hat{H}_i(h, a_i)$  and  $s_{-i} \in S_{-i}(h')$ . Therefore,

$$S_{-i}(h) \subseteq \bigcup_{h' \in \hat{H}_i(h, a_i)} S_{-i}(h').$$

We conclude that  $\{S_{-i}(h')\}_{h' \in \hat{H}_i(h, a_i)}$  is a partition. ■

We say that a belief  $\mu_i \in \Delta(\Sigma_{-i})$  is **supported by Dirac models** if

$$\mu_i(\{\delta_{s_{-i}} : s_{-i} \in S_{-i}\}) = 1.$$

Then

$$\forall s_{-i} \in S_{-i}, \mu_i(\delta_{s_{-i}}) = p_{\mu_i}(s_{-i})$$

and

$$\forall h \in \hat{H}_i(\mu_i), \forall s_{-i} \in S_{-i}(h), \mu_i(\delta_{s_{-i}}|h) = p_{\mu_i}(s_{-i}|h).$$

If  $\mu_i$  is supported by Dirac models,

$$V_i(a_i|h; s_i, \mu_i, \phi_i) = \phi_i^{-1} \left( \sum_{s_{-i} \in S_{-i}(h)} p_{\mu_i}(s_{-i}|h) \phi_i(U_i((s_i|h, a_i), s_{-i})) \right).$$

Since  $\phi_i$  is strictly increasing, maximizing  $V_i(a_i|h; s_i, \mu_i, \phi_i)$  is the same as maximizing

$$\phi_i(V_i(a_i|h; s_i, \mu_i, \phi_i)) = \sum_{s_{-i} \in S_{-i}(h)} p_{\mu_i}(s_{-i}|h) \phi_i(U_i((s_i|h, a_i), s_{-i}))$$

and we can focus on the latter expected value. Under a belief supported by Dirac models,  $\phi_i(V_i(s_i|h; s_i, \mu_i, \phi_i))$  is a weighted sum of the expected values given by  $s_i$  at the next information sets (including the terminal ones), where the weights are the predictive probabilities of reaching them.

**Lemma 7** Fix any  $i \in I$  and a belief supported by Dirac models  $\hat{\mu}_i$ . Then, for every  $s_i \in S_i$  and  $h \in H_i(\hat{\mu}_i)$ ,

$$\phi_i(V_i(s_i|h; s_i, \hat{\mu}_i, \phi_i)) = \sum_{h' \in \hat{H}_i(h, s_i, h)} p_{\hat{\mu}_i}(S_{-i}(h')|h) \phi_i(V_i(s_i|h'|h'; s_i, \hat{\mu}_i, \phi_i)).$$

**Proof.** Fix  $s_i \in S_i$  and  $h \in H_i(\hat{\mu}_i)$  arbitrarily. By Remark 6 and taking into account that  $\hat{\mu}_i$  is supported by Dirac models, we can write

$$\begin{aligned}
& \phi_i(V_i(s_{i,h}|h; s_i, \hat{\mu}_i, \phi_i)) = \sum_{s_{-i} \in S_{-i}(h)} p_{\hat{\mu}_i}(s_{-i}|h) \phi_i(U_i(s_{i|h}, s_{-i})) \\
&= \sum_{h' \in \hat{H}_i(h, s_{i,h})} \sum_{s_{-i} \in S_{-i}(h')} p_{\hat{\mu}_i}(s_{-i}|h) \phi_i(U_i(s_{i|h}, s_{-i})) \\
&= \sum_{h' \in \hat{H}_i(h, s_{i,h}) : p_{\hat{\mu}_i}(S_{-i}(h')|h) > 0} p_{\hat{\mu}_i}(S_{-i}(h')|h) \sum_{s_{-i} \in S_{-i}(h')} \frac{p_{\hat{\mu}_i}(s_{-i}|h)}{p_{\hat{\mu}_i}(S_{-i}(h')|h)} \phi_i(U_i(s_{i|h}, s_{-i})).
\end{aligned}$$

By the chain rule, for every  $h' \in \hat{H}_i(h, s_i)$  with  $p_{\hat{\mu}_i}(S_{-i}(h')|h) > 0$  and for every  $s_{-i} \in S_{-i}(h')$ ,

$$\frac{p_{\hat{\mu}_i}(s_{-i}|h)}{p_{\hat{\mu}_i}(S_{-i}(h')|h)} = p_{\hat{\mu}_i}(s_{-i}|h').$$

Furthermore, for every  $h' \in \hat{H}_i(h, s_{i,h})$  and  $s_{-i} \in S_{-i}(h')$ ,  $U_i(s_{i|h}, s_{-i}) = U_i(s_{i|h'}, s_{-i})$ , because both  $s_{i|h}$  and  $s_{i|h'}$  make  $h'$  reachable and make the same choices before  $h'$  (by perfect recall), at  $h'$ , and after  $h'$  (by definition of replacement); thus,

$$\begin{aligned}
& \sum_{s_{-i} \in S_{-i}(h')} \frac{p_{\hat{\mu}_i}(s_{-i}|h)}{p_{\hat{\mu}_i}(S_{-i}(h')|h)} \phi_i(U_i(s_{i|h}, s_{-i})) \\
&= \sum_{s_{-i} \in S_{-i}(h')} p_{\hat{\mu}_i}(s_{-i}|h') \phi_i(U_i(s_{i|h'}, s_{-i})) = \phi_i(V_i(s_{i,h'}|h'; s_i, \hat{\mu}_i, \phi_i))
\end{aligned}$$

and we obtain the desired result. ■

Iterating the previous formula and using the chain rule, one can prove the following:

**Corollary 14** *Fix  $i \in I$  and a belief  $\hat{\mu}_i$  supported by Dirac models. Then, for every  $s_i \in S_i$  and  $h \in H_i(\hat{\mu}_i)$ ,*

$$\phi_i(V_i(s_{i,h}|h; s_i, \hat{\mu}_i, \phi_i)) = \sum_{s_{-i} \in S_{-i}} p_{\hat{\mu}_i}(s_{-i}|h) \phi_i(U_i(s_{i|h}, s_{-i})).$$

Thus, maximizing the conditional value of a strategy under a belief  $\hat{\mu}_i$  supported by Dirac models is the same as maximizing the subjective expectation of the transformed utility  $\hat{u}_i = \phi_i \circ U_i$ . Applying Proposition 2 to  $\hat{\mu}_i$  and  $\hat{u}_i$  we obtain the following:

**Lemma 8** *For every  $i \in I$  and  $\hat{\mu}_i \in \Delta(\Sigma_{-i})$ , if  $\hat{\mu}_i$  is a belief supported by Dirac models, then the set of  $(\hat{\mu}_i, \phi_i)$ -unimprovable and  $(\hat{\mu}_i, \phi_i)$ -sequentially optimal strategies of  $i$  coincide.*

Note that, for every belief  $\mu_i \in \Delta(\Sigma_{-i})$  there is a unique belief  $\hat{\mu}_i$  supported by Dirac models with the same predictives as  $\mu_i$ :

$$\forall s_{-i} \in S_{-i}, \hat{\mu}_i(\delta_{s_{-i}}) = p_{\mu_i}(s_{-i}).$$

This implies that

$$\forall h \in H_i(\mu_i), \forall s_{-i} \in S_{-i}(h), \hat{\mu}_i(\delta_{s_{-i}}|h) = p_{\mu_i}(s_{-i}|h).$$

We show that, for an ambiguity averse agent,  $\hat{\mu}_i$  is the “most pessimistic” belief with the same predictive as  $\mu_i$  in the following sense: for every  $h \in H_i(\mu_i) = H_i(\hat{\mu}_i)$ , the value at  $h \in H_i(\mu_i)$  of a  $(\mu_i, \phi_i)$ -unimprovable strategy is at least as high as the value at  $h$  of a  $(\hat{\mu}_i, \phi_i)$ -unimprovable strategy. Formally:

**Lemma 9** *Fix  $i \in I$ ,  $\mu_i \in \Delta(\Sigma_{-i})$ , and a concave second-order utility function  $\phi_i$ . Let  $\hat{\mu}_i \in \Delta(\Sigma_{-i})$  denote the belief supported by Dirac models with the same predictive as  $\mu_i$ , and let  $s_i^*$  be  $(\mu_i, \phi_i)$ -unimprovable; then,*

$$\forall s_i \in S_i, \forall h \in \hat{H}_i(\mu_i), V_i(s_i^*|h; s_i^*, \mu_i, \phi_i) \geq V_i(s_{i,h}|h; s_i, \hat{\mu}_i, \phi_i).$$

**Proof.** First note that  $\hat{H}_i(\mu_i)$ —with the obvious precedence relation—is a directed forest and its leaves are also leaves in  $\hat{H}_i$  as well, that is, if  $h$  is a leaf in  $\hat{H}_i(\mu_i)$  then  $h$  is a set of terminal nodes ( $h \in \{f_i^{-1}(m) : m \in f_i(Z)\}$ ). Furthermore,  $\hat{H}_i(\mu_i) = \hat{H}_i(\hat{\mu}_i)$  because  $p_{\mu_i}(S_{-i}(h)) = p_{\hat{\mu}_i}(S_{-i}(h))$  for every  $h \in \hat{H}_i$ . We prove the result by induction on the depth of  $h$  within  $\hat{H}_i(\mu_i)$ .

*Basis step.* Let  $h = f_i^{-1}(m)$  be a terminal information set in  $\hat{H}_i(\mu_i)$ . Then the weak inequality holds trivially as an equality: in particular, by observable payoffs

$$\forall z \in h, V_i(s_{i,h}^*|h; s_i^*, \mu_i, \phi_i) = u_i(z) = V_i(s_{i,h}|h; s_i, \mu_i, \phi_i).$$

*Inductive step.* Let  $h$  have at least one follower in  $\hat{H}_i(\mu_i)$  and suppose, by way of induction, that for every  $h' \in \hat{H}_i(\mu_i)$  with  $h \prec h'$ ,

$$V_i(s_{i,h'}^*|h'; s_i^*, \mu_i, \phi_i) \geq V_i(s_{i,h'}|h'; s_i, \hat{\mu}_i, \phi_i). \quad (\text{I.H.})$$

Consider the replacement plan  $s_i^h = (s_{i|h}^*, s_{i,h})$ , that is,

$$\forall h' \in H_i, (s_i^h)_{h'} = (s_{i|h}^*, s_{i,h})_{h'} = \begin{cases} s_{i,h} & \text{if } h' = h, \\ \alpha_i(h', h) & \text{if } h' \prec h, \\ s_{i,h'}^* & \text{if } h' \not\prec h. \end{cases}$$

Since  $s_i^*$  is  $(\mu_i, \phi_i)$ -unimprovable

$$V_i(s_{i,h}^*|h; s_i^*, \mu_i, \phi_i) \geq V_i(s_{i,h}|h; s_i^*, \mu_i, \phi_i).$$

Next we prove that

$$V_i(s_{i,h}|h; s_i^*, \mu_i, \phi_i) \geq V_i(s_{i,h}|h; s_i, \hat{\mu}_i, \phi_i),$$

which implies the thesis.

To ease notation, write

$$\hat{H}_i(h, a_i, \mu_i) = \hat{H}_i(h, a_i) \cap \hat{H}_i(\mu_i)$$

for the collection of information sets that immediately follow  $h$  after  $a_i$  and can be reached with positive probability under  $\mu_i$ ;  $\hat{H}_i(h, a_i, \sigma_{-i})$  is similarly defined (thus,  $\hat{H}_i(h, a_i, \sigma_{-i}) = \hat{H}_i(h, a_i, \delta_{\sigma_{-i}})$ ). By Remark 6, the definitions of  $V_i$  and  $s_i^h$ , Jensen's inequality, the chain rule for  $\sigma_{-i}(\cdot|\cdot)$ , Bayes rule for  $\mu_i(\cdot|\cdot)$ , Lemma 7, and the Inductive Hypothesis,

$$\begin{aligned} & \phi_i(V_i(s_{i,h}|h; s_i^*, \mu_i, \phi_i)) \stackrel{\text{(R6,def)}}{=} \int_{\Sigma_{-i}(h)} \phi_i \left( \sum_{h' \in \hat{H}_i(h, s_{i,h})} \sum_{s_{-i} \in S_{-i}(h')} \sigma_{-i}(s_{-i}|h) U_i(s_i^h, s_{-i}) \right) \mu_i(d\sigma_{-i}|h) \\ &= \int_{\Sigma_{-i}(h)} \phi_i \left( \sum_{h' \in \hat{H}_i(h, s_{i,h}, \sigma_{-i})} \sigma_{-i}(S_{-i}(h')|h) \sum_{s_{-i} \in S_{-i}(h')} \frac{\sigma_{-i}(s_{-i}|h)}{\sigma_{-i}(S_{-i}(h')|h)} U_i(s_i^h, s_{-i}) \right) \mu_i(d\sigma_{-i}|h) \\ & \stackrel{\text{(Jen,ch)}}{\geq} \int_{\Sigma_{-i}(h)} \sum_{h' \in \hat{H}_i(h, s_{i,h}, \sigma_{-i})} \sigma_{-i}(S_{-i}(h')|h) \phi_i \left( \sum_{s_{-i} \in S_{-i}(h')} \sigma_{-i}(s_{-i}|h') U_i(s_i^h, s_{-i}) \right) \mu_i(d\sigma_{-i}|h) \\ & \stackrel{\text{(def)}}{=} \sum_{h' \in \hat{H}_i(h, s_{i,h}, \mu_i)} \int_{\Sigma_{-i}(h')} \sigma_{-i}(S_{-i}(h')|h) \phi_i(U_i(s_i^*|h', \sigma_{-i}|h')) \mu_i(d\sigma_{-i}|h) \\ &= \sum_{h' \in \hat{H}_i(h, s_{i,h}, \mu_i)} p_{\mu_i}(S_{-i}(h')|h) \int_{\Sigma_{-i}(h')} \frac{\sigma_{-i}(S_{-i}(h')|h)}{p_{\mu_i}(S_{-i}(h')|h)} \phi_i(U_i(s_i^*|h', \sigma_{-i}|h')) \mu_i(d\sigma_{-i}|h) \\ & \stackrel{\text{(Bayes)}}{=} \sum_{h' \in \hat{H}_i(h, s_{i,h}, \mu_i)} p_{\mu_i}(S_{-i}(h')|h) \int_{\Sigma_{-i}(h')} \phi_i(U_i(s_i^*|h', \sigma_{-i}|h')) \mu_i(d\sigma_{-i}|h) \\ & \stackrel{\text{(def)}}{=} \sum_{h' \in \hat{H}_i(h, s_{i,h}, \mu_i)} p_{\mu_i}(S_{-i}(h')|h) \phi_i(V_i(s_i^*|h'; s_i^*, \mu_i, \phi_i)) \\ & \stackrel{\text{(I.H.)}}{\geq} \sum_{h' \in \hat{H}_i(h, s_{i,h}, \mu_i)} p_{\mu_i}(S_{-i}(h')|h) \phi_i(V_i(s_{i,h'}|h'; s_i, \hat{\mu}_i, \phi_i)) \stackrel{\text{(L7)}}{=} \phi_i(V_i(s_{i,h}|h; s_i, \hat{\mu}_i, \phi_i)). \end{aligned}$$

Since  $\phi_i$  is strictly increasing,  $V_i(s_{i,h}|h; s_i^*, \mu_i, \phi_i) \geq V_i(s_{i,h}|h; s_i, \hat{\mu}_i, \phi_i)$ . ■

**Proof of Theorem 6.** Let  $z^* \in \zeta(\text{symSCE}(\Gamma, f, \bar{\phi}))$ . Then, there is a symmetric SCE  $(s_i^*)_{i \in I}$  of  $(\Gamma, f, \bar{\phi})$  such that  $z^* = \zeta((s_i^*)_{i \in I})$ . Let  $(s_i^*)_{i \in I}$  be justified by the profile of confirmed beliefs  $(\mu_i)_{i \in I}$ . For each  $i \in I$ , let  $m_i^* = f_i(z^*)$ , and let  $\hat{\mu}_i \in \Delta(\Sigma_{-i})$  be the belief supported by Dirac models with the same predictive as  $\mu_i$ , so that  $H_i(\mu_i) = H_i(\hat{\mu}_i)$ . Compute by folding back a  $(\hat{\mu}_i, \bar{\phi}_i)$ -unimprovable strategy  $\hat{s}_i^*$ ; let  $\hat{s}_i$  be the strategy that coincides with equilibrium strategy  $s_i^*$  on each path observationally equivalent to the equilibrium path, and with  $\hat{s}_i^*$  on the other paths, that is,

$$\forall h \in H_i, \hat{s}_{i,h} = \begin{cases} s_{i,h}^* & \text{if } h \prec f_i^{-1}(m_i^*), \\ \hat{s}_{i,h}^* & \text{if } h \not\prec f_i^{-1}(m_i^*). \end{cases}$$

By construction,  $\zeta((\hat{s}_i)_{i \in I}) = \zeta((s_i^*)_{i \in I}) = z^*$ . For each  $i \in I$ , since  $\mu_i$  is a confirmed belief and  $\hat{\mu}_i$  has the same predictive as  $\mu_i$ ,

$$p_{\hat{\mu}_i}(S_{-i}(f^{-1}(m_i^*))) = p_{\mu_i}(S_{-i}(f^{-1}(m_i^*))) = 1.$$

Therefore, each belief in profile  $(\hat{\mu}_i)_{i \in I}$  is confirmed under both  $(s_i^*)_{i \in I}$  and  $(\hat{s}_i)_{i \in I}$ . We are going to prove that each  $\hat{s}_i$  is  $(\hat{\mu}_i, \bar{\phi}_i)$ -sequentially optimal, which implies that  $(\hat{s}_i)_{i \in I}$  is an SCE of  $(\Gamma, f, \bar{\phi})$  and that Lemma 6 applies. Therefore, there is an equivalent symmetric SCE  $(s_i)_{i \in I}$  of the more ambiguity averse game  $(\Gamma, f, \phi)$  such that  $\zeta((s_i)_{i \in I}) = z^*$ , as desired.

*Sequential optimality of  $\hat{s}_i$  under  $(\bar{\phi}_i, \hat{\mu}_i)$ .* We show that  $\hat{s}_i$  is  $(\bar{\phi}_i, \hat{\mu}_i)$ -unimprovable. Since  $\hat{\mu}_i$  is supported by Dirac models, Lemma 8 implies that  $\hat{s}_i$  is also  $(\hat{\mu}_i, \bar{\phi}_i)$ -sequentially optimal.

By belief confirmation and perfect recall, the collection of information sets over which  $s_i^*$  and  $\hat{s}_i$  coincide is given by those that are possible (indeed, certain) under  $(\hat{s}_i, \hat{\mu}_i)$ , that is,

$$H_i(\hat{\mu}_i) \cap H_i(\hat{s}_i) = \{h \in H_i : h \prec f_i^{-1}(m_i^*)\},$$

because  $i$  expects message  $m_i^*$  with certainty given that he plans to play  $\hat{s}_i$ .

By (ex post) perfect recall, for all  $h, h' \in H_i(\hat{\mu}_i)$  such that  $h \not\prec f_i^{-1}(m_i^*)$  (hence  $h \notin H_i(\hat{s}_i)$ ),  $h \prec h'$  implies  $h' \not\prec f_i^{-1}(m_i^*)$ . Therefore,

$$\forall h \in H_i(\hat{\mu}_i) \setminus H_i(\hat{s}_i), \hat{s}_{i,h} = \hat{s}_{i,h}^*$$

and

$$\forall h \in H_i(\hat{\mu}_i) \setminus H_i(\hat{s}_i), V_i(\cdot | h; \hat{s}_i, \hat{\mu}_i, \bar{\phi}_i) = V_i(\cdot | h; \hat{s}_i^*, \hat{\mu}_i, \bar{\phi}_i).$$

By  $(\hat{\mu}_i, \bar{\phi}_i)$ -unimprovability of  $\hat{s}_i^*$ , this implies

$$\forall h \in H_i(\hat{\mu}_i) \setminus H_i(\hat{s}_i), \hat{s}_{i,h} \in \arg \max_{a_i \in A_i(h)} V_i(a_i | h; \hat{s}_i, \hat{\mu}_i, \bar{\phi}_i). \quad (16)$$

Next consider any  $h \in H_i(\hat{\mu}_i) \cap H_i(\hat{s}_i)$ , hence any  $h \prec f_i^{-1}(m^*)$ . Belief confirmation (for both  $\mu_i$  and  $\hat{\mu}_i$ ) and payoffs observability imply

$$u_i(z^*) = V_i(s_{i,h}^* | h; s_i^*, \mu_i, \bar{\phi}_i).$$

By definition of  $\hat{s}_i$ ,

$$V_i(\hat{s}_{i,h} | h; \hat{s}_i, \hat{\mu}_i, \bar{\phi}_i) = u_i(z^*) = V_i(s_{i,h}^* | h; s_i^*, \mu_i, \bar{\phi}_i).$$

By Lemma 9,

$$V_i(s_{i,h}^* | h; s_i^*, \mu_i, \bar{\phi}_i) \geq V_i(\hat{s}_{i,h}^* | h; \hat{s}_i^*, \hat{\mu}_i, \bar{\phi}_i).$$

Hence:

$$V_i(\hat{s}_{i,h} | h; \hat{s}_i, \hat{\mu}_i, \bar{\phi}_i) \geq V_i(\hat{s}_{i,h}^* | h; \hat{s}_i^*, \hat{\mu}_i, \bar{\phi}_i).$$

By definition of  $\hat{s}_i$  and  $(\hat{\mu}_i, \bar{\phi}_i)$ -unimprovability of  $\hat{s}_i^*$

$$\forall a_i \in A_i(h) \setminus \{\hat{s}_{i,h}\}, V_i(\hat{s}_{i,h}^* | h; \hat{s}_i^*, \hat{\mu}_i, \bar{\phi}_i) \geq V_i(a_i | h; \hat{s}_i^*, \hat{\mu}_i, \bar{\phi}_i) = V_i(a_i | h; \hat{s}_i, \hat{\mu}_i, \bar{\phi}_i),$$

where the equality holds because  $\hat{s}_{i,h} = \hat{s}_{i,h}^*$  and  $\hat{s}_i$  coincides with  $\hat{s}_i^*$  after deviations from an expected path. Collecting these equalities and inequalities, we obtain

$$\forall h \in H_i(\hat{\mu}_i) \cap H_i(\hat{s}_i), \hat{s}_{i,h} \in \arg \max_{a_i \in A_i(h)} V_i(a_i | h; \hat{s}_i, \hat{\mu}_i, \bar{\phi}_i). \quad (17)$$

Eq.s (16)-(17) imply that  $\hat{s}_i$  is  $(\hat{\mu}_i, \bar{\phi}_i)$ -unimprovable, hence—by Lemma 8—sequentially optimal and the thesis follows from Lemma 6.  $\blacksquare$

## 10.5 Full Unimprovability and Rationalizable SCE

For the reader's convenience we recall some notation and definitions of Section 7.  $\bar{H}_i = H_i \cup \{\{\emptyset\}\}$  is the extended collection of information sets that includes the initial information set  $\{\emptyset\}$  even if  $i$  is not a first mover ( $\bar{H}_i = H_i$  if  $i$  is a first mover). With this, we let  $\bar{H}_i(\mu_i) = H_i(\mu_i) \cup \{\{\emptyset\}\}$  be the subcollection of reachable information sets given prior  $\mu_i$ . A CPS  $\bar{\mu}_i(\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i}) \subseteq [\Delta(\Sigma_{-i})]^{\bar{H}_i}$  for player  $i$  specifies an initial belief  $\bar{\mu}_i(\cdot | \{\emptyset\}) \in \Delta(\Sigma_{-i})$  and a conditional belief  $\bar{\mu}_i(\cdot | h)$  for each  $h \in H_i$ . Full  $(\bar{\mu}_i(\cdot), \phi_i)$ -unimprovability requires value-maximization over actions at each  $h \in H_i$ , that is, at the information sets where  $i$  is active. The initial belief  $\bar{\mu}_i(\cdot | \{\emptyset\})$  matters to compare full  $(\bar{\mu}_i(\cdot), \phi_i)$ -unimprovability with  $(\mu_i, \phi_i)$ -unimprovability, because in the latter  $\mu_i$  has to be interpreted as an initial belief. Furthermore  $\bar{\mu}_i(\cdot | \{\emptyset\})$  captures how  $i$  strategically analyzes the game before playing it.



### 10.5.1 Equivalence of SCE and fully unimprovable SCE

**Proof of Proposition 10.** First note that, for every prior  $\mu_i$  on  $\Sigma_{-i}$ , one can find a CPS  $\bar{\mu}_i(\cdot|\cdot)$  on  $(\Sigma_{-i}, \bar{H}_i)$  such that  $\mu_i(\cdot|h) = \bar{\mu}_i(\cdot|h)$  for all information sets  $h \in \bar{H}_i(\mu_i)$ : Let  $\bar{\mu}_i(\cdot|\{\emptyset\}) = \mu_i$  and derive  $\bar{\mu}_i(\cdot|h)$  by conditioning for all  $h \in H_i(\mu_i)$ . Next, for every  $h \in H_i \setminus H_i(\mu_i)$  whose immediate predecessor  $h$  in  $(\bar{H}_i, \preceq)$  belongs to  $\bar{H}_i(\mu_i)$ , fix some  $\bar{\mu}_i(\cdot|h) \in \Delta(\Sigma_{-i}(h))$  such that  $p_{\bar{\mu}_i(\cdot|h)}(S_{-i}(h')) > 0$  for all the information sets  $h'$  that weakly follow  $h$ , and derive  $\bar{\mu}_i(\cdot|h')$  from  $\bar{\mu}_i(\cdot|h)$  by conditioning. One can check that the constructed array  $(\bar{\mu}_i(\cdot|h))_{h \in \bar{H}_i} \in [\Delta(\Sigma_{-i})]^{\bar{H}_i}$  is a CPS.

Fix an SCE  $\bar{\sigma}$  justified by confirmed beliefs  $(\mu_{s_i})_{i \in I, s_i \in \text{Supp} \bar{\sigma}_i}$ . For each  $i \in I$  and  $s_i \in \text{Supp} \bar{\sigma}_i$ , let  $\bar{\mu}_{s_i}(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$  be a CPS such that  $\bar{\mu}_{s_i}(\cdot|h) = \mu_{s_i}(\cdot|h)$  for every  $h \in H_i(\mu_{s_i})$  (see above). Now construct a new strategy  $\hat{s}_i(s_i)$  so that (1)  $\hat{s}_i(s_i)(h) = s_i(h)$  for all  $h \in H_i(\mu_{s_i})$ , and (2)  $\hat{s}_i(s_i)$  is derived by folding back on  $H_i \setminus H_i(\mu_i)$  given  $\bar{\mu}_{s_i}(\cdot|\cdot)$ . Since  $s_i$  is  $(\mu_{s_i}, \phi_i)$ -unimprovable,  $\hat{s}_i(s_i)$  must be *fully*  $(\bar{\mu}_{s_i}(\cdot|\cdot), \phi_i)$ -unimprovable. By construction,  $(s_i, \mu_i)$  and  $(\hat{s}_i(s_i), \bar{\mu}_{s_i}(\cdot|\{\emptyset\}))$  imply the same probabilities of terminal nodes, because they reach the same information sets,  $H_i(s_i) \cap H_i(\mu_{s_i}) = H_i(\hat{s}_i(s_i)) \cap H_i(\bar{\mu}_{s_i}(\cdot|\{\emptyset\}))$ , where  $s_i$  and  $\hat{s}_i(s_i)$  prescribe the same actions.

By ex post perfect recall and the self-confirming conditions for  $\bar{\sigma}$ , for every  $i \in I$  and  $s_i \in \text{Supp} \bar{\sigma}_i$ ,  $(s_i, \mu_{s_i})$  and  $(s_i, \bar{\sigma}_{-i})$  yield the same probabilities of reaching information sets of  $i$ :  $p_{\mu_{s_i}}(S_{-i}(h)) = \bar{\sigma}_{-i}(S_{-i}(h))$  for every  $h \in H_i(s_i)$ . Therefore, by construction,  $p_{\bar{\mu}_{s_i}}(S_{-i}(h)|\{\emptyset\}) = \bar{\sigma}_{-i}(S_{-i}(h))$  for every  $i \in I$ ,  $s_i \in \text{Supp} \bar{\sigma}_i$ , and  $h \in H_i(s_i)$ . Now, for every  $i \in I$ , consider the pushforward measure  $\hat{\sigma}_i = \bar{\sigma}_i \circ \hat{s}_i^{-1}$ , that is,

$$\forall s_i \in S_i, \hat{\sigma}_i(s_i) = \sum_{s'_i: \hat{s}_i(s'_i) = s_i} \bar{\sigma}_i(s'_i).$$

By construction  $\sigma$  and  $\hat{\sigma}$  yield the same distribution over terminal nodes, because they reach the same information sets and, for each  $i \in I$ , the pure strategies in the support of  $\sigma_i$  take the same actions as the associated pure strategies in the support of  $\hat{\sigma}_i$  at all reachable information sets. Furthermore, the profile  $(\hat{\sigma}_i, (\bar{\mu}_{\hat{s}_i}(\cdot|\cdot))_{\hat{s}_i \in \text{Supp} \hat{\sigma}_i})_{i \in I}$ , where  $\bar{\mu}_{\hat{s}_i}(\cdot|\cdot) = \bar{\mu}_{s_i}(\cdot|\cdot)$  for some  $s_i \in \hat{s}_i^{-1}(\hat{s}_i)$ ,<sup>46</sup> also satisfies the confirmed beliefs condition on top of the full unimprovability condition. Therefore  $\hat{\sigma}$  is a fully unimprovable SCE.  $\blacksquare$

### 10.5.2 Monotonicity of the symmetric RSCE correspondence

Recall that in the analysis of symmetric RSCE, we assume that there are no chance moves.

**Proof of Remark 5** (Only if) Let  $\bar{B}_i = \bigcap_{k \in \mathbb{N}} B_i^k$  for each  $i \in I$ ; it can be checked that  $(\bar{B}_i)_{i \in I}$  satisfies the required property.

<sup>46</sup>It can be checked that it does not matter which  $s_i \in \hat{s}_i^{-1}(\hat{s}_i)$  we pick.

(If) We show by induction that if  $(\bar{B}_i)_{i \in I} \in \times_{i \in I} 2^{S_i \times M}$  has the required property, then  $\bar{B}_i \subseteq B_i^k$  for every  $i \in I$  and  $k \in \mathbb{N}_0$ . The claim is trivially true for  $k = 0$ . Suppose it is true for  $k \in \mathbb{N}_0$ . Fix  $i \in I$  and  $(\hat{s}_i, \hat{m}_i) \in \bar{B}_i$  arbitrarily. Then  $(\hat{s}_i, \hat{m}_i) \in B_i^k$  (inductive hypothesis), and there is  $p_i(\cdot) \in \Delta^{\bar{H}_i}(S_{-i})$  with  $\hat{s}_i \in r_i(p_i(\cdot), \phi_i)$  such that eq. (10) holds. By the inductive hypothesis,  $\bar{B}_{-i} \subseteq B_{-i}^k$ ; hence

$$\begin{aligned} & p_i \left( F_{\hat{s}_i}^{-1}(\hat{m}_i) \cap \left\{ s_{-i} : (s_j, F_j(\hat{s}_i, s_{-i}))_{j \neq i} \in B_{-i}^k \right\} \mid \{\emptyset\} \right) \\ & \geq p_i \left( F_{\hat{s}_i}^{-1}(\hat{m}_i) \cap \left\{ s_{-i} : (s_j, F_j(\hat{s}_i, s_{-i}))_{j \neq i} \in \bar{B}_{-i} \right\} \mid \{\emptyset\} \right), \end{aligned}$$

and (10) implies

$$p_i \left( F_{\hat{s}_i}^{-1}(\hat{m}_i) \cap \left\{ s_{-i} : (s_j, F_j(\hat{s}_i, s_{-i}))_{j \neq i} \in B_{-i}^k \right\} \mid \{\emptyset\} \right) = 1.$$

Therefore,  $(\hat{s}_i, \hat{m}_i) \in B_i^{k+1}$ . ■

**Proof of Lemma 2.** The statement is trivially true for  $k = 0$ . Suppose, by way of induction, that

$$\text{symSCE}^k(\Gamma, f, \phi) = \left\{ \bar{s} : (\bar{s}_i, F_i(\bar{s}))_{i \in I} \in B^k \right\}.$$

We first show that, for every fixed  $\bar{s}$  in the above set and  $i \in I$

$$F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i} = F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \left\{ s_{-i} : (s_j, F_j(\bar{s}_i, s_{-i}))_{j \in I \setminus \{i\}} \in B_{-i}^k \right\}. \quad (18)$$

(Proof of  $\subseteq$ ) By definition of section and the inductive hypothesis,

$$\begin{aligned} \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i} &= \left\{ s_{-i} : (\bar{s}_i, s_{-i}) \in \text{symSCE}^k(\Gamma, f, \phi) \right\} \\ &= \left\{ s_{-i} : ((\bar{s}_i, F_i(\bar{s}_i, s_{-i})), (s_j, F_j(\bar{s}_i, s_{-i}))_{j \in I \setminus \{i\}}) \in B^k \right\} \\ &\subseteq \left\{ s_{-i} : (s_j, F_j(\bar{s}_i, s_{-i}))_{j \neq i} \in B_{-i}^k \right\}. \end{aligned}$$

Hence

$$F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i} \subseteq F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \left\{ s_{-i} : (s_j, F_j(\bar{s}_i, s_{-i}))_{j \in I \setminus \{i\}} \in B_{-i}^k \right\}.$$

(Proof of  $\supseteq$ ) Let  $\hat{s}_{-i} \in F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \left\{ s_{-i} : (s_j, F_j(\bar{s}_i, s_{-i}))_{j \in I \setminus \{i\}} \in B_{-i}^k \right\}$ . Since  $\hat{s}_{-i} \in F_{\bar{s}_i}^{-1}(F_i(\bar{s}))$  and  $(\bar{s}_i, F_i(\bar{s})) \in B_i^k$ , then  $F_i(\bar{s}_i, \hat{s}_{-i}) = F_i(\bar{s})$  and

$$((\bar{s}_i, F_i(\bar{s}_i, \hat{s}_{-i})), (\hat{s}_j, F_j(\bar{s}_i, \hat{s}_{-i}))_{j \in I \setminus \{i\}}) \in B^k.$$

Hence,  $\hat{s}_{-i} \in \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i}$ . Thus

$$F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \left\{ s_{-i} : (s_j, F_j(\bar{s}_i, s_{-i}))_{j \in I \setminus \{i\}} \in B_{-i}^k \right\} \subseteq F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i}.$$

This completes the proof of (18).  $\square$

Now, let  $\bar{s} \in \text{symSCE}^{k+1}(\Gamma, f, \phi)$ ; then – by definition –  $\bar{s} \in \text{symSCE}^k(\Gamma, f, \phi)$  and the inductive hypothesis implies that  $((\bar{s}_i, F_i(\bar{s})))_{i \in I} \in B^k$ . Let  $(p_i(\cdot|\cdot))_{i \in I}$  be as in the definition of  $\text{symSCE}^{k+1}(\Gamma, f, \phi)$ . Then, by eq. (18),

$$p_i \left( F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \left\{ s_{-i} : (s_j, F_j(\bar{s}_i, s_{-i}))_{j \in I \setminus \{i\}} \in B_{-i}^k \right\} \mid \{\emptyset\} \right) = 1$$

for every  $i \in I$ , and  $(\bar{s}_i, F_i(\bar{s}))_{i \in I} \in B^{k+1}$ . Similarly, let  $(\bar{s}_i, F_i(\bar{s}))_{i \in I} \in B^{k+1}$ ; then, by eq. (18),

$$p_i \left( F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i} \mid \{\emptyset\} \right) = 1$$

for every  $i \in I$ , and  $\bar{s} \in \text{symSCE}^{k+1}(\Gamma, f, \phi)$ .  $\blacksquare$

Our results about monotonicity of the RSCE correspondence rely on the following result of *monotonicity of the justifiability correspondence*: Consider a decision problem under uncertainty  $(\hat{A}, \hat{S}, \hat{u})$  where  $\hat{A}$  and  $\hat{S}$  are finite sets of actions and states, and  $\hat{u} : \hat{A} \times \hat{S} \rightarrow \mathbb{R}$  is a vN-M utility function. Fix a non-empty and *compact* set of distributions  $\hat{\Sigma} \subseteq \Delta(\hat{S})$ . Let  $\mathcal{IC} \subseteq \mathbb{R}^{\hat{S}}$  denote the set of *strictly increasing* and *continuous* functions. For each  $\phi \in \mathcal{IC}$  and  $\mu \in \Delta(\hat{\Sigma})$  define the set of  $(\mu, \phi)$ -best replies:

$$\hat{r}(\mu, \phi) = \arg \max_{a \in \hat{A}} \int_{\hat{\Sigma}} \phi(\mathbb{E}_{\sigma} [\hat{u}(a, \cdot)]) \mu(d\sigma).$$

Similarly, for each  $p \in \Delta(\hat{S})$  we write

$$\hat{r}(p, \phi) = \arg \max_{a \in \hat{A}} \sum_{s \in \hat{S}} \phi(\hat{u}(a, s)) p(s),$$

which is the special case when  $\mu$  is supported by Dirac beliefs and  $p$  is the corresponding predictive belief. If an action is a  $(\mu, \phi)$ -best reply we say that it is  **$\phi$ -justified** by  $\mu$ ; we say that it is  **$\phi$ -justifiable** if is  $\phi$ -justified by some  $\mu$ . Battigalli et al. (2016a) proved that the  $\phi$ -justifiability correspondence is monotone with respect to concave and strictly increasing transformations (see also Weinstein, 2016):

**Lemma 10** *For all  $\bar{\phi}, \tau \in \mathcal{IC}$ , if  $\tau \in \mathcal{IC}$  is a concave transformation then*

$$\bigcup_{\mu \in \Delta(\hat{\Sigma})} \hat{r}(\mu, \tau \circ \bar{\phi}) \supseteq \bigcup_{\bar{\mu} \in \Delta(\hat{\Sigma})} \hat{r}(\bar{\mu}, \bar{\phi}).$$

Since  $\hat{S} \cong \{\delta_s : s \in \hat{S}\} \subseteq \Delta(\hat{S})$ , letting  $\hat{\Sigma} = \{\delta_s : s \in \hat{S}\}$  in Lemma 10 we obtain the following:

**Corollary 15** For all  $\bar{\phi}, \tau \in \mathcal{IC}$ , if  $\tau \in \mathcal{IC}$  is a concave transformation then

$$\bigcup_{p \in \Delta(\hat{S})} \hat{r}(p, \tau \circ \bar{\phi}) \supseteq \bigcup_{\bar{p} \in \Delta(\hat{S})} \hat{r}(\bar{p}, \bar{\phi}).$$

From now on, whenever we refer to games in which every player moves at most once along every path, strategies will be omitted from the formulas of values because, in this class of games, the value of an action at an information set depends only on the action itself and not on the overall strategy of the agent.

**Proof of Theorem 11.** To prove the result, we will show that

$$\forall k \in \mathbb{N}, \text{symSCE}^k(\Gamma, f, \bar{\phi}) \subseteq \text{symSCE}^k(\Gamma, f, \phi).$$

Then the claim follows from Lemma 2. The statement is trivially true for  $k = 0$ . Suppose, by way of induction, that

$$\text{symSCE}^k(\Gamma, f, \bar{\phi}) \subseteq \text{symSCE}^k(\Gamma, f, \phi). \quad (\text{I.H.})$$

Fix  $\bar{s} \in \text{symSCE}^{k+1}(\Gamma, f, \bar{\phi})$  and  $i \in I$  arbitrarily. By definition of  $\text{symSCE}^{k+1}$  and the inductive hypothesis, there is  $\bar{p}_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(S_{-i})$  such that  $\bar{s}_i$  is fully  $(\bar{p}_i(\cdot|\cdot), \bar{\phi}_i)$ -unimprovable and

$$\text{Supp} \bar{p}_i(\cdot|\{\emptyset\}) \subseteq \left( F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \bar{\phi})_{\bar{s}_i} \right) \stackrel{(\text{I.H.})}{\subseteq} \left( F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i} \right).$$

We construct  $p_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$  such that  $p_i(\cdot|h) = \bar{p}_i(\cdot|h)$  for each  $h \in \bar{H}_i(\bar{p}_i(\cdot|\{\emptyset\}))$  and  $\bar{s}_i$  is fully  $(p_i(\cdot|\cdot), \phi_i)$ -unimprovable. Since

$$\text{Supp} p_i(\cdot|\{\emptyset\}) = \text{Supp} \bar{p}_i(\cdot|\{\emptyset\}) \subseteq \left( F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \phi)_{\bar{s}_i} \right)$$

and the construction holds for each  $i$ , this implies that  $\bar{s} \in \text{symSCE}^{k+1}(\Gamma, f, \phi)$ .

(Construction of  $p_i(\cdot|\cdot)$ ) Since  $i$  moves at most once on every path,  $H_i(s_i) = H_i$ ; furthermore, the value of any action  $a_i \in A_i(h)$  ( $h \in H_i$ ) is independent of  $i$ 's strategy. To construct  $p_i(\cdot|\cdot)$ , keep the same initial belief as  $\bar{p}_i$ :  $p_i(\cdot|\{\emptyset\}) = \bar{p}_i(\cdot|\{\emptyset\})$ . By symmetry (pure equilibrium), no chance moves, confirmed beliefs and ex-post perfect recall, there is a unique  $h \in H_i(\bar{p}_i(\cdot|\{\emptyset\})) = H_i(p_i(\cdot|\{\emptyset\}))$  that  $i$  expects to reach with probability 1 under  $\bar{p}_i$ . Thus, let  $p_i(\cdot|h) = \bar{p}_i(\cdot|\{\emptyset\}) = \bar{p}_i(\cdot|h)$  for  $h \in H_i(\bar{p}_i(\cdot|\{\emptyset\}))$  even if  $h \neq \{\emptyset\}$ . It follows that for the unique  $h \in H_i(p_i(\cdot|\{\emptyset\}))$  and for every  $a_i \in A_i(h)$ ,

$$\begin{aligned} V_i(\bar{s}_{i,h}|h; p_i, \phi_i) &\stackrel{(\text{L4})}{=} U_i(\bar{s}) \\ &\stackrel{(\text{L4})}{=} V_i(\bar{s}_{i,h}|h; p_i, \bar{\phi}_i) \\ &\stackrel{(\text{uprv.})}{\geq} V_i(a_i|h; p_i, \bar{\phi}_i) \\ &\stackrel{(\text{L5})}{\geq} V_i(a_i|h; p_i, \phi_i), \end{aligned}$$

where the equalities follow from  $h \in H_i(\bar{p}_i(\cdot | \{\emptyset\})) = H_i(p_i(\cdot | \{\emptyset\}))$  and (given the observability of payoffs) Lemma 4, the first inequality follows from  $(p_i(\cdot | \{\emptyset\}), \bar{\phi}_i)$ -unimprovability of  $\bar{s}_i$  (uprv.) and the second one from Lemma 5.

Now, consider any  $h \in H_i \setminus H_i(\bar{p}_i(\cdot | \{\emptyset\}))$ . Since  $\bar{s}_i$  is fully  $(\bar{p}_i(\cdot | \cdot), \bar{\phi}_i)$ -unimprovable, we have

$$\bar{s}_{i,h} \in \arg \max_{a_i \in A_i(h)} V_i(a_i | h; \bar{p}_i, \bar{\phi}_i) = \arg \max_{a_i \in A_i(h)} \mathbb{E}_{\bar{p}_i(\cdot | h)} [\bar{\phi}_i \circ U_{h,a_i}].$$

By Corollary 15 there exists some  $p_{i,h} \in \Delta(S_{-i}(h))$  such that

$$\bar{s}_{i,h} \in \arg \max_{a_i \in A_i(h)} \mathbb{E}_{p_{i,h}} [\varphi \circ \bar{\phi}_i \circ U_{h,a_i}] = \arg \max_{a_i \in A_i(h)} \mathbb{E}_{p_{i,h}} [\phi_i \circ U_{h,a_i}].$$

Let  $p_i(\cdot | h) = p_{i,h}$ . By the one-move assumption,  $h$  has no strict predecessors or followers in  $H_i$ . Then, the array  $(p_i(\cdot | h))_{h \in \bar{H}_i}$  is a CPS. By construction  $\bar{s}_i$  is fully  $(p_i(\cdot | \cdot), \phi_i)$ -unimprovable.  $\square$

We conclude that  $\bar{s} \in \text{symSCE}^{k+1}(\Gamma, f, \phi)$ .  $\blacksquare$

**Proof of Theorem 12.** Throughout the proof, for each  $s = (s_j)_{j \in I}$ , let  $[s] = \times_{j \in I} [s_j]$  and  $[s_{-i}] = \times_{j \neq i} [s_j]$ . To prove the result, we will show that for every  $k \in \mathbb{N}$  and  $\bar{s} \in \text{symSCE}^k(\Gamma, f, \bar{\phi})$ , there exists  $s \in \text{symSCE}^k(\Gamma, f, \phi)$  with  $s \in [\bar{s}]$ . Then the claim follows from Lemma 2. The statement is trivially true for  $k = 0$ .

Suppose, by way of induction, that the statement is true for  $k$ . Let

$$\bar{s} = (\bar{s}_i)_{i \in I} \in \text{symSCE}^{k+1}(\Gamma, f, \bar{\phi}) \subseteq \text{symSCE}^k(\Gamma, f, \bar{\phi})$$

(the inclusion holds by definition). By the inductive hypothesis, for every  $i \in I$ , there exists some  $s^* = (s_i^*)_{i \in I} \in \text{symSCE}^k(\Gamma, f, \phi)$  such that  $s^* \in [\bar{s}]$ . Since  $\bar{s} \in \text{symSCE}^{k+1}(\Gamma, f, \bar{\phi})$ , for every  $i \in I$  there is some CPS  $\bar{p}_i(\cdot | \cdot) \in \Delta^{\bar{H}_i}(S_{-i})$  such that  $\bar{s}_i$  is fully  $(\bar{p}_i(\cdot | \cdot), \bar{\phi}_i)$ -unimprovable and

$$\text{Supp} \bar{p}_i(\cdot | \{\emptyset\}) \subseteq \left( F_{\bar{s}_i}^{-1}(F_i(\bar{s})) \cap \text{symSCE}^k(\Gamma, f, \bar{\phi})_{\bar{s}_i} \right).$$

It can be checked that, for any  $s = (s_i, s_{-i}) \in \text{symSCE}^k(\Gamma, f, \phi)$ , if  $s_i \in [s_i^*]$ , then  $(s_i^*, s_{-i}) \in \text{symSCE}^k(\Gamma, f, \phi)$  as well. Then, by the inductive hypothesis, for every  $s_{-i} \in \text{symSCE}^k(\Gamma, f, \bar{\phi})_{\bar{s}_i}$ , there exists  $s'_{-i} \in \text{symSCE}^k(\Gamma, f, \phi)_{s_i^*}$  such that  $s'_{-i} \in [s_{-i}]$ . Moreover, since strategic feedback depends only on the realization equivalence classes of strategies, for each  $s_{-i} \in F_{\bar{s}_i}^{-1}(F_i(\bar{s}))$  and  $s'_{-i} \in [s_{-i}]$ , we have  $s'_{-i} \in F_{s_i^*}^{-1}(F_i(s^*))$ . So, we can construct a belief  $p_i^* \in \Delta(S_{-i})$  with

$$\text{Supp} p_i^* \subseteq \left( F_{s_i^*}^{-1}(F_i(s^*)) \cap \text{symSCE}^k(\Gamma, f, \phi)_{s_i^*} \right)$$

such that for each  $\hat{s}_{-i} \in \text{Supp} p_i^*$ ,

$$p_i^*([\hat{s}_{-i}]) = \bar{p}_i([\hat{s}_{-i}] | \{\emptyset\}). \quad (19)$$

Since  $\bar{p}_i(\cdot|\{\emptyset\})$  is the predictive probability isomorphic to a belief over Dirac models, by Corollary 14 player  $i$ 's preferences are dynamically consistent. Thus,<sup>47</sup>

$$\bar{s}_i \in \arg \max_{s_i \in S_i} \sum_{s_{-i}} \bar{\phi}_i(U_i(s_i, s_{-i}) \bar{p}_i(s_{-i}|\{\emptyset\})).$$

But then, by eq. (19), we also have

$$\bar{s}_i \in \arg \max_{s_i \in S_i} \sum_{s_{-i}} \bar{\phi}_i(U_i(s_i, s_{-i}) p_i^*(s_{-i})). \quad (20)$$

Consider the decision problem  $(\hat{A}, \hat{S}, \hat{u})$  where  $\hat{A} = S_i$ ,  $\hat{S} = F_{s_i^*}^{-1}(F_i(s^*)) \cap \text{symSCE}^k(\Gamma, f, \phi)_{s_i^*}$ , and  $\hat{u}$  is the restriction of  $\bar{\phi}_i \circ U_i$  on  $\hat{A} \times \hat{S}$ . By eq. (20),  $\bar{s}_i$  is  $\bar{\phi}_i$ -justified by  $p_i^* \in \Delta(\hat{S})$ . By Corollary 15, there exists  $p_i \in \Delta(\hat{S})$  that  $\phi_i$ -justifies  $\bar{s}_i$ . Let  $p_i(\cdot|\{\emptyset\}) = p_i$ . Then

$$p_i(\cdot|\{\emptyset\}) \in \Delta\left(F_{s_i^*}^{-1}(F_i(s^*)) \cap \text{symSCE}^k(\Gamma, f, \phi)_{s_i^*}\right)$$

and

$$\bar{s}_i \in \arg \max_{s_i \in S_i} \sum_{s_{-i}} \phi_i(U_i(s_i, s_{-i}) p_i(s_{-i}|\{\emptyset\})).$$

By dynamic consistency of player  $i$ 's preferences and standard arguments, this implies that for every  $s_i \in [\bar{s}_i]$ , for every  $h' \in H_i(p_i(\cdot|\{\emptyset\})) \cap H_i(\bar{s}_i)$ ,

$$s_{i,h'} \in \arg \max_{a_i \in A_i(h')} \sum_{s_{-i}} \phi_i(U_i((s_i|_{h'}, a_i), s_{-i}|h') p_i(s_{-i}|h')),$$

where  $p_i(s_{-i}|h')$  is derived by conditioning.

Next, fix  $h \in H_i(\bar{s}_i) \setminus H_i(p_i(\cdot|\{\emptyset\}))$  with  $h' \in H_i(p_i(\cdot|\{\emptyset\}))$  for all  $h' \prec h$ . By dynamic consistency, we can frame the continuation of the game as a (static) decision problem  $(\hat{A}, \hat{S}, \hat{u})$ , with  $\hat{A} = S_i(h)$ ,  $\hat{S} = S_{-i}(h)$  and  $\hat{u}$  is the restriction of  $\bar{\phi}_i \circ U_i$  on  $\hat{A} \times \hat{S} = S_i(h) \times S_{-i}(h)$ . By the same argument as above, since  $\bar{s}_i$  is  $\bar{\phi}_i$ -justified by  $\bar{p}_i(\cdot|h)$ , Corollary 15 implies that  $\bar{s}_i$  is  $\phi_i$ -justified by some  $p_i(\cdot|h) \in \Delta(S_{-i}(h))$ . By dynamic consistency of player  $i$ 's preferences and standard arguments, this implies that for every  $s_i \in [\bar{s}_i]$ , for every  $h' \in H_i(p_i(\cdot|h)) \cap H_i(\bar{s}_i)$  with  $h \preceq h'$ ,

$$s_{i,h'} \in \arg \max_{a_i \in A_i(h')} \sum_{s_{-i}} \phi_i(U_i((s_i|_{h'}, a_i), s_{-i}|h') p_i(s_{-i}|h')),$$

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<sup>47</sup>Recall that, for every  $p \in \Delta(S)$  and every strictly increasing function  $\phi$ ,

$$\arg \max_{s_i \in S_i} \phi^{-1}(\mathbb{E}_p[\phi(U_{i,s_i})]) = \arg \max_{s_i \in S_i} \mathbb{E}[\phi(U_{i,s_i})].$$

where  $p_i(s_{-i}|h')$  is derived from  $p_i(s_{-i}|h)$  by conditioning.

Repeating iteratively the operation at all information sets that are not reached with positive probability under the probability measures already constructed, one can finally construct a CPS  $\hat{p}_i(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$  such that  $\hat{p}_i(\cdot|h) = p_i(\cdot|h)$  for each  $h \in H_i(\bar{s}_i)$  with  $p_i(S_{-i}(h)|h') = 0$  for all  $h' \prec h$ . Since every  $s_i \in [\bar{s}_i]$  is  $(\hat{p}_i(\cdot|\cdot), \phi_i)$ -unimprovable at every  $h \in H_i(\bar{s}_i)$ , there exists a fully  $(\hat{p}_i(\cdot|\cdot), \phi_i)$ -unimprovable  $s_i \in [\bar{s}_i] = [s_i^*]$ . Let  $s = (s_i)_{i \in I}$ . By construction,

$$\hat{p}_i(\cdot|\{\emptyset\}) \in \Delta \left( F_{s_i^*}^{-1}(F_i(s^*)) \cap \text{symSCE}^k(\Gamma, f, \phi)_{s_i^*} \right).$$

By  $s_i \in [s_i^*]$  and  $s \in [s^*]$ ,  $F_{s_i^*}^{-1}(F_i(s^*)) = F_{s_i}^{-1}(F_i(s))$ . So,  $s \in \text{symSCE}^1(\Gamma, f, \phi)$ . If  $k > 0$ , for all  $s'_{-i} \in S_{-i}$ ,  $(s_i^*, s'_{-i}) \in \text{symSCE}^1(\Gamma, f, \phi)$  if and only if  $(s_i, s'_{-i}) \in \text{symSCE}^1(\Gamma, f, \phi)$ . So,  $\text{symSCE}^1(\Gamma, f, \phi)_{s_i^*} = \text{symSCE}^1(\Gamma, f, \phi)_{s_i}$ . But then,  $\hat{p}_i(\cdot|\{\emptyset\}) \in \Delta \left( F_{s_i}^{-1}(F_i(s)) \cap \text{symSCE}^1(\Gamma, f, \phi)_{s_i} \right)$ , and so  $s \in \text{symSCE}^2(\Gamma, f, \phi)$ . Inductively,

$$\hat{p}_i(\cdot|\{\emptyset\}) \in \Delta \left( F_{s_i}^{-1}(F_i(s)) \cap \text{symSCE}^k(\Gamma, f, \phi)_{s_i} \right),$$

and so  $s \in \text{symSCE}^{k+1}(\Gamma, f, \phi)$ . ■

### 10.5.3 Monotonicity of the RSCE correspondence

We first use Lemma 10 to prove a preliminary monotonicity result for the set of fully unimprovable SCEs. Here we reintroduce chance moves.

**Lemma 11** *Fix two games with observable payoffs where no player moves more than once,  $(\Gamma, f, \phi)$  and  $(\Gamma, f, \bar{\phi})$ , so that  $(\Gamma, f, \phi)$  features more ambiguity aversion than  $(\Gamma, f, \bar{\phi})$ . Then  $\text{SCE}^1(\Gamma, f, \bar{\phi}) \subseteq \text{SCE}^1(\Gamma, f, \phi)$ .*

**Proof of Lemma 11.** Let  $\bar{\sigma} \in \text{SCE}^1(\Gamma, f, \bar{\phi})$ . Fix  $i \in I$ ,  $s_i \in \text{Supp} \bar{\sigma}_i$  and let  $\bar{\mu}_{s_i}(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$  be such that  $s_i$  is fully  $(\bar{\mu}_{s_i}(\cdot|\cdot), \bar{\phi}_i)$ -unimprovable and

$$\text{Supp} \bar{\mu}_{s_i}(\cdot|\{\emptyset\}) \subseteq \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) \stackrel{(\text{def.})}{=} \left\{ \sigma_{-i} \in \Sigma_{-i} : \hat{F}_i(s_i, \sigma_{-i}) = \hat{F}_i(s_i, \bar{\sigma}_{-i}) \right\}.$$

We will construct  $\mu_{s_i}(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$  such that  $\mu_{s_i}(\cdot|h) = \bar{\mu}_{s_i}(\cdot|h)$  for each  $h \in \bar{H}_i(\bar{\mu}_{s_i}(\cdot|\{\emptyset\}))$ , and  $s_i$  is fully  $(\mu_{s_i}(\cdot|\cdot), \phi_i)$ -unimprovable. Since

$$\text{Supp} \bar{\mu}_{s_i}(\cdot|\{\emptyset\}) = \text{Supp} \mu_{s_i}(\cdot|\{\emptyset\}) \subseteq \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}),$$

this implies that  $\bar{\sigma} \in \text{SCE}^1(\Gamma, f, \phi)$ .

Recall that  $H_i(s_i) = H_i$  because  $i$  moves at most once on every path. To construct  $\mu_{s_i}(\cdot|\cdot)$ , keep the same prior belief:  $\mu_{s_i}(\cdot|\{\emptyset\}) = \bar{\mu}_{s_i}(\cdot|\{\emptyset\})$ . For each  $h \in H_i(\bar{\mu}_{s_i}(\cdot|\{\emptyset\})) =$

$H_i(\mu_{s_i}(\cdot | \{\emptyset\}))$  define  $\mu_{s_i}(\cdot | h)$  by conditioning. Hence, for each  $h \in \bar{H}_i(\bar{\mu}_{s_i}(\cdot | \{\emptyset\}))$ ,  $\mu_{s_i}(\cdot | h) = \bar{\mu}_{s_i}(\cdot | h)$ . Thus, for every  $a_i \in A_i(h)$ ,

$$\begin{aligned} V_i(s_{i,h}|h; \mu_{s_i}, \phi_i) &\stackrel{(L4)}{=} U_i(s_i, \bar{\sigma}_{-i}|h) \\ &\stackrel{(L4)}{=} V_i(s_{i,h}|h; \mu_{s_i}, \bar{\phi}_i) \\ &\stackrel{(\text{uprv.})}{\geq} V_i(a_i|h; \mu_{s_i}, \bar{\phi}_i) \\ &\stackrel{(L5)}{\geq} V_i(a_i|h; \mu_{s_i}, \phi_i), \end{aligned}$$

where the equalities follow from Lemma 4, the first inequality from full  $(\bar{\mu}_{s_i}(\cdot | \cdot), \bar{\phi}_i)$ -unimprovability of  $s_i$  (uprv.), and the second one from Lemma 5.

Now, consider any  $h \in H_i \setminus H_i(\bar{\mu}_{s_i}(\cdot | \{\emptyset\}))$ . Since  $s_i$  is fully  $(\bar{\mu}_{s_i}(\cdot | \cdot), \bar{\phi}_i)$ -unimprovable and  $\bar{\mu}_{s_i}(\Sigma_{-i}(h)|h) = 1$  by condition (1) of Definition 7, we have

$$\begin{aligned} s_{i,h} &\in \arg \max_{a_i \in A_i(h)} \int_{\Sigma_{-i}} \bar{\phi}_i(U_i((s_i|h), a_i), \sigma_{-i}|h) \bar{\mu}_{s_i}(d\sigma_{-i}|h) \\ &= \arg \max_{a_i \in A_i(h)} \int_{\Sigma_{-i}(h)} \bar{\phi}_i \left( \sum_{s_{-i} \in S_{-i}(h)} \sigma_{-i}(s_{-i}|h) u_i(\zeta(a_i, s_{-i})) \right) \bar{\mu}_{s_i}(d\sigma_{-i}|h), \end{aligned} \quad (21)$$

where we abuse notation and write  $\zeta(a_i, s_{-i})$ , because  $i$  does not move before  $h$ , nor after  $h$ , hence the terminal node reached depends only on  $s_{-i} \in S_{-i}(h)$  and the action chosen by  $i$  at  $h$ . Next, we consider the problem of choice under uncertainty  $(\hat{A}, \hat{S}, \hat{u})$  where  $\hat{A} = A_i(h)$ ,  $\hat{S} = S_{-i}(h)$ ,  $\hat{u}(a_i, s_{-i}) = u_i(\zeta(a_i, s_{-i}))$ . Recall that we assumed *observable deviators*, hence

$$S_{-i}(h) = \times_{j \in I_0 \setminus \{i\}} S_j(h).$$

Under this assumption, each conditional measure  $\sigma_{-i}(\cdot | h)$  is a product measure. Therefore, the set of conditional measures given  $h$  is a subset of  $\Sigma_{-i}(h)$ . With this, consider the following compact set of product distributions

$$\Sigma_{-i|h} = \{\sigma_{-i} \in \Sigma_{-i} : \sigma_{-i}(S_{-i}(h)) = 1\} \subseteq \Sigma_{-i}(h),$$

that will play the role of subset  $\hat{\Sigma}$  in Lemma 10. The following map associates each  $\sigma_{-i} \in \Sigma_{-i}(h)$  with the corresponding updated distribution defined in eq. (1):

$$\begin{aligned} \varsigma_{h,-i} : \Sigma_{-i}(h) &\rightarrow \Sigma_{-i|h} \\ \sigma_{-i} &\mapsto \mathbf{1}_{S_{-i}(h)}(\cdot) \frac{\sigma_{-i}(\cdot)}{\sigma_{-i}(S_{-i}(h))} \end{aligned}$$

$[\mathbf{1}_{S_{-i}(h)}]$  denotes the indicator function of set  $S_{-i}(h)$ . Note that  $\Sigma_{-i}(h)$  is a (relatively) open subset of the Polish space  $\Sigma_{-i}$  and  $\varsigma_{h,-i}$  is continuous. Furthermore, the restriction



$\varsigma_{h,-i}|_{\Sigma_{-i}|h}$  is the identity on  $\Sigma_{-i}|h$ , because  $\sigma_{-i}(S_{-i}(h)) = 1$  implies

$$\sigma_{-i}(s_{-i}|h) = \frac{\sigma_{-i}(s_{-i})}{\sigma_{-i}(S_{-i}(h))} = \sigma_{-i}(s_{-i})$$

for each  $s_{-i}$ . Therefore  $\varsigma_{h,-i}$  is also onto and  $\varsigma_{h,-i}$  is a measurable surjection that yields the *onto* pushforward map

$$\begin{aligned} \hat{\varsigma}_{h,-i} : \Delta(\Sigma_{-i}(h)) &\rightarrow \Delta(\Sigma_{-i}|h) \\ \mu_i &\mapsto \mu_i \circ \varsigma_{h,-i}^{-1}. \end{aligned}$$

Let  $\bar{\mu}_{s_i|h} = \bar{\mu}_{s_i}(\cdot|h) \circ \varsigma_{h,-i}^{-1} \in \Delta(\Sigma_{-i}|h)$ , that is,

$$\forall E \in \mathcal{B}(\Sigma_{-i}|h), \quad \bar{\mu}_{s_i|h}(E) = \bar{\mu}_{s_i}(\varsigma_{h,-i}^{-1}(E)|h).$$

For every  $a_i \in A_i(h)$ , we have:

$$\begin{aligned} \int_{\Sigma_{-i}(h)} \bar{\phi}_i \left( \sum_{s_{-i} \in S_{-i}(h)} \sigma_{-i}(s_{-i}|h) v_i(\gamma(\zeta(a_i, s_{-i}))) \right) \bar{\mu}_{s_i}(\mathrm{d}\sigma_{-i}|h) = \\ \int_{\Sigma_{-i}|h} \bar{\phi}_i \left( \sum_{s_{-i} \in S_{-i}(h)} \sigma_{-i}(s_{-i}|h) \hat{u}(a_i, s_{-i}) \right) \bar{\mu}_{s_i|h}(\mathrm{d}\sigma_{-i}(\cdot|h)). \end{aligned}$$

Lemma 10, eq. (21), and the above equality imply that  $s_{i,h}$  is  $\phi_i$ -justified by some belief  $\mu_i|h \in \Delta(\Sigma_{-i}|h)$ , that is,

$$s_{i,h} \in \arg \max_{a_i \in A_i(h)} \int_{\Sigma_{-i}|h} \phi_i \left( \sum_{s_{-i} \in S_{-i}(h)} \sigma_{-i}(s_{-i}|h) \hat{u}(a_i, s_{-i}) \right) \mu_i|h(\mathrm{d}\sigma_{-i}(\cdot|h)).$$

Now go back to a belief on  $\Sigma_{-i}(h)$ : Since the pushforward map  $\hat{\varsigma}_{h,-i}$  is onto we can find some belief  $\mu_{s_i}(\cdot|h)$  with  $\mu_{s_i}(\cdot|h) \in \hat{\varsigma}^{-1}(\mu_i|h) \subseteq \Delta(\Sigma_{-i}(h))$  such that

$$\begin{aligned} s_{i,h} &\in \arg \max_{a_i \in A_i(h)} \int_{\Sigma_{-i}(h)} \phi_i \left( \sum_{s_{-i} \in S_{-i}(h)} \sigma_{-i}(s_{-i}|h) \hat{u}(a_i, s_{-i}) \right) \mu_{s_i}(\mathrm{d}\sigma_{-i}(\cdot|h)) \\ &= \arg \max_{a_i \in A_i(h)} \int_{\Sigma_{-i}(h)} \phi_i(U_i(a_i, \sigma_{-i}|h)) \mu_{s_i}(\mathrm{d}\sigma_{-i}|h). \end{aligned}$$

We can do this for every off-path information set  $h \in H_i \setminus H_i(\bar{\mu}_{s_i}(\cdot|\{\emptyset\}))$  and obtain an array  $(\mu_{s_i}(\cdot|h))_{h \in \bar{H}_i} \in \times_{h \in \bar{H}_i} \Delta(\Sigma_{-i}(h))$  (recall that  $\mu_{s_i}$  coincides with  $\bar{\mu}_{s_i}$  on  $H_i(\bar{\mu}_{s_i}(\cdot|\{\emptyset\}))$ ) that satisfies condition (1) of the definition of CPS by construction. Since

every player moves at most once on every path, condition (2) trivially holds. Finally, by construction,  $s_i$  is fully  $(\mu_i(\cdot|\cdot), \phi_i)$ -unimprovable. ■

**Comment:** What matters in the previous proof is that the range  $\varsigma_{h,-i}(\Sigma_{-i}(h))$  of the continuous map  $\varsigma_{h,-i}$  is a compact subset of  $\Delta(S_{-i}(h))$  even though  $\Sigma_{-i}(h)$  is not closed. We used the assumption of observable deviators to prove this fact. There are simple examples of games without observable deviators where  $\varsigma_{h,-i}(\Sigma_{-i}(h))$  is not compact and therefore we cannot apply Lemma 10. However, we have not yet found examples of games without observable deviators where players move at most once and monotonicity of  $\phi \mapsto SCE^1(\Gamma, f, \bar{\phi})$  (hence, monotonicity of RSCE) does not hold.

**Proof of Theorem 13.** Lemma 11 shows that the result holds for  $k = 1$ . Suppose by way of induction that  $SCE^k(\Gamma, f, \bar{\phi}) \subseteq SCE^k(\Gamma, f, \phi)$ . Fix  $\bar{\sigma} \in SCE^{k+1}(\Gamma, f, \bar{\phi})$ ,  $i \in I$ ,  $s_i \in \text{Supp}\bar{\sigma}_i$  arbitrarily and let  $\bar{\mu}_{s_i}(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$  be a confirmed CPS that  $\bar{\phi}_i$ -justifies  $s_i$  (that is,  $s_i$  is fully  $(\bar{\mu}_{s_i}(\cdot|\cdot), \bar{\phi})$ -unimprovable) with

$$\bar{\mu}_{s_i} \left( \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) \cap \psi_{-i}(\text{proj}_{\Sigma_{-i}}(\{\bar{\sigma}_0\} \times SCE^k(\Gamma, f, \bar{\phi}))) | \{\emptyset\} \right) = 1.$$

Since  $\bar{\sigma} \in SCE^{k+1}(\Gamma, f, \bar{\phi}) \subseteq SCE^k(\Gamma, f, \bar{\phi})$ , the inductive hypothesis implies  $\bar{\sigma} \in SCE^k(\Gamma, f, \phi)$ , so there is  $\hat{\mu}_{s_i}(\cdot|\cdot) \in \Delta^{\hat{H}_i}(\Sigma_{-i})$  such that  $s_i$  is fully  $(\hat{\mu}_{s_i}(\cdot|\cdot), \phi_i)$ -unimprovable. Let  $\mu_{s_i}(\cdot|\cdot) \in \Delta^{\bar{H}_i}(\Sigma_{-i})$  be such that  $\mu_{s_i}(\cdot|h) = \bar{\mu}_{s_i}(\cdot|h)$  for each  $h \in \bar{H}_i(\bar{\mu}_{s_i}(\cdot|\{\emptyset\}))$  and  $\mu_{s_i}(\cdot|h) = \hat{\mu}_{s_i}(\cdot|h)$  for each  $h \in H_i \setminus \bar{H}_i(\bar{\mu}_{s_i}(\cdot|\{\emptyset\}))$ . By same argument employed in the proof of Lemma 11,  $\mu_{s_i}(\cdot|\cdot)$  is a CPS and  $s_i$  is  $(\mu_{s_i}(\cdot|\cdot), \phi_i)$ -unimprovable. Moreover, by construction and the inductive hypothesis,

$$\begin{aligned} & \mu_{s_i} \left( \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) \cap \psi_{-i}(\text{proj}_{\Sigma_{-i}}(\{\bar{\sigma}_0\} \times SCE^k(\Gamma, f, \phi))) | \{\emptyset\} \right) \\ & \geq \mu_{s_i} \left( \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) \cap \psi_{-i}(\text{proj}_{\Sigma_{-i}}(\{\bar{\sigma}_0\} \times SCE^k(\Gamma, f, \bar{\phi}))) | \{\emptyset\} \right) \\ & = \bar{\mu}_{s_i} \left( \hat{\Sigma}_{-i}(s_i, \bar{\sigma}_{-i}) \cap \psi_{-i}(\text{proj}_{\Sigma_{-i}}(\{\bar{\sigma}_0\} \times SCE^k(\Gamma, f, \bar{\phi}))) | \{\emptyset\} \right) = 1. \end{aligned}$$

Therefore  $s_i$  is  $\phi_i$ -justified by a confirmed CPS that initially believes  $\text{proj}_{\Sigma_{-i}}(\{\bar{\sigma}_0\} \times SCE^k(\Gamma, f, \phi))$ . This holds for every  $i \in I$  and  $\text{Supp}\bar{\sigma}_i$ , thus,  $\bar{\sigma} \in SCE^{k+1}(\Gamma, f, \phi)$ . ■

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