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Working Paper n. 660

This Version: March, 2020

IGIER – Università Bocconi, Via Guglielmo Röntgen 1, 20136 Milano –Italy
http://www.igier.unibocconi.it

Information, Liquidity, and Dynamic Limit Order Markets*

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March 1, 2020

Abstract

This paper describes price discovery and liquidity provision in a dynamic limit order market with asymmetric information and non-Markovian learning. Investors condition on information in both the current limit order book and also, unlike in previous research, on the prior order history when deciding whether to provide or take liquidity. Our analysis shows that the information content of the prior order history can be substantial. Surprisingly, the information content of equilibrium orders can differ from order direction and aggressiveness.

JEL classification: G10, G20, G24, D40

Keywords: Limit order markets, asymmetric information, liquidity, market microstructure

*We have benefited from helpful comments from Pierpaolo Battigalli, Alexander David, Sandra Fortini, Thierry Foucault, Paolo Giacomazzi, Burton Hollifield, Phillip Illeditsch, Alfred Lehar, Stefan Lewellen, Katya Malinova, Marco Ottaviani, Uday Rajan, Ioanid Roşu, Tom Ruchti, Gunther Wuyts, Hiaoxiong Zhu, and seminar participants at Carnegie Mellon University, the University of Calgary, and at the 2018 NYU Microstructure Conference, 2018 WFA meeting, 2018 SAFE Market Microstructure Conference, and 2019 AFA. We are grateful to Fabio Sist for his significant contribution to the computer code for our model.

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The aggregation of asymmetric information and the dynamics of liquidity supply and demand are closely intertwined in financial markets. In dealer markets, informed and uninformed investors trade via market orders that take liquidity, while dealers provide liquidity and extract information from the arriving order flow (e.g., as in Kyle (1985) and Glosten and Milgrom (1985)). However, in limit order markets — the dominant form of securities market organization today — the relation between who has information and who is trying to learn it and who supplies and demands liquidity is not well understood theoretically.[1] Recent empirical research highlights the role of informed traders as liquidity suppliers as well as liquidity takers. O'Hara (2015) argues that fast informed traders use both market and limit orders. Fleming, Mizrach, and Nguyen (2017) and Brogaard, Hendershott, and Riordan (2019) find that limit orders reveal information.[2] Thus, understanding how and when informed, and uninformed, investors choose to trade using limit and market orders is important for market liquidity, price discovery, and investor welfare.

Our paper presents the first rational expectations model of a dynamic limit order market with asymmetric information and history-dependent Bayesian learning. In particular, learning is not constrained to be Markovian in the limit order book. In addition, we model a trading day with market opening and closing effects to investigate non-stationarity in intraday market dynamics. Our analysis leads to two main sets of results:

- Several standard intuitions about adverse selection can fail when informed and uninformed investors endogenously choose between limit and market orders. For example, increased adverse selection does not always worsen market liquidity as in Kyle (1985). Rather, liquidity can improve if informed traders trade more aggressively by submitting more limit orders at the inside quotes than by using market orders. In addition, the equilibrium information content of arriving orders can be opposite both order direction and order aggressiveness.

- Learning dynamics are non-Markovian in that the order history has information in addition to the current state of the limit order book.[3] In addition, the incremental

[1]See Jain (2005) about the prevalence of limit order markets. See Parlour and Seppi (2008) for a survey of theoretical models of limit order markets. See Rindi (2008) and Boulatov and George (2013) for models of informed traders as liquidity providers.

[2]Kacperczyk and Pagnotta (2019) and Garriott and Riordan (2019) find that directly identified informed traders empirically use limit orders frequently. Gencay, Mahmoodzadeh, Rojcek, and Tseng (2016) investigate brief episodes of extreme quotation behavior in the U.S. equity market (bursts in liquidity provision that happen several hundreds of time a day for actively traded stocks) and find that limit orders during these bursts significantly impact prices.

[3]To be clear about terminology, we say a stochastic process followed by a set of variables x is *non-Markovian* if the conditional probability distributions $Pr[x_s|\, x_t, x_{t-1}, \ldots]$ and $Pr[x_s|\, x_t]$ are different for some times t and $s > t$. If a summary function $a(x_{t-1}, \ldots)$ exists such that $Pr[(x_s, a(x_{s-1}, \ldots))|\,(x_t, a(x_{t-1}, \ldots)), (x_{t-1}, a(x_{t-2}, \ldots)), \ldots] = Pf[(x_s, a(x_{s-1}, \ldots))|(x_t, a(x_{t-1}, \ldots))]$, then we say the augmented process $\{x, a\}$ is Markovian but not that the unaugmented process $\{x\}$ is Markovian.

informational price-impact of arriving limit and market orders can vary conditional on time of day, the current standing limit order book, and the prior order history.

Much of our analysis takes the form of showing how the relationships between adverse selection, liquidity, price discovery, and welfare are induced by investor trading behavior and how their trading behavior changes with market conditions. For order submission and market quality, key drivers are i) the differential incentives for informed and uninformed investors to use limit orders as noted by Bloomfield, O'Hara, and Saar (2005), ii) the increased use of limit orders at aggressive limit prices by informed investors when adverse selection is stronger, and iii) the effect on investor order submissions when the standing limit order book is unbalanced with more depth on one side than the other. Our price discovery results follow from the fact that the information revealed by arriving orders depends on the mixture of informational and uninformed motives for submitting different market and limit orders. This mixture depends on i) the exogenous ex ante distributions over potential information and private gains-from-trade, ii) endogenous prior learning about realized value-shocks conditional on the history of past orders, iii) the endogenous state of the standing limit order book that defines the trading opportunities available to investors when they arrive, iv) the amount of time remaining over the day for limit orders to be executed, and v) endogenous investor choices about how to trade given changing trade-offs between trade profitability and execution probability given investors' realized information and private values. Despite the rich behavior that emerges in our model, however, our model structure is designed to be quite parsimonious.

Dynamic limit order markets without price discovery are studied in a large theoretical literature. This includes Foucault (1999), Parlour (1998), Foucault, Kadan, and Kandel (2005), Goettler, Parlour, and Rajan (2005) and Roşu (2009). Ait-Sahalia and Saglam (2017) model limit-order strategies for HFT market makers who do not have private valuation information. There is some prior theoretical research that allows informed traders to supply liquidity. Kumar and Seppi (1994) is a static model in which optimizing informed and uninformed investors use profiles of multiple limit and market orders to trade. Kaniel and Liu (2006) and Brolley and Malinova (2017) extend the Glosten and Milgrom (1985) dealership market to allow informed traders to post limit orders. Goettler, Parlour, and Rajan (2009) allow informed and uninformed traders to post limit or market orders, but their model is stationary and assumes Markovian learning in the standing limit order book and the most recent trade. Roşu (2016b) studies a steady-state limit order market equilibrium in continuous-time also assuming Markovian learning with some additional information-processing restrictions. These last two papers are closest to ours. Our model differs from them in two ways: First, they assume restrictions on Markovian learning in order to study dynamic trading strategies with order cancellation, whereas we simplify the strategy space in order to investigate non-Markovian learning given full order histories. Second, we model a non-stationary trading day with opening and closing effects. Market opens and closes are important daily events in the dynamics of liquidity in financial markets. Bloomfield et al. (2005) show in an experimental market analysis of limit orders that liquidity from informed traders changes over the day and sometimes exceeds liquidity from uninformed traders. Our model provides equilibrium examples of such behavior.

Our work also has implications for empirical research on limit order markets. We show not only that limit orders can have informational content — as documented in Fleming, Mizrach, and Nguyen (2019) and Brogaard et al. (2019) — but that orders' information content can vary over time with the state of the standing limit order book and the intraday order history. Yueshen and Zhang (2019) use a filtering approach to show that the price impact of orders is time-varying. Our results identify explanatory variables that may account for some of this intraday time-variation.

A growing literature investigates the relation between information and trading speed (e.g., Biais, Foucault, and Moinas (2015); Foucault, Hombert, and Roşu (2016); and Roşu (2016a)). However, these models assume Kyle or Glosten-Milgrom market structures and, thus, cannot consider the roles of informed and uninformed traders as endogenous liquidity providers and demanders. We argue that understanding price discovery dynamics in limit order markets is an essential precursor to understanding speedbumps and cross-market competition in real-world limit order markets.

1 Model

We consider a limit order market in which a risky asset is traded at N discrete times $t_j \in \{t_1, \ldots, t_N\}$ over a trading day. The fundamental value of the asset at the end of the day after time t_N is

$$\tilde{v} = v_0 + \Delta = \begin{cases} \bar{v} = v_0 + \delta & with\ Pr(\bar{v}) = \frac{1}{3} \\ v_0 & with\ Pr(v_0) = \frac{1}{3} \\ \underline{v} = v_0 - \delta & with\ Pr(\underline{v}) = \frac{1}{3} \end{cases} \tag{1}$$

where v_0 is the ex ante expected asset value, and Δ is a symmetrically distributed value shock. The limit order market allows for trading through two types of orders: Limit orders are price-contingent orders that are collected in a limit order book. Market orders are executed immediately at the best available price in the limit order book. The limit order book has a price grid with four prices, $P_i \in \{A_2, A_1, B_1, B_2\}$, two each on the ask and bid sides of the market. The tick size is equal to $\kappa > 0$, and the ask prices are $A_1 = v_0 + 0.5\kappa$, $A_2 = v_0 + 1.5\kappa$, and, by symmetry, the bid prices are $B_1 = v_0 - 0.5\kappa$, $B_2 = v_0 - 1.5\kappa$. For simplicity, we normalize the tick size to $\kappa = 1$.

Order execution follows time and price priority. At each time t_j, seven possible actions x_{t_j} are available to investors: One possibility is to submit a market order $MBA_{i_{t_j}}$ or $MSB_{i_{t_j}}$ to buy or sell immediately at the best available ask $A_{i_{t_j}}$ or bid $B_{i_{t_j}}$ (indexed by i_{t_j}) in the limit order book at time t_j. A subscript $i_{t_j} = 1$ indicates that the best standing quote at time t_j is at an inside price A_1 or B_1, and $i_{t_j} = 2$ means the best quote is at an outside price A_2 or B_2. Alternatively, the investor can submit one of four possible limit orders LBB_i and LSA_i to buy or sell at the different prices B_i and A_i on the bid or ask side of the book. A subscript $i = 1$ denotes an aggressive limit order posted at the inside quote, and $i = 2$ is a less aggressive limit order at the outside quotes.[4] Yet another alternative is to do nothing (NT).

Two types of investors trade in the market. The first are a sequence of arriving active traders with potential gains-from-trade due to private information and/or random private values. One active investor arrives at each time t_j. They are risk-neutral and asymmetrically informed. The active investor arriving at time t_j is informed with probability α and uninformed with probability $1 - \alpha$. Informed investors know the realized value shock Δ perfectly. A generic informed investor is denoted as I. When we want to make explicit the specific information known by the informed investor, then we denote the informed investor as $I_{\bar{v}}$ if the value shock is positive ($\Delta = \delta$), as $I_{\underline{v}}$ if the shock is negative ($\Delta = -\delta$), and as I_{v_0} if the shock is zero ($\Delta = 0$). Informed investors arriving at different times during the day all have the identical asset-value information (i.e., there is only one realized Δ in the day). Uninformed investors do not know Δ, but they use Bayes' Rule and their knowledge of the equilibrium to learn about Δ from the observable order history over time. Uninformed investors are denoted as U.

An investor arriving at time t_j may also have an additive random personal private value β_{t_j}. Non-informational private-value trading motives include preference shocks, hedging needs, and taxation. The absence of non-informational trading motives would lead to the Milgrom and Stokey (1982) no-trade result. The sequence of arriving active investors is independently distributed over time in terms of whether investors are informed or uninformed and in terms of their individual private values β_{t_j}. Our analysis considers two model specifications for private values. In both specifications, the β_{t_j} value at time t_j for uninformed investors is drawn from a truncated-Normal distribution, $Tr[\mathcal{N}(\mu, \sigma^2)]$, with support over

[4] For tractability, it is assumed investors cannot post buy limit orders at A_1 and sell limit orders at B_1. This is one way in which the investor action space is simplified in our model.

the interval $[-10, 10]$, which corresponds to private valuations of up to plus or minus 10 ticks (see distribution in Figure 1).[5] The mean, $\mu = 0$, is a neutral private value. The parameter σ determines the dispersion of investor private values β_{t_j} and, thus, the probability of large private gains-from-trade due to extreme private valuations. In our first model, informed investors have no private-value motives in that their β_{t_j}s are all equal to 0. In the second specification, the β_{t_j}s for informed investors are random and are independently drawn from the same truncated-Normal distribution $Tr[\mathcal{N}(\mu, \sigma^2)]$ as the uninformed investors.

The second type of investors in the market are a group of passive liquidity providers with no active motive to trade. These investors, who we call the *trading crowd*, submit limit orders to provide liquidity. By assumption, the crowd just posts single limit orders at the outside prices A_2 and B_2. In particular, the crowd is always willing to provide liquidity at the outside prices, since, given a parametric assumption $\delta \leq 1.5$, only uninformed investors submit market orders at those quotes. The market opens with an initial book submitted from the crowd. After the order-submission by the arriving active investor at each time t_j, the crowd replenishes the book at the outside prices, as needed, when either side of the book is empty. Otherwise, if there are limit orders on both sides of the book, the crowd does nothing. The trading crowd effectively establishes a maximal bid-ask spread in the market. Including liquidity from a crowd seems reasonable since, given our parametric restriction, it only involves "zero intelligence" behavior in the sense that it does not require complex optimization or belief updating by the crowd, and they never lose money on their trades (i.e., since at the outside quotes they only trade with uninformed investors given our parametric assumption $\delta < 1.5$ on value shocks relative to the tick size). Excluding such liquidity would, therefore, make the book unreasonably thin. Since the maximal bid-ask spread from the crowd is large relative to the tick size, there is room for further liquidity supply from limit orders from arriving informed and uninformed investors. The goal of our model is to understand the dynamics of liquidity supply and demand due to active investor trading within this maximal bid-ask spread set by the crowd.[6]

For tractability, we make four additional simplifying assumptions. First, limit orders cannot be modified or canceled after submission. Thus, each arriving investor has one opportunity to submit an order. Second, there is no quantity decision. Orders are to buy or sell one share. Third, arriving active investors can only submit one single order. Fourth, limit orders by the active investors have priority over limit orders from the crowd. This departure

[5] We expect similar results for other private-value distributions with a relatively wide support.

[6] The maximal bid-ask spread could arise in other ways. For example, the maximal bid-ask spread of a more sophisticated crowd that could do Bayesian updating could still be wider than one tick given sufficient adverse selection. Alternatively, the trading crowd can be endogenized as HFT investors in a Budish, Cramton, and Shim (2015) style model by adding picking-off risk due to immediate public intraday value shocks to the model.

from time priority for active traders relative to the crowd (but not other active traders) means arriving active investors always have a choice whether to use market orders or different limit orders while keeping the limit order book dynamics simple with only a small number of possible limit prices.[7] Taken together, these assumptions let us express the action set for arriving active investors at time t_j as $X_{t_j} = \{MSB_{i_{t_j}}, LSA_1, LSA_2, NT, LBB_2, LBB_1, MBA_{i_{t_j}}\}$, where each of the orders is for one share.[8]

Our model is intentionally non-stationary over the trading day in order to capture market opening and closing effects and intraday dynamics. When the market opens at t_1, the only standing limit orders in the book are those at prices A_2 and B_2 from the trading crowd.[9] At the end of the day all unexecuted limit orders are cancelled. The state of the limit order book at a generic time t_j during the day is

$$L_{t_j} = [q_{t_j}^{A_2}, q_{t_j}^{A_1}, q_{t_j}^{B_1}, q_{t_j}^{B_2}] \tag{2}$$

where $q_{t_j}^{A_i}$ and $q_{t_j}^{B_i}$ indicate the total depths at prices A_i and B_i at time t_j. The limit order book changes over time due to the arrival of new limit orders (which augment the depth of the book) and market orders (which remove depth from the book) from arriving informed and uninformed investors and due to limit-order submissions from the crowd. The resulting dynamics are

$$L_{t_j} = L_{t_{j-1}} + Q_{t_j} + C_{t_j} \quad j = 1, \ldots, N \tag{3}$$

where Q_{t_j} is the change in the book due to an arriving investor's action $x_{t_j} \in X_{t_j}$ at t_j:[10]

$$Q_{t_j} = [Q_{t_j}^{A_2}, Q_{t_j}^{A_1}, Q_{t_j}^{B_1}, Q_{t_j}^{B_2}] = \begin{cases} [-1,0,0,0] & \text{if } x_{t_j} = MBA_2 \\ [0,-1,0,0] & \text{if } x_{t_j} = MBA_1 \\ [+1,0,0,0] & \text{if } x_{t_j} = LSA_2 \\ [0,+1,0,0] & \text{if } x_{t_j} = LSA_1 \\ [0,0,0,0] & \text{if } x_{t_j} = NT \\ [0,0,+1,0] & \text{if } x_{t_j} = LBB_1 \\ [0,0,0,+1] & \text{if } x_{t_j} = LBB_2 \\ [0,0,-1,0] & \text{if } x_{t_j} = MSB_1 \\ [0,0,0,-1] & \text{if } x_{t_j} = MSB_2 \end{cases} \tag{4}$$

[7] In a richer model, we could assume the crowd submits limit orders at prices three ticks from the unconditional common value v_0 and that their limit orders also have time priority.

[8] The action space X_{t_j} of orders that can be submitted at time t_j includes market orders at the standing best bid or offer at time t_j. The index i_{t_j} in $MSB_{i_{t_j}}$ and $MBA_{i_{t_j}}$ reflects the fact that the best bid or offer at time t_j are not at fixed price levels but rather depends on the incoming state of the limit order book at the inside and outside prices at t_j. There is no time script in the limit order notation LSA_1, ... because these are just limit orders at particular fixed prices $A_1, \ldots$ in the price grid.

[9] In practice, daily opening limit order books include uncancelled orders from the previous day and new limit orders from opening auctions. For simplicity, we abstract from these interesting features of markets.

[10] There are nine alternatives in (4) because we allow separately for cases in which the best bid and ask for market sells and buys at time t_j are at the inside and outside quotes.

where "+1" with a limit order denotes the arrival of an additional order at a particular limit price, and "−1" with a market order denotes execution against a limit order at the standing best-bid-or-offer limit price, and where C_{t_j} is the change in the limit order book due to any limit orders submitted by the crowd:

$$C_{t_j} = \begin{cases} [1,0,0,0] & \text{if } q^{A_2}_{t_{j-1}} + Q^{A_2}_{t_j} = 0 \\ [0,0,0,1] & \text{if } q^{B_2}_{t_{j-1}} + Q^{B_2}_{t_j} = 0. \\ [0,0,0,0] & \text{otherwise.} \end{cases} \tag{5}$$

A potentially important source of information at time t_j is the observed history of orders at prior times $t_1, .., t_{j-1}$. When traders arrive in the market, they observe the history of market activity up through the current standing limit order book at the time they arrive. However, since orders from the crowd have no incremental information beyond that in the arriving active-investor orders, we exclude them from the notation for the portion of the order history used for informational updating of investor beliefs, which we denote by $\mathscr{L}_{t_{j-1}} = \{Q_{t_1}, \ldots, Q_{t_{j-1}}\}$.

Investors trade using optimal order-submission strategies given their information and any private-value motive. If an uninformed investor arrives at time t_j, then his order x_{t_j} is chosen to maximize his expected terminal payoff:

$$\max_{x \in X_{t_j}} w^U(x \,|\beta_{t_j}, \mathscr{L}_{t_{j-1}}) \;=\; \begin{cases} \Big[v_0 + E[\Delta \,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] + \beta_{t_j} - p(x)\Big]\; Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}}) & \text{if } x \text{ is a buy order} \\ 0 & \text{if } x \text{ is a } NT \\ \Big[p(x) - (v_0 + E[\Delta \,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] + \beta_{t_j})\Big]\; Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}}) & \text{if } x \text{ is a sell order} \end{cases} \tag{6}$$

where $p(x)$ is the price at which order x trades. If x is a market order, then the execution price $p(x)$ is the best standing quote on the other side of the market at time t_j. If x is a non-marketable limit order, then $p(x)$ is its limit price. The expression $\theta^x_{t_j}$ in (6) denotes the set of future trading states in which an order x submitted at time t_j is executed.[11] Conditioning on $\theta^x_{t_j}$ matters for limit orders because the sequences of subsequent orders that may or may not result in the execution of earlier limit orders can be correlated with the asset-value shock Δ. For example, subsequent buy market orders may be more likely given positive Δ shocks if value shocks are sufficiently large (i.e., if $\delta >> \kappa/2$, so that informed investors are willing to cross the inside bid-ask spread using market orders). The probability of outside limit orders being undercut by subsequent aggressive inside limit orders by informed investors is also different given good or bad news. Market orders, of course, execute with probability 1. Uninformed investors rationally take the relation between future orders and Δ into account

[11] A market orders x_{t_j} is executed immediately at time t_j and so is executed for sure.

when forming their expectation $E[\Delta\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$ of what the asset will be worth in states in which their limit orders are executed. Uninformed investors in (6) use the prior order history $\mathscr{L}_{t_{j-1}}$ in two ways: It affects their beliefs about limit-order execution probabilities $Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})$ and their execution-state-contingent asset-value expectations $E[\Delta\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$.

An informed investor who arrives at t_j chooses an order x_{t_j} to maximize her expected payoff:

$$\max_{x \in X_{t_j}} w^I(x\,|\,v, \beta_{t_j}, \mathscr{L}_{t_{j-1}}) \;=\; \begin{cases} \big[v + \beta_{t_j} - p(x)\big]\; Pr(\theta^x_{t_j}|v, \mathscr{L}_{t_{j-1}}) & \text{if } x \text{ is a buy order} \\ 0 & \text{if } x \text{ is a } NT \\ \big[p(x) - (v + \beta_{t_j})\big]\; Pr(\theta^x_{t_j}|v, \mathscr{L}_{t_{j-1}}) & \text{if } x \text{ is a sell order} \end{cases} \quad (7)$$

The only uncertainty for informed investors is whether any limit orders they submit will be executed. Their belief about order-execution probabilities $Pr(\theta^x_{t_j}\,|\,v, \mathscr{L}_{t_{j-1}})$ are conditioned on both the trading history up through the current book and on their knowledge about the ending asset value. Informed traders condition on $\mathscr{L}_{t_{j-1}}$, not to learn about the value shock Δ (which they already know) or about later investor private values β_{t_j} (which are independent over time), but rather because the order history is an input in the trading behavior of later uninformed investors (with whom they might trade in the future) and, thus, also in the trading behavior of later informed investors (against whom they compete and who will also take history-contingent uninformed-investor learning behavior into account when deciding whether to undercut earlier limit orders).[12]

The optimization problem in (6) defines sets of actions $x_{t_j} \in X_{t_j}$ that are optimal for the uninformed investor at different times t_j given different private-value factors β_{t_j} and order histories $\mathscr{L}_{t_{j-1}}$. Optimal orders can be unique, or there may be multiple orders that make uninformed investors equally well-off. The *optimal order-submission strategy* for uninformed investors is a probability function $\varphi^U_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}})$ that is zero if the order x is suboptimal and equals a mixing probability over optimal orders. If an optimal order x is unique, then $\varphi_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}}) = 1$. Mixed strategies are also allowed. Similarly, the optimization problem in (7) leads to an optimal order-submission strategy $\varphi^I_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})$ for informed investors at time t_j given their private value β_{t_j}, their knowledge about the asset value v, and the order history $\mathscr{L}_{t_{j-1}}$.

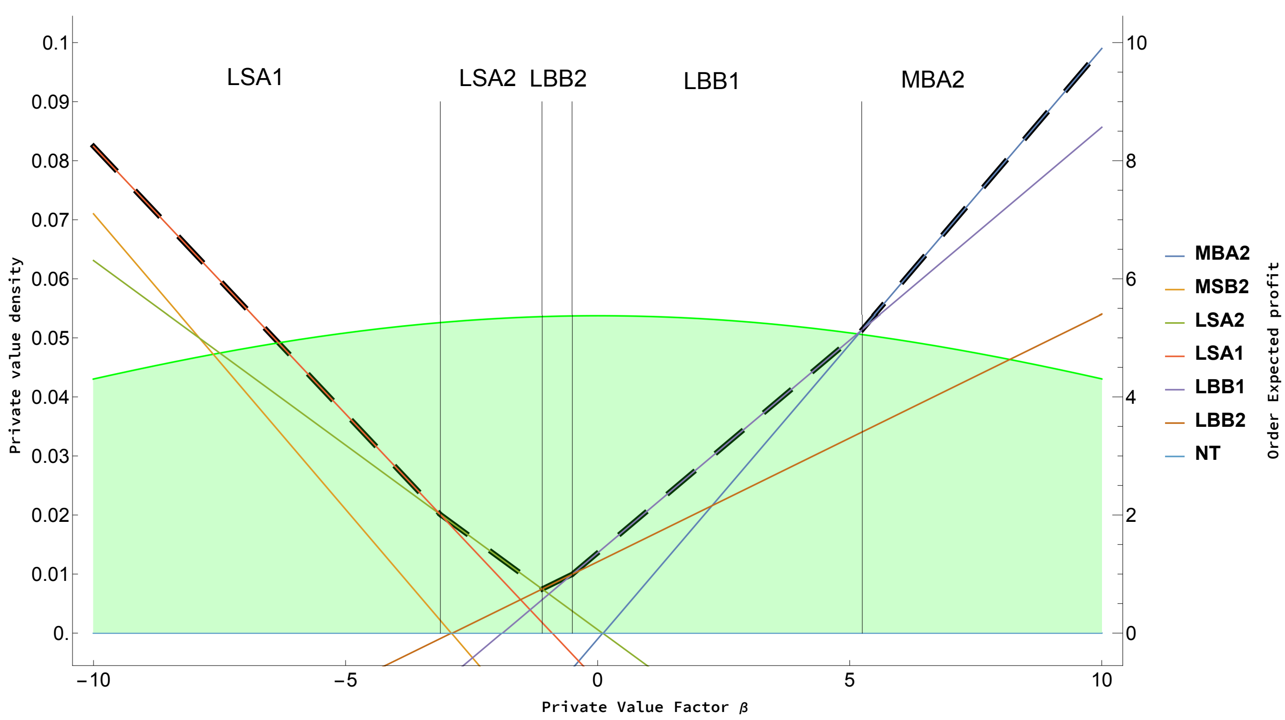
Our model has four sources of potential order-flow randomness. First, orders are random due to the random arrival of informed and uninformed investors. Second, they are random due to the asset-value shock Δ that informed traders condition on. Third, orders are random

[12]Recall here that in one specification of our model, only uninformed investors have random private valuations, while in a second richer specification both informed and uninformed investors have random private valuations.

due to randomness in investors' personal private values β_{t_j}. This is illustrated in Figure 1 for a numerical example of our model that is considered in detail in Section 2.2. The plot shows where the order-submission probabilities come from for an informed investor $I_{\bar{v}}$ at time t_1 by superimposing the upper envelope of the expected payoffs for the different optimal orders at time t_1 for the case of good news about a positive value shock Δ on the truncated Normal β_{t_j} distribution. It shows how different β_{t_j} subranges correspond to discrete sets of optimal orders delimited by β_{t_j} thresholds. Similar constructions exist at other dates for other informed and uninformed investors. Fourth and lastly, some equilibria involve order randomness due to mixed strategies $\varphi^U_{t_j}$ and $\varphi^I_{t_j}$.

To summarize, our model captures the following economic drivers of trading in a dynamic limit order market with adverse selection: First, trading by uninformed investors provides camouflage for trading by the informed investors. Second, investors trade off gains from trading immediacy and price improvement when deciding between submitting market vs. limit orders and in their choice of using more vs. less aggressive limit orders. Third, there is dynamic competition between informed investor to trade on their common private information over time. Fourth, there is competition between informed and uninformed investors in liquidity provision.

Figure 1: $\boldsymbol{\beta}$ Distribution and Upper Envelope for Informed Investor $\boldsymbol{I_{\bar{v}}}$ at time $\boldsymbol{t_1}$. This figure shows the private value $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ distribution superimposed on the plot of the expected payoffs the informed investor $I_{\bar{v}}$ with good news at time t_1 for each equilibrium order type MBA_2, MSB_2, LSA_2, LSA_1, LBB_1, LBB_2, NT, (solid colored lines) when the total book (including crowd limit orders) opens $L_{t_0} = [1\ 0\ 0\ 1]$. The dashed lines show the investor's upper envelope for the optimal orders. The vertical black lines show the β-thresholds at which two adjacent optimal strategies yield the same expected payoffs. LSA_1 is the optimal order for values of β below the first vertical black line; LSA_2 is the optimal order for the values of β between the first and the second vertical lines; and so forth. The market parameters are $\alpha = 0.8$, $\delta = 1.4$, $\mu = 0$, $\sigma = 15$, and $\kappa = 1$.



1.1 Equilibrium

An equilibrium is a set of mutually consistent beliefs and optimal strategies for uninformed and informed investors for each time t_j, given each order history $\mathscr{L}_{t_{j-1}}$, private-value factor β_{t_j}, and (for informed traders) private information v. This section explains what "mutually consistent" means and then gives a formal definition of an equilibrium.

A central feature of our model is asymmetric information. By observing orders over time, uninformed traders infer information about the asset value v and use it in their order-submission strategies. More precisely, uninformed traders rationally learn from the trading history about the probability that v will go up, stay constant, or go down. However, investors cannot learn about the private values (β_{t_j}) of investors at future times t_j or about the information status (I or U) of future traders since, by assumption, these are both independent over time. Informed investors do not need to learn about v since they know it directly. However, they do condition their orders on v both because v is the final stock value and also because v tells them what type of informed investors I_v will arrive in the future (along with the uninformed U traders). Informed investors also condition on the order history $\mathscr{L}_{t_{j-1}}$, since $\mathscr{L}_{t_{j-1}}$ affects the trading behavior of future investors.[13]

The underlying *economic state* in our model is the realization of the asset value v and a sequence of investors who arrive in the market. The investor who arrives at time t_j is described by two characteristics: Their status as being informed or uninformed, I or U, and their private-value factor β_{t_j}. The underlying economic state is exogenously chosen over time by Nature. More formally, it follows an exogenous stochastic process described by the model parameters δ, α, μ, and σ. An asset value and a sequence of arriving investors together with a pair of strategy functions — which we denote here as $\Phi = \{\varphi^U_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}}), \varphi^I_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})\}$ — induce a sequence of trading actions x_{t_j} which — together with the predictable actions of the trading crowd — results in a sequence of observable changes in the state L_{t_j} of the limit order book which results in a sequence of order paths $\mathscr{L}_{t_j}$. Thus, the stochastic process generating paths of order histories is induced by the economic state process and the strategy functions. First, the order-path process determines the unconditional probabilities of different paths $Pr(\mathscr{L}_{t_j})$ and the conditional probabilities $Pr(Q_{t_j}|\mathscr{L}_{t_{j-1}})$ of particular order book changes Q_{t_j} due to arriving investors given a prior history $\mathscr{L}_{t_{j-1}}$. Second, the endogenous order-path process also determines the order-execution probabilities $Pr(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$, with v, and $Pr(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$ for informed and

[13] The order history $\mathscr{L}_{t_{j-1}}$ is an input in the learning problem of future uninformed investors and, thus, is an input in their order-submission strategy. In addition, since future informed investors know that $\mathscr{L}_{t_{j-1}}$ can affect trading by uninformed investors, it also enters the order-submission strategies of future informed investors. Thus, for both reasons current informed investors condition on $\mathscr{L}_{t_{j-1}}$.

uninformed investors for each possible orders x at time t_j. These probabilities are computed by listing all of the possible underlying economic states (i.e., value and investor sequence realizations), mechanically applying the order-submission rules Φ, identifying the order path outcomes, and then taking expectations across paths.

Let ℓ denote the set of all histories $\{\mathscr{L}_{t_j} : j = 1, \ldots, N-1\}$ of available orders of lengths up to $N-1$ trading periods. A $N-1$ period long history is the longest history an order-submission strategy can depend on in our N-period model. Certain order paths in ℓ are *possible* in that they have positive probability, $Pr(\mathscr{L}_{t_j}) > 0$, given the strategy functions Φ, but other paths may be *feasible* in that they consist of available orders in the action choice sets X_{t_j} but not possible in that $Pr(\mathscr{L}_{t_j}) = 0$ for Φ. Let $\ell^{\,in,\Phi}$ denote the subset of all possible order paths $\mathscr{L}_{t_j}$ in ℓ with positive probability given order strategies Φ, and let $\ell^{\,off,\Phi}$ denote the complementary set of order paths that are feasible but not possible given Φ. This notation will be useful when discussing "equilibrium" beliefs on order paths that have positive probability and "off equilibrium" beliefs on paths that have zero probability given investor strategies. The strategy functions Φ are defined for all paths in ℓ. In particular, this includes the possible paths in $\ell^{\,in,\Phi}$ given Φ and also the paths in $\ell^{\,off,\Phi}$. As a result, the probabilities $Pr(Q_{t_j}|\mathscr{L}_{t_{j-1}})$, $Pr(\theta^x_{t_j}|\,v, \mathscr{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\,\mathscr{L}_{t_{j-1}})$ are always well-defined, because the continuation trading process going forward — even after an unexpected order-arrival event (i.e., a path $\mathscr{L}_{t_{j-1}} \in \ell^{\,off,\Phi}$) — is still well-defined.

The stochastic process for order paths and its relation to the underlying economic state also determine the uninformed-investor expectations $E[v\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$ of the terminal asset value given the previous order history $\mathscr{L}_{t_{j-1}}$ and conditional on future execution in states $\theta^x_{t_j}$ for orders x at time t_j. In particular, beliefs and expectations for uninformed investors involve backward conditioning on the prior order history $\mathscr{L}_{t_{j-1}}$ and forward conditioning on the endogenous set of future states $\theta^x_{t_j}$ in which orders x are executed. These beliefs and expectations are determined as follows:

- Step 1: The conditional probabilities $\pi^v_{t_j} = Pr(v|\mathscr{L}_{t_j})$ for final asset values $v = \bar{v}, v_0$ or $\underline{v}$ given a possible order history $\mathscr{L}_{t_j} \in \ell^{\,in,\Phi}$ up through time t_j are given by Bayes' Rule. At time t_1, this probability is

$$\begin{aligned} \pi^v_{t_1} &= \frac{Pr(v, \mathscr{L}_{t_1})}{Pr(\mathscr{L}_{t_1})} = \frac{Pr(\mathscr{L}_{t_1}|v)Pr(v)}{Pr(\mathscr{L}_{t_1})} = \frac{Pr(Q_{t_1}|v)Pr(v)}{Pr(Q_{t_1})} \\ &= \frac{Pr(Q_{t_1}|v, I)Pr(I) + Pr(Q_{t_1}|U)Pr(U)}{Pr(Q_{t_1})} Pr(v) \\ &= \frac{E^\beta[\varphi^I_{t_1}(x_{t_1}|\beta_{t_1}, v)|v, I]\alpha + E^\beta[\varphi^U_{t_1}(x_{t_1}|\beta_{t_1})|U](1-\alpha)}{Pr(Q_{t_1})} \pi^v_{t_0} \end{aligned} \tag{8}$$

where the prior is the unconditional probability $\pi^v_{t_0} = Pr(v)$, x_{t_1} is the order at time t_1 that leads to the order book change Q_{t_1}, and E^β denotes an expectation with respect to the random private value β_{t_1}.[14] At times $t_j > t_1$, the history-conditional probabilities are given recursively by[15]

$$\begin{aligned}
\pi^v_{t_j} &= \frac{Pr(v, \mathscr{L}_{t_j})}{Pr(\mathscr{L}_{t_j})} = \frac{Pr(v, Q_{t_j}, \mathscr{L}_{t_{j-1}})}{Pr(Q_{t_j}, \mathscr{L}_{t_{j-1}})} = \frac{\begin{pmatrix} Pr(Q_{t_j}|v, \mathscr{L}_{t_{j-1}}, I) Pr(I|\mathscr{L}_{t_{j-1}}) Pr(v|\mathscr{L}_{t_{j-1}}) \\ + Pr(Q_{t_j}|v, \mathscr{L}_{t_{j-1}}, U) Pr(U|\mathscr{L}_{t_{j-1}}) Pr(v|\mathscr{L}_{t_{j-1}}) \end{pmatrix}}{Pr(Q_{t_j}|\mathscr{L}_{t_{j-1}})} \\
&= \frac{E^\beta[\varphi^I_{t_j}(x_{t_j}|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})|v, I, \mathscr{L}_{t_{j-1}}]\ \alpha + E^\beta[\varphi^U_{t_j}(x_{t_j}|\beta_{t_j}, \mathscr{L}_{t_{j-1}})|U, \mathscr{L}_{t_{j-1}}]\ (1-\alpha)}{Pr(Q_{t_j}|\mathscr{L}_{t_{j-1}})} \pi^v_{t_{j-1}} \quad (9)
\end{aligned}$$

Given these probabilities, the expected asset value conditional on the order history is

$$E[\tilde{v}|\mathscr{L}_{t_{j-1}}] = \pi^{\bar{v}}_{t_{j-1}}\, \bar{v} + \pi^{v_0}_{t_{j-1}}\, v_0 + \pi^{\underline{v}}_{t_{j-1}} \underline{v} \quad (10)$$

- Step 2: Conditional probabilities $\pi^v_{t_j}$ for a "feasible but not possible in equilibrium" order history $\mathscr{L}_{t_j} \in \ell^{\,off,\Phi}$ containing a limit order book change Q_{t_j} that is inconsistent with the strategies Φ at time t_j are set as follows:

  1. If the priors are fully revealing in that $\pi^v_{t_{j-1}} = 1$ for some v, then $\pi^v_{t_j} = \pi^v_{t_{j-1}}$ for all v.
  2. If the priors are not fully revealing at time t_j, then $\pi^v_{t_j} = 0$ for any v for which $\pi^v_{t_{j-1}} = 0$ and the probabilities $\pi^v_{t_j}$ for the remaining v's can be any non-negative numbers such that $\pi^{\bar{v}}_{t_j} + \pi^{v_0}_{t_j} + \pi^{\underline{v}}_{t_j} = 1$.
  3. Thereafter, until any next unexpected trading event, the subsequent probabilities $\pi^v_{t_{j'}}$ for $j' > j$ are updated according to Bayes' Rule as in (9).

- Step 3: The execution-contingent conditional probabilities $\hat{\pi}^v_{t_j} = Pr(v|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j})$ of a final asset value v conditional on a prior path $\mathscr{L}_{t_{j-1}}$ and on execution of a limit order x submitted at time t_j is

$$\begin{aligned}
\hat{\pi}^v_{t_j} &= \frac{Pr(\mathscr{L}_{t_{j-1}}) Pr(v|\mathscr{L}_{t_{j-1}}) \Pr(\theta^x_{t_{j-1}}|v, \mathscr{L}_{t_{j-1}})}{Pr(\theta^{x t_j}, \mathscr{L}_{t_{j-1}})} \quad (11) \\
&= \frac{\Pr(\theta^x_{t_j}|v, \mathscr{L}_{t_{j-1}})}{Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})} \pi^v_{t_{j-1}}
\end{aligned}$$

[14]A trader's information status (I or U) is independent of the asset value v, so $Pr(I|v) = Pr(I)$ and $Pr(U|v) = Pr(U)$. Furthermore, uninformed traders have no private information about v, so the probability $Pr(Q_{t_1}|U)$ with which they take a trading action Q_{t_1} does not depend on v. Note that our notation allows for different distributions over β_{t_1} in the expectation E^β conditional on I and U investors.

[15]A trader's information status is again independent of v, and it is also independent of the past trading history $\mathscr{L}_{t_1}$. While the probability with which an uninformed trader takes a trading action Q_{t_1} may depend on the past order history $\mathscr{L}_{t_j}$, it does not depend directly on v which uninformed traders do not know.

This holds when adjusting for a future execution contingency both when the probabilities $\pi^v_{t_{j-1}}$ given the prior history $\mathscr{L}_{t_{j-1}}$ are for possible paths in $\ell^{\,in,\Phi}$ (from (8) and (9) in Step 1) and also for feasible but not possible paths in $\ell^{\,off,\Phi}$ (from Step 2). These execution-contingent probabilities $\hat{\pi}^v_{t_j}$ are used to compute the execution-contingent conditional expected value:

$$E[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] = \hat{\pi}^{\bar{v}}_{t_j}\,\bar{v} + \hat{\pi}^{v_0}_{t_j}\,v_0 + \hat{\pi}^{\underline{v}}_{t_j}\,\underline{v} \tag{12}$$

used by uninformed traders to compute expected payoffs for limit orders. In particular, the probabilities in (12) are the execution-contingent probabilities $\hat{\pi}^v_{t_j}$ from (11) rather than the probabilities $\pi^v_{t_j}$ from (9) that just condition on the prior trading history but not on the future states in which the limit order is executed.

Given these updating dynamics, we can now define an equilibrium.

Definition. A *Perfect Bayesian Nash Equilibrium* of the trading game in our model is a collection $\{\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}}), \varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}}), Pr^*(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}}), Pr^*(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}}), E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]\}_{j=1}^N$ of order-submission strategies, execution-probability functions, and execution-contingent conditional expected asset-value functions such that:

- The equilibrium execution probabilities $Pr^*(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr^*(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$ are consistent with the equilibrium order-submission strategies $\{\varphi^{U,*}_{t_{j+1}}(x|\beta_{t_{j+1}}, \mathscr{L}_{t_j}), \ldots, \varphi^{U,*}_{t_5}(x|\beta_{t_5}, \mathscr{L}_{t_4})\}$ and $\{\varphi^{I,*}_{t_{j+1}}(x|\beta_{t_{j+1}}, v, \mathscr{L}_{t_j}), \ldots, \varphi^{I,*}_{t_5}(x|\beta_{t_5}, v, \mathscr{L}_{t_4})\}$ after time t_j.

- The execution-contingent conditional expected asset values $E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]\}$ agree with Bayesian updating equations (8), (9), (11), and (12) in Steps 1 and 3 when the order x is consistent with the equilibrium strategies $\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}})$ and $\varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})$ at date t_j and with the off-equilibrium updating in Step 2 when x is an off-equilibrium action inconsistent with the equilibrium strategies.

- The positive-probability supports of the equilibrium strategy functions $\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}})$ and $\varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})$ (i.e., the orders with positive probability in equilibrium) are subsets of the sets of optimal orders for uninformed and informed investors from their optimization problems (6) and (7) and the equilibrium execution probabilities and outcome-contingent conditional asset-value expectation functions $Pr^*(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$, $Pr^*(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$, and $E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$.

Our equilibrium concept differs from the Markov Perfect Bayesian Equilibrium used in Goettler et al. (2009). Beliefs and strategies in our model are path-dependent; that is to say, traders use Bayes Rule given the full prior order history when they arrive in the market. In contrast, Goettler et al. (2009) restricts Bayesian updating to the current state of the limit order book and does not allow for conditioning on the previous order history. Roşu (2016b) also assumes a Markov Perfect Bayesian Equilibrium. The quantitative importance of the order history is considered in Section 2.

To help with intuition, Appendix A illustrates the order-submission and Bayesian updating mechanics for a particular realized equilibrium path in the extensive form of the trading game. Appendix B explains the algorithm used to compute equilibria in our model.

2 Results

This section presents results about how trading decisions of informed and uninformed investors and the learning process for uninformed investors affect market liquidity, price discovery, and investor welfare. Section 2.1 first considers a model specification in which only uninformed investors have random private-value trading motives. Section 2.2 considers a second specification that generalizes the analysis and shows the robustness of our findings and extends them when informed investors also have private-value motives. Throughout our analysis, there are $N = 5$ trading rounds and the truncated Normal distribution for private values $Tr[\mathscr{N}(\mu, \sigma^2)]$ has a mean $\mu = 0$, dispersion $\sigma = 15$, and support $[-10, 10]$. Our main results explain how various features of investors' endogenous order-submission strategies can combine to overturn standard intuitions about adverse selection, information revelation, liquidity, and welfare in limit order markets and describe non-Markovian aspects of price discovery.

Our model is non-stationary with start-up effects at the beginning of the day and terminal horizon effects at the market close, much like actual trading days. Thus, we report results for two time windows. The first is the market open at time t_1. The second is over the middle of the trading day from times t_2 through t_4. When the market opens at time t_1, there are time-dependent incentives to provide liquidity: The opening book L_{t_0} is thin (with limit orders only from the crowd), and there is the maximum time for future investors to arrive to hit limit orders from t_1. There are also time-dependent disincentives for limit orders: Information asymmetries are maximal at time t_1, since there has been no learning through the trading process. Also, there is the maximal time for early less aggressive outside limit orders (at A_2 and B_2) to be undercut by later more aggressive inside limit orders (at A_1 and B_1). Over the day, information is revealed (lessening adverse selection costs), but the standing book can also become fuller (due to competition in liquidity provision from earlier limit orders with time priority), and the remaining time for market orders to arrive and execute limit orders becomes shorter. Comparing these two time windows shows how non-stationary market dynamics change over the day. The market close at t_5 is also important, but trading then is straightforward. At the end of the day, investors only submit market orders (or do not trade), because the execution probability for new limit orders at t_5 is zero given our assumption that unfilled limit orders are canceled once the market closes. Our choice of $N = 5$ trading rounds in a day is a compromise between computational tractability and having time mid-day for relatively less constrained endogenous choices between market and limit orders at times t_2 through t_4 away from the immediate mechanical effects of the relatively thin book at the market open at t_1 and the end-of-day market orders at t_5.

Our model lets us investigate three questions: First, who provides and takes liquidity, and how does the amount of adverse selection affect investor decisions to take and provide liquidity? Second, how does market liquidity vary with different amounts of adverse selection? Third, how does the information content of different types of orders depend on an order's direction, aggressiveness, intraday timing, and on the prior order history?

We present numerical comparative statics and other analyses for four different combinations of parameters with high and low informed-investor arrival probabilities ($\alpha = 0.8$ and 0.2) and high and low value-shock volatilities ($\delta = 1.4$ and 0.2). The value-shock volatility δ controls the magnitude of adverse selection in the market: A large δ means the private information of informed investors is potentially large. We call markets with $\delta = 1.4$ *high-volatility* markets and markets with $\delta = 0.2$ *low-volatility* markets because of the size of arriving information δ relative to the $\kappa = 1$ tick size. In high-volatility markets with $\delta = 1.4$, the final asset value v given good or bad news is inside the outside quotes A_2 or B_2. This has two implications: First, providing liquidity at the outside quotes is always profitable for the crowd, even when the market is fully revealing. Second, when informed-investor private-value realizations β_{t_j} are sufficiently small (e.g., zero), informed investors in our high-volatility market (given $\delta < 1.5$) never use market orders to trade at the outside quotes, but they do potentially use market orders to trade with limit orders at the inside quotes. In contrast, in low-volatility markets, v is always between the inside quotes A_1 and B_1, and so market orders are never profitable for informed investors with small (e.g., zero) private values β_{t_j}. A real-world example of a low-volatility trading environment are days on which a heteroscedastic stocks' stochastic volatility process has low volatility. Another example are certain futures contracts where a large amount of bid-ask bounce indicates that their customized price grids are large relative to the underlying information flow.[16] One fur-

[16]We thank Rob Almgren for bringing this empirical fact about futures to our attention.

ther example is algorithmic informed trading on small valuation inferences from correlated securities.

The informed-investor arrival probability α controls the amount of informed trading. It has four effects: First, it controls the amount of informational competition faced by informed investors: When α is large, it is more likely an arriving informed investor at a time t_j will face competition from other informed investors with the same private information who may arrive later in the day (who might undercut outside limit orders posted at t_j) and who have already arrived earlier in the day (i.e., the incoming book at time t_j reflects the decisions of any earlier informed investors to hit earlier attractive inside limit orders or to post their own limit orders). Second, α affects adverse selection faced by uninformed investors via the probability that uninformed traders trade with informed investors as counterparties and via the probability of prior information revelation by prior informed trading. Third, when α is high, future arriving investor trading motives are more correlated over time. Fourth, α affects the potential gains-from-trade in the market if informed and uninformed traders have different β_{t_j} distributions.

2.1 Uninformed traders with random private-value motives

In our first model specification, only uninformed U traders have random private values β_{t_j}. Informed I investors have fixed neutral private values $\beta_{t_j} = 0$. Thus, as in Kyle (1985), there is a clear differentiation between investors who speculate on private information and those trading for purely non-informational reasons. Unlike Kyle (1985), informed and uninformed investors here can choose to trade using limit or market orders rather than being restricted to just market orders.

2.1.1 Trading strategies

The equilibrium trading behavior of investors is complex because of endogenous equilibrium interactions and forward- and backward-looking belief updating and because the dynamic optimization problems in (6) and (7) are non-stationary (given the reduction in the remaining time for limit order execution as time passes over the day) and state-contingent (based on valuation information revealed by the prior order history and on the state of the stochastically evolving standing limit order book when investors arrive in the market). However, despite this complexity, a few basic motives can be identified that drive order-submission decisions for the different investors: First, directionally-informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors trade both to speculate on their private information and also potentially to profit from providing liquidity to uninformed investors. Second, informed I_{v_0} investors with neutral information trade to profit from liquidity provision to uninformed investors given their private knowledge that there was no valuation shock so that the ex ante expectation $E[v]$ is correct. Lastly, uninformed U investors trade to capture their personal private-value gains-from-trade (if their $|\beta_{t_j}|$ is large) and to provide liquidity to other uninformed investors but at the risk of adverse selection with informed investors (if their $|\beta_{t_j}|$ is small).

We begin by investigating who supplies and takes liquidity and how these decisions change with the amount of adverse selection and information competition. Our starting point follows from first principles:

Proposition 1 Directionally-informed investors with $\beta_{t_j} = 0$ only trade using limit orders when the value-shock volatility δ is small, but they use both market and limit orders when δ is sufficiently large. In addition, their trading strategies are affected differently by changes in adverse selection due to the value-shock size δ vs. changes in the arrival probability α of informed investors.

Proof: Consider first the effect of the value-shock δ on informed-investor order submissions given any fixed $\alpha > 0$. If the value-shock δ is sufficiently close to zero, then directionally-informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors with good or bad news never use market orders, since the terminal asset value v is always between the inside bid and ask prices A_1 and B_1 given a discrete tick size. However, once δ is sufficiently large, investors with good and bad news start to use market orders for their guaranteed execution. Thus, the set of orders used by directionally-informed investors can change when δ changes. This is true for all informed-investor arrival probabilities $\alpha > 0$. In contrast, consider the effect of the informed-investor arrival probability α on informed-investor order submissions given a fixed $\delta > 0$. If the value-shocks δ are close to zero, informed investors with good or bad news never use market orders for any informed-investor arrival probability α. They are unwilling to pay a large tick size to trade on their small information. Instead, they use limit orders to supply liquidity asymmetrically depending on the direction of their information. Thus, the set of orders used by directionally-informed investors in low-volatility markets never changes to include market orders when α changes. $\square$

Our analysis continues by describing several specifics about investor trading behavior that, in addition to their own interest, will drive our later results about liquidity, price discovery, and welfare. Table 1 reports results about trading early in the day at time t_1 using a 2×2 format. Each of the four cells corresponds to a different combination of parameters. Comparing cells horizontally shows the effect of a change in the value-shock size δ while holding the arrival probability α for informed traders fixed. Comparing cells vertically shows the effect of a change in the informed-investor arrival probability while holding the value-shock size fixed. In each cell corresponding to a set of parameters, there are four columns reporting conditional results for informed investors with good news, neutral news, and bad news about the asset ($I_{\overline{v}}$, I_{v_0}, $I_{\underline{v}}$) and for an uninformed investor (U) and a fifth column with the unconditional market results ($Uncond$). The table reports order-submission probabilities and several market-quality metrics. Specifically, we report expected bid-ask spreads conditioning on the three informed-investor types $E[Spread\,|I_v]$ and on the uninformed trader $E[Spread\,|U]$, the unconditional expected market spread $E[Spread]$, and expected depths at the inside prices (A_1 and B_1) and the total at both prices ($A_1 + A_2$ and $B_1 + B_2$) on each side of the market. As we shall see, our results are symmetric for the directionally informed investors $I_{\overline{v}}$ and $I_{\underline{v}}$ on the buy and sell sides of the market. In addition, we report the probability-weighted contributions to the different investors' welfare (i.e., expected gains-from-trade) from limit and market orders respectively, and investor total expected welfare.[17] Table C1 in Appendix C provides additional results about execution probabilities for the different orders ($P^{EX}(x_{t_1})$) and also the uninformed investor's updated expected asset value $E[v|x_{t_1}]$ given different types of buy orders x_{t_1} at time t_1.

Table 2 shows average results for times t_2 through t_4 during the day using a similar 2×2 format. The averages are across time. Comparing results for time t_1 with the averages for t_2 through t_4 shows intraday variation in the trading process. There is no table for time t_5, because only market orders are used to trade at the market close.

Consider first directionally-informed $I_{\overline{v}}$ and $I_{\underline{v}}$ investors. Two properties have important impacts on trading dynamics and on market-quality and order-informativeness in the price-discovery process. First, directionally-informed investors tend to trade more aggressively in high-volatility markets in which their private information is large relative to the tick size. This *aggressive-trading property* is intuitive since larger potential payoffs make price improvement less important relative to trade execution on speculative trades. The most aggressive way to trade is via market orders, which take liquidity. However, the next most aggressive way to trade is via limit orders at the inside prices. Thus, in some cases greater trading aggressivenss means greater liquidity provision by inside limit orders rather than liquidity taking by market orders. Second, directionally-informed investors switch from speculation to liquidity provision on the other side of the market if the speculative side of the standing limit order book is too deep when they arrive. In particular, a deep standing

[17]Let $W^U(\beta_{t_j})$ and $W^I(v, \beta_{t_j})$ denote value functions when (6) and (7) are evaluated at time t_j using optimal strategies for the uninformed and informed investors. The total ex ante welfare for the uninformed investor at t_j is $E^\beta[W^U(\beta_{t_j})]$ where the expectation is taken over the uninformed investor's random β_{t_j}. The corresponding expected welfare for informed investors given v is $W^I(v, \beta_{t_j})$ in our first model specification where $\beta_{t_j} = 0$ for informed investors and $E^\beta[W^I(v, \beta_{t_j})]$ in our second specification in Section 2.2 where the expectation is taken over the informed investor's random β_{t_j}.

book of competing limit orders with time priority from earlier investors reduces the execution probability, and thus expected profitability, of further speculative limit orders (e.g., limit buys given good news $\bar{v}$), but if there is little depth on the other side of the book, then directionally-informed investors may start submitting limit orders to profit from liquidity provision on the other side of the book (e.g., limit sells at limit prices above $\bar{v}$). However, if the standing book is too deep on both sides, then they do not submit any order (NT). We call these the *unbalanced deep-book effect* and the *two-sided deep-book effect.*

The aggressive-trading property can be seen in Table 1 at time t_1 where $I_{\bar{v}}$ and $I_{\underline{v}}$ investors only post limit orders at the less-aggressive outsides quotes A_2 and B_2 in the two low-volatility parameterizations on the right (with $\delta = 0.2$ and $\alpha = 0.2$ or 0.8) but use aggressive limit orders at the inside quotes A_1 and B_1 — either as a pure-strategy or in a mixed-strategy along with outside limit orders — in the two high-volatility parameterization on the left (with $\delta =$ 1.4 and the same two α values). Next, consider the average order-submission probabilities at times t_2 through t_4 in Table 2. In the two low-volatility parameterizatons on the right, informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors supply liquidity via limit orders on both sides of the market — either to speculate on their information or to provide liquidity due to the unbalanced deep-book effect — with order-submission probabilities that are somewhat skewed at the inside quote in the direction of their small amount of private information ($\delta = 0.2$).[18] Now consider increasing volatility and moving to the two high-volatility parameterizations on the left. If the informed-investor arrival probability is low ($\alpha = 0.2$), higher value volatility δ causes directionally-informed investors to increase their probability of using aggressive inside limit orders (the trading aggressiveness property for limit orders) and also to start using market orders to speculate their information. However, when the informed-investor arrival probability is high ($\alpha = 0.8$), increased trading aggressiveness by the $I_{\bar{v}}$ and $I_{\underline{v}}$ investors in the high-volatility market takes the form of reduced speculative limit orders and increased market orders at times t_2 through t_4. However the biggest effects of higher asset volatility in the high α case are their increased use of NT and increased liquidity provision via outside limit orders on the opposite side of their information (due to the two-sided and unbalanced book effects). With a high $\alpha = 0.8$, informed investors face more competition on one or both sides of the limit order book from earlier investors with the same information. This can cause the book to become "full" for purposes of speculation and/or liquidity provision.

Next, consider the uninformed U investors. One obvious fact in Table 1 is that, in these parameterizations, uninformed-trader behavior at t_1 changes more when α changes than

[18]The high probability of an outside LSA_2 limit sell by an $I_{\bar{v}}$ investor later in the day when $\alpha = 0.8$ is in part due to the high probability that the standing book will already have a LBB_2 limit buy from an $I_{\bar{v}}$ investor at t_1.

when δ changes. This is because uniformed traders, given their potentially large private-value gains-from-trade, tend here to be more concerned about execution probability (controlled by α with its negative impact on the mix of investor with potentially large private-value trading motives) than the relatively modest adverse selection costs (even with $\delta = 1.4$).

An equilibrium trading interaction here is noteworthy. Note that informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors use aggressive limit orders at the inside quotes at time t_1 with a lower probability in the upper-left (high adverse-selection) parameterization than in the lower-left (less-intense adverse selection) parameterization (0.433 vs 1.0). At first glance, this might seem counter-intuitive since informational competition from future informed investors (and the possibility of early outside limit orders being undercut by later inside limit orders) is greater when the informed-investor arrival probability α is large (i.e., $\alpha = 0.8$ vs 0.2). The explanation is that in the high-δ/high-α parametrization, uninformed U investors use aggressive inside limit orders at t_1 with only a very small probability relative to the high-δ/low-α market (0.03 vs 0.393). As a result, there is much less potential camouflage in equilibrium from uninformed U investors posting inside limit orders in the upper-left parametrization. Table C1 in Appendix C shows that, as a result, the execution probabilities for the highly revealing inside limit orders in the high-δ/high-α market are much lower than for the less-informative inside limit orders in the high-δ/low-α market (0.090 vs 0.735).[19]

Lastly, consider the neutrally-informed I_{v_0} investors. There are three points to note: First, neutrally-informed investors use more inside limit orders at times t_2 through t_4 than directionally-informed traders in some parameterizations (high-α/high-δ and low-α/low-δ). This is consistent with the intuition of Bloomfield, O'Hara and Saar (BOS 2005), who find in laboratory experiments that informed investors provide liquidity via limit orders when mispricing is small (i.e., as here when $v = v_0$). However, the *BOS effect* does not obtain in other parametrizations where it is either negligible or is reversed (high-α/low-δ and low-α/high-δ) due other causal effects. Second, the use of inside limit orders by the neutrally-informed I_{v_0} investors increases significantly at times t_2 through t_4 when the probability α of informed traders increases. This is an example of increased informational competition. Third, neutrally-informed I_{v_0} investors respond differently to adverse selection than uninformed U investors because the I_{v_0} investors have an advantage in that there is no adverse selection risk for them. They know the value shock Δ is 0 and, thus, that the unconditional valuation v_0 is correct. This is another variation on the BOS intuition. We see this effect at times t_2 through t_4 where I_{v_0} investors consistently use inside limit orders more than U investors. Overall, Tables 1 and 2 show there is variation across parameterizations

[19] Uninformed investors with extreme private values still hit limit orders when the market is fully revealing.

in the considerations driving neutrally-informed I_{v_0} investor trading.

2.1.2 Market quality

Market liquidity changes when the magnitude of adverse selection (δ) and the amount of informed trading (α) in a market change. A standard intuition, as in Kyle (1985), is that liquidity deteriorates given more adverse selection. Roşu (2016b) also finds worse liquidity (a wider bid-ask spread) given higher value volatility in his limit order market. However, we show the standard intuition is not always true when informed investors endogenously choose whether to supply liquidity via limit orders or take liquidity via market orders.

Observation 1 Liquidity can potentially improve when adverse selection increases.

In particular, markets can become more liquid when the value-shock volatility δ increases from being small (0.2) to being large (1.4) relative the price tick size.

The impact of adverse selection on market liquidity follows directly from three intuitions about the trading strategies in Section 2.1.1. First, changes in market parameterizations (i.e., δ and α) that make directionally-informed investors trade more aggressively (i.e., that reduce their use of outside limit orders at A_2 and B_2) can improve liquidity if their stronger trading interest migrates to aggressive inside limit orders at A_1 and B_1 rather than to market orders. This is the aggressive trading effect for limit orders. Second, neutrally-informed investors have a comparative advantage in providing liquidity over uninformed investors since I_{v_0} investors know the unconditional asset value is correct. This is a version of the *Bloomfield-O'Hara-Saar effect*. Third, liquidity can change due to a *composition effect* when changes in α change the mix of informed and uninformed investors, since different types of investors affect liquidity differently. Informed I_{v_0} investors with neutral news are natural liquidity providers. Their impact on liquidity comes from whether they supply liquidity at the inside (A_1 and B_1) or outside (A_2 and B_2) prices. In contrast, directionally informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors and uninformed U traders affect liquidity depending on whether and how they opportunistically take or supply liquidity. All three effects can contribute to overturning the standard intuition about adverse selection and liquidity.

Table 1: Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.4 and 0.2). The private-value parameters are $\mu = 0$ and $\sigma = 15$, the tick size is $\kappa = 1$, and there are $N = 5$ trading dates. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at time t_1, and expected welfare of investors arriving at t_1. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar v}$,I_{v_0},$I_{\underline v}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.4$					$\delta = 0.2$				
		$I_{\bar v}$	I_{v_0}	$I_{\underline v}$	U	$Uncond.$	$I_{\bar v}$	I_{v_0}	$I_{\underline v}$	U	$Uncond.$
$\alpha = 0.8$	LSA_2	0	0.500	0.567	0.087	0.302	0	0.500	1.000	0.052	0.410
	LSA_1	0	0	0.433	0.030	0.122	0	0	0	0.079	0.016
	LBB_1	0.433	0	0	0.030	0.122	0	0	0	0.079	0.016
	LBB_2	0.567	0.500	0	0.087	0.302	1.000	0.500	0	0.052	0.410
	MBA_2	0	0	0	0.383	0.077	0	0	0	0.369	0.074
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.383	0.077	0	0	0	0.369	0.074
	NT	0	0	0	0	0	0	0	0	0	0
	E[Spread $\|\cdot$]	2.567	3.000	2.567	2.940	2.757	3.000	3.000	3.000	2.842	2.968
	E[Depth A_2+A_1 $\|\cdot$]	1.000	1.500	2.000	1.117	1.423	1.000	1.500	2.000	1.131	1.426
	E[Depth A_1 $\|\cdot$]	0	0	0.433	0.030	0.122	0	0	0	0.079	0.016
	E[Depth B_1 $\|\cdot$]	0.433	0	0	0.030	0.122	0	0	0	0.079	0.016
	E[Depth B_1+B_2 $\|\cdot$]	2.000	1.500	1.000	1.117	1.423	2.000	1.500	1.000	1.131	1.426
	E[Welfare LO $\|\cdot$]	0.367	0.614	0.367	0.089	0.377	0.846	0.688	0.846	0.153	0.665
	E[Welfare MO $\|\cdot$]	0	0	0	3.413	0.683	0	0	0	3.390	0.678
	E[Welfare $\|\cdot$]	0.367	0.614	0.367	3.502	1.060	0.846	0.688	0.846	3.543	1.343
$\alpha = 0.2$	LSA_2	0	0.500	0	0.056	0.078	0	0.500	1.000	0.063	0.150
	LSA_1	0	0	1.000	0.393	0.381	0	0	0	0.397	0.318
	LBB_1	1.000	0	0	0.393	0.381	0	0	0	0.397	0.318
	LBB_2	0	0.500	0	0.056	0.078	1.000	0.500	0	0.063	0.150
	MBA_2	0	0	0	0.051	0.041	0	0	0	0.040	0.032
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.051	0.041	0	0	0	0.040	0.032
	NT	0	0	0	0	0	0	0	0	0	0
	E[Spread $\|\cdot$]	2.000	3.000	2.000	2.213	2.237	3.000	3.000	3.000	2.206	2.365
	E[Depth A_2+A_1 $\|\cdot$]	1.000	1.500	2.000	1.449	1.459	1.000	1.500	2.000	1.460	1.468
	E[Depth A_1 $\|\cdot$]	0	0	1.000	0.393	0.381	0	0	0	0.397	0.318
	E[Depth B_1 $\|\cdot$]	1.000	0	0	0.393	0.381	0	0	0	0.397	0.318
	E[Depth B_1+B_2 $\|\cdot$]	2.000	1.500	1.000	1.449	1.459	2.000	1.500	1.000	1.460	1.468
	E[Welfare LO $\|\cdot$]	2.618	1.471	2.618	3.379	3.150	2.268	1.497	2.268	3.595	3.279
	E[Welfare MO $\|\cdot$]	0	0	0	0.803	0.643	0	0	0	0.642	0.514
	E[Welfare $\|\cdot$]	2.618	1.471	2.618	4.182	3.793	2.268	1.497	2.268	4.238	3.793

Table 2: Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.4 and 0.2) averaged over times t_2 through t_4. The private-value parameters are $\mu = 0$ and $\sigma = 15$, the tick size is $\kappa = 1$, and there are $N = 5$ trading dates. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1), the total depths on each side of the market after order submissions, and expected arriving investor welfare averaged over times t_2 through t_4. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.4$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	LSA_2	0.444	0.248	0.050	0.024	0.203	0.399	0.255	0.108	0.026	0.209
	LSA_1	0	0.246	0.254	0.149	0.163	0.192	0.239	0.288	0.064	0.205
	LBB_1	0.254	0.246	0	0.149	0.163	0.288	0.239	0.192	0.064	0.205
	LBB_2	0.050	0.248	0.444	0.024	0.203	0.108	0.255	0.399	0.026	0.209
	MBA_2	0	0	0	0.289	0.058	0	0	0	0.347	0.069
	MBA_1	0.045	0	0	0.031	0.018	0	0	0	0.058	0.012
	MSB_1	0	0	0.045	0.031	0.018	0	0	0	0.058	0.012
	MSB_2	0	0	0	0.289	0.058	0	0	0	0.347	0.069
	NT	0.207	0.013	0.207	0.012	0.116	0.013	0.010	0.013	0.011	0.012
$\alpha = 0.8$											
	E[Spread $\vert\cdot$]	2.059	2.260	2.059	2.194	2.140	2.269	2.275	2.269	2.738	2.364
	E[Depth A_2+A_1 $\vert\cdot$]	1.849	2.311	2.514	1.830	2.146	2.165	2.300	2.433	1.608	2.161
	E[Depth A_1 $\vert\cdot$]	0.001	0.370	0.944	0.405	0.432	0.226	0.362	0.506	0.131	0.318
	E[Depth B_1 $\vert\cdot$]	0.944	0.370	0.001	0.405	0.432	0.506	0.362	0.226	0.131	0.318
	E[Depth B_1+B_2 $\vert\cdot$]	2.514	2.311	1.849	1.830	2.146	2.433	2.300	2.165	1.608	2.161
	E[Welfare LO $\vert\cdot$]	0.088	0.130	0.088	0.928	0.267	0.299	0.133	0.299	0.055	0.206
	E[Welfare MO $\vert\cdot$]	0.041	0	0.041	2.969	0.616	0	0	0	3.538	0.708
	E[Welfare $\vert\cdot$]	0.129	0.130	0.129	3.897	0.883	0.299	0.133	0.299	3.592	0.914
	LSA_2	0.103	0.390	0.601	0.083	0.139	0.375	0.389	0.443	0.093	0.155
	LSA_1	0	0.094	0.201	0.069	0.075	0.044	0.096	0.116	0.066	0.070
	LBB_1	0.201	0.094	0	0.069	0.075	0.116	0.096	0.044	0.066	0.070
	LBB_2	0.601	0.390	0.103	0.083	0.139	0.443	0.389	0.375	0.093	0.155
	MBA_2	0	0	0	0.214	0.171	0	0	0	0.218	0.175
	MBA_1	0.070	0	0	0.131	0.110	0	0	0	0.120	0.096
	MSB_1	0	0	0.070	0.131	0.110	0	0	0	0.120	0.096
	MSB_2	0	0	0	0.214	0.171	0	0	0	0.218	0.175
	NT	0.026	0.033	0.026	0.005	0.009	0.022	0.030	0.022	0.005	0.009
$\alpha = 0.2$											
	E[Spread $\vert\cdot$]	2.116	2.122	2.116	2.359	2.311	2.212	2.173	2.212	2.478	2.422
	E[Depth A_2+A_1 $\vert\cdot$]	1.459	2.096	2.549	1.572	1.665	3.066	3.026	3.066	2.442	1.680
	E[Depth A_1 $\vert\cdot$]	0.239	0.440	0.704	0.324	0.351	0.346	0.414	0.442	0.262	0.290
	E[Depth B_1 $\vert\cdot$]	0.704	0.440	0.239	0.324	0.351	0.442	0.414	0.346	0.262	0.290
	E[Depth B_1+B_2 $\vert\cdot$]	2.549	2.096	1.459	1.572	1.665	2.257	2.091	1.932	1.576	1.680
	E[Welfare LO $\vert\cdot$]	1.228	0.609	1.228	0.477	0.586	1.206	0.654	1.206	0.500	0.605
	E[Welfare MO $\vert\cdot$]	0.278	0	0.278	3.467	2.811	0	0	0	3.417	2.734
	E[Welfare $\vert\cdot$]	1.506	0.609	1.506	3.944	3.397	1.206	0.654	1.206	3.917	3.338

The two measures of liquidity we consider are the expected bid-ask spread and the expected depth at the inside prices. In Table 1, liquidity improves at time t_1 when the value-shock volatility δ increases (comparing parameterizations horizontally with α fixed). This happens, contrary to the standard intuition, because the informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors submit limit orders at the inside quotes in these high-volatility markets, whereas they only use limit orders at the outside quotes in the low-volatility markets. Violations of the standard adverse-selection intuition also occur on average at times t_2 through t_4 in Table 2. Once again, liquidity improves when δ increases for both the low and high α values. However, the underlying causes are different: When α is low (0.2), high-volatility markets are more liquid due both to the increased average use of inside limit orders (0.201 vs 0.044 + 0.116) by directionally-informed investors at t_2 through t_4 (i.e., the aggressive trading effect for limit orders) and due to a mechanical *liquidity carry-over effect* from a deeper book at t_1. In contrast, when α is high (0.8), directionally-informed investors in the high-volatility market use fewer inside limit orders at times t_2 through t_4 (0.254 vs 0.192 + 0.288), but average liquidity is still better due to the carry-over effect of greater depth from the book at time t_1.

Next, consider the effect of the amount of informed trading α on liquidity (comparing parameterizations vertically with δ fixed). At t_1, liquidity is decreasing in α, as per the standard intuition, due to the camouflage effect (with $\delta = 1.4$) and greater adverse selection for uninformed investors (with $\delta = 0.2$). However, at times t_2 through t_4, for both high and low value-shock volatility, a higher probability α of informed investors increases liquidity. Liquidity improves because i) neutrally-informed I_{v_0} investors consistently increase their use of limit orders at the inside prices due to increased informational competition and ii) there is a *composition effect* with more informed investors, who tend to use inside limit orders more than uninformed investors at later dates.

Our results show that the relation between adverse selection and liquidity in limit order markets is more subtle than the standard intuition. In particular, it is the ability of investors to choose endogenously whether to supply or demand liquidity and at what prices that can overturn the standard intuition. Goettler et al. (2009) also investigate a market with informed traders with no private-value motives and uninformed with private-value motives. In their model, when volatility increases, informed traders reduce their liquidity provision and increase their demand for liquidity; with the opposite holding for uninformed traders. Our results show the effects are more nuanced.

2.1.3 Welfare

Tables 1 and 2 also report results about investor welfare. First, we clearly see the importance of limit orders for informed investors. Even in parameterizations in which informed traders use market orders, most of their expected gains-from-trade come from limit orders. Second, the equilibrium impact of greater value-shock volatility on directionally-informed investor welfare is mixed, whereas a partial-equilibrium intuition might suggest it should be positive. With a low $\alpha = 0.2$ at times t_2 through t_4, increasing δ increases directionally-informed investor welfare (1.506 up from 1.206). However, with a high $\alpha = 0.8$, increasing δ leads to lower average welfare for directionally-informed investors at times t_2 through t_4 (0.129 down from 0.299). This is due partly to their increased probability of NT (0.207 up from 0.013) due to the reduced use of market orders by uninformed investors and because the book can fill up in the direction of the information resulting in directionally-informed investors providing liquidity opposite their information using low-profit outside limit orders. Third and perhaps more surprisingly, uniformed U investor utility can increase when the value volatility δ is larger at t_2 through t_4. This is due to an increased trading demand by informed investors that leads to more trading that allows more uninformed investors to capture more of their potential gains-from-trade, which are large in the parameterizations here relative to the still small increased adverse selection. The net effect is that it is possible, in low-α parameterizations, to overturn another standard intuition about adverse selection.

Observation 2 It is possible for greater adverse selection due to greater value-shock volatility δ to increase total unconditional investor welfare.

In contrast, total welfare is reduced when the arrival probability α of informed investors increases. This is because in this model specification only uninformed U investors have private values (i.e., informed trading leads to zero-sum net transfers between investors).

2.1.4 Information content of orders

Real-world traders and empirical researchers are interested in the information content of different types of orders for price discovery.[20] Informativeness of an order x_{t_1} at time t_1 is measured here as the Bayesian revision $E[v|\, x_{t_1}] - E[v]$ in the uninformed investor's expectation of the terminal value v after an order x_{t_1} at time t_1. Informativeness at later dates t_2 through t_4 is the Bayesian revision $E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}]$ for different given types of orders x_{t_j} at time t_j relative to the incoming expectation conditional on the preceding in-equilibrium order-flow history $\mathscr{L}_{t_{j-1}}$.

[20]Fleming et al. (2017) extend the VAR estimation approach of Hasbrouck (1991) to estimate the price impacts of limit orders as well as market orders. See also Brogaard et al. (2019).

A necessary condition for an order to be informative is that informed investors use it. However, the magnitude of order informativeness is determined by the mix of equilibrium probabilities with which informed and uninformed traders use an order. If uninformed traders use the same orders as informed investors, they add noise to the price discovery process, and orders become less informative. In our model, the mix of information– and noise–based orders depends on the exogenous market parameterization (the underlying proportion α of informed investors and value-shock volatility δ) and on investors' endogenous order-submission strategies.

We expect different market and limit orders to have different information content. A natural conjecture is that the sign of the information revision associated with an order should agree with the order direction (e.g., buy market and limit orders should lead to positive valuation revisions). Another natural conjecture is that the magnitude of information revisions should be greater for more aggressive orders. Surpisingly, however, the order-sign and order-aggressiveness conjectures need not always hold.

Observation 3 Order informativeness is not always increasing in order aggressiveness.

Observation 4 The direction of order informativeness can be opposite the order sign.

These results are another consequence of how informed investors trade on their information. In addition, the relative informativeness of different market and limit orders change in high-volatility and low-volatility markets.

The order-aggressiveness violation is immediate from first principles in Proposition 1 for (aggressive) market orders versus (less aggressive) limit orders in low-volatility ($\delta = 0.2$) markets in which informed investors avoid market orders all together. However, order-aggressiveness violations for market orders can also obtain with high volatility. In addition, the order-aggressiveness conjecture can fail for aggressive limit orders at the inside quotes (A_1 and B_1) relative to less-aggressive limit orders at the outside quotes (A_2 and B_2).

Figure 2 shows the informativeness of different types of market and limit buy orders. Each row has four plots showing the informativeness of different types of orders at different times during the day for each of the four market parameterizations. In particular, the informativeness of a given order type may change over time and may differ conditional on different preceding order histories. The vertical heights of the individual dots in the plots indicate the informativeness of given orders at particular times given different preceding histories.[21] The associated probabilities can differ for the different dots. The rectangles

[21] A sequence of equilibrium orders might be produced by more than one investor-arrival sequence. Thus, individual dots correspond to sets of investor-arrival sequences. The horizontal spacing of the dots is simply for ease of viewing.

show the range of our informativeness metrics across paths. The vertical height of the blue squares indicate the probability-weighted average informativeness of a given type of order across all prior in-equilibrium paths at a point in time. The results are symmetric for sell orders.

The results in Figure 2 point to a variety of properties about order informativeness. First, the most obvious point is the heterogeneity in the information content of a given order both at different times during the day and also conditional on different prior order histories. For example, plot I(c) shows the Bayesian revisions for a LBB_1 limit buy order at the inside quotes B_1 in the high-δ/high-α market. At time t_1, an inside LBB_1 order is highly revealing with a Bayesian revision relative to the unconditional expectation of 1.331. This follows from the fact in Table 1 that informed $I_{\bar{v}}$ investors with good news use LBB_1 orders with a much higher probability than uninformed U investors at time t_1. However, at later dates an LBB_1 limit order has different information content depending on the prior history.[22] Over time the number of equilibrium paths grows by definition, but, in addition, the amount of equilibrium informational heterogeneity across paths also grows (i.e., the number of dots associated with individual paths grows). Moreover, there are an increasing number of paths with zero Bayesian revisions. One reason this happens is that the number of fully revealing prior order histories is non-decreasing over time.

Second, Figure 2 shows that the order-aggressiveness conjecture can fail in a variety of ways. While the conjecture can fail for individual paths, we focus here on even stronger results where the order-aggressiveness conjecture fails in expectation across paths. One example is that the expected Bayesian revisions across-paths (the small solid squares) for limit buys are frequently larger (i.e., higher) than for market buys. This follows immediately from Proposition 1 for low-volatility markets ($\delta = 0.2$). However, the conjecture also fails in high-volatility markets ($\delta = 1.4$) where informed investors do use market orders. For example, in the high-δ/high-α market, the average revision at t_3 for inside limit orders in Plot I(c) is larger than for market orders at the inside quotes in Plot I(b). This is even more

[22] The fact that inside limit orders are also submitted by uninformed U investors with a small, but still positive, probability at t_1 explains the outcomes (dots) in Figure 2 with extremely large positive valuation swings for MBA_1 market buys at t_2 and later in the day. These are paths where a LSA_1 limit order submitted by an uninformed U investor at t_1 was interpreted as almost revealing bad news (given the high probability that such orders are submitted by $I_{\underline{v}}$ investors), but then a subsequent MBA_1 market buy fully reveals good news since in this parameterization only informed $I_{\bar{v}}$ investors submit MBA_1 market buys on such paths. This outcome is rare given the low probability of uninformed investors arriving and submitting inside limit orders at t_1 in this parameterization. A similarly unlikely but possible sequence of events explains the outcome (dot) associated with a large negative valuation swing given an outside LBB_2 limit buy at time t_4. In this case, an inside LBB_1 limit buy was submitted by an uninformed investor at t_1 (and was interpreted as almost revealing good news) and was followed by an inside LSA_1 limit sell at t_2 (which did not contract bad news since uninformed investors submit such orders at t_2) and then the standing LBB_1 order was hit by a market sell at t_3 (which was interpreted as an uninformed-investor order) In this case, only an informed investor with bad news submits an outside LBB_2 limit buy at t_4 (to provide liquidity since a limit sell would not execute given the standing limit sell at A_1) which fully reveals bad news, leading to the large valuation swing.

true at t_2 and t_3 for MBA_1 market orders in Plot II(b) versus LBB_1 and LBB_2 limit orders in II(c) and II(d) in the high-δ/low-α market.

The order-aggressiveness conjecture can also fail for aggressive vs. less-aggressive limit orders. Plots II(c) and II(d) for the high–δ/low–α market show that less-aggressive LBB_2 limit buys at t_2 and t_3 have larger average Bayesian revisions than the aggressive LBB_1 limit buys. Having shown that the aggressiveness conjecture can fail, we also note that it does not always fail. For example, the average Bayesian revisions for aggressive LBB_1 limit orders at times t_1 through t_4 in Plot I(c) are larger than for the less-aggressive LBB_2 limit orders in Plot I(d).

Third, Figure 2 shows that violations of the order-sign conjecture are rare but possible. Buy orders are associated with non-negative Bayesian revisions for most individual paths and, thus, in expectation. However, there are exceptions. We see this for a few paths in Plot I(d) at t_4. More dramatically, the order-sign conjecture fails in expectation for LBB_2 limit buys at time t_2 in the low-δ/high-α parameterization in Plot III(d). This is, in part, a consequence of the fact in Table 2 that investors use outside limit orders to provide liquidity opposite their information (e.g., limit sells at A_2 given good news $\bar{v}$ since $A_2 > \bar{v}$ here) more than to trade with their information (e.g., limit sells at A_2 given bad news $\underline{v}$).[23] Violations of the order-sign conjecture are even more frequent in Section 2.2 below when informed investors also have private-value motives to trade.

Figure 2 also has one further implication:

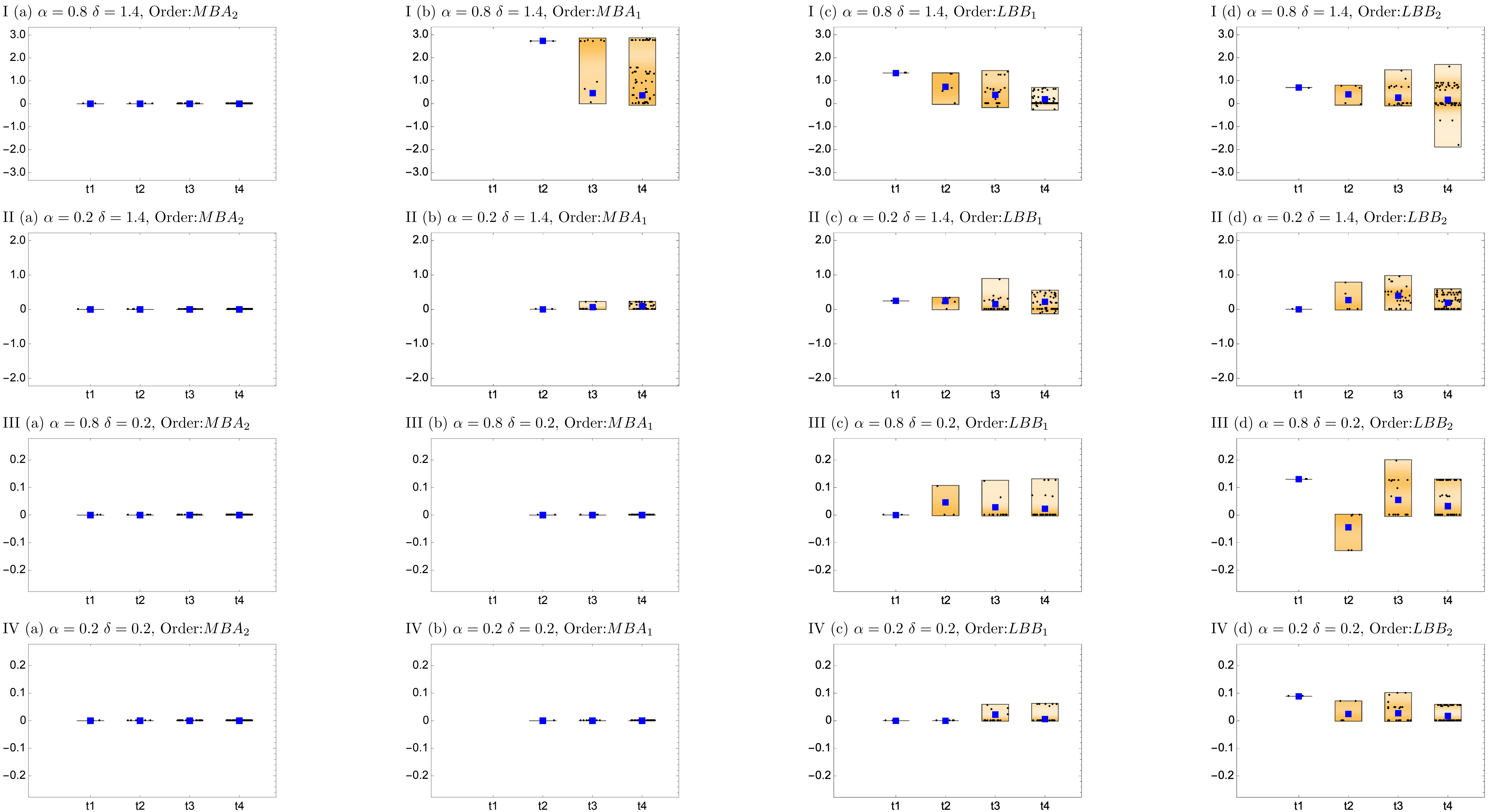
Observation 5 The information content of market orders can differ depending on the limit prices at which they are executed.

In the two high-variance markets, market orders executed at the inside prices reveal information whereas those at the outside prices do not.[24] This implies that empirical price-impact estimation should treat inside and outside market orders separately and not pool them.

[23]Note that directionally-informed investors submit outside limit orders opposite their information more than with their information due to unbalanced deep books in the low-δ/high-α market in Table 2. However, this is not sufficient for an order-sign violation. We also see more outside limit order submissions opposite directionally-informed investors' information in other market parameterizations in Table 2 without order-sign violations in Figure 2. What matters for order-sign violation is not the total probability of limit order submission opposite an investor's information, but rather the joint distribution of the order information content and order submissions. For example, if a market is already fully-revealing on the preceding order path, then there is no information in any subsequent limit orders.

[24]A similar result easily obtains, more generally, in any limit order market in which an arbitrary private-value distribution has a continuous support that extends beyond an arbitrary bounded private-information support (i.e., uninformed trading motives are potentially larger than speculative motives) where the discrete price grid includes prices inside the private-information distribution. In such a generalized market, there will be two qualitatively different categories of market orders: Market orders executed at extreme standing bids and asks outside of the private-information support will only be used by uninformed trades with extreme private values, and, thus, will, a priori, have zero information content. In contrast, market orders executed at standing bids and asks inside the private-information support could potentially come from informed or uninformed investors, and so their information content depends on the specifics of investors' endogenous trading strategies.

Figure 2: Order Informativeness for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 to t_4. This figure shows the path-contingent Bayesian value-forecast revisions $E[v|\mathscr{L}_{t_j-1}, x_{t_j}] - E[v|\mathscr{L}_{t_j-1}]$, which shows the change in the uninformed traders's expected value of the fundamental conditional on different actions x_{t_j}. We only consider orders when they are equilibrium orders for the trading periods. Each dot indicates an equilibrium revision, the rectangles indicate the maximum and the minimum, and the squares denote probability-weighed cross-path means. The plots are grouped by their respective market parameterizations (δ, α).



2.1.5 Non-Markovian learning

This section investigates the role of the order history in Bayesian learning. A major difference between our model and Goettler et al. (2009) and Roşu (2016b) is their assumption that learning is Markovian in that the standing limit order book L_{t_j} is a sufficient statistic at times $t_j > t_1$ for the information content of the full prior order history $\mathscr{L}_{t_j}$. In contrast, we show the prior order history can have substantial information about the asset value v in excess of the information in the standing limit order book.[25] [26]

Observation 6 Price discovery can be significantly non-Markovian.

The plots in Figure 3 measure the non-Markov information content of order histories by

$$E[v|\,\mathscr{L}_{t_j}(L_{t_j})] - E[v|\,L_{t_j}], \tag{13}$$

which is the incremental information in the uninformed investors' expected asset value conditional on an order history path $\mathscr{L}_{t_j}(L_{t_j})$ ending with a particular limit order book L_{t_j} at time t_j net of the corresponding expectation conditional on just the ending book L_{t_j}. In particular, we are interested in books L_{t_j} that can be preceded in equilibrium by more than one different prior history. If learning is Markov, then order histories $\mathscr{L}_{t_j}(L_{t_j})$ preceding a book L_{t_j} should convey no additional information beyond L_{t_j}; in which case our metric in (13) should be zero. Individual dots in the plots indicate the incremental information content of particular histories preceding different orders submitted at each of the different dates.[27] The standing book L_{t_j} may reflect some information at each time t_j about the path of past active investor actions. However, i) the book does not necessarily fully reveal the timing of past orders, and ii) past crowd actions can partially obscure past active-investor actions, since the crowd replenishes the book when it is depleted at the outside prices by active investors. Each plot is for a different market parameterization. For brevity, the plots include all possible books L_{t_j}, rather than having separate plots for individual books.

The main result from Figure 3 is that there is substantial incremental information in the preceding order histories $\mathscr{L}_{t_j}(L_{t_j})$ after conditioning on the standing limit order book L_{t_j}. As expected, variation in the conditioning information in the preceding order histories in Figure 3 is greater when the shock volatility δ is greater (note the difference in vertical

[25] The evidence of path-contingent order informativeness in Figure 2 by itself does not necessarily imply non-Markovian learning. Markovian learning is still possible if the incoming book L_{t_j} at time t_j summarizes the information content of the full order history $\mathscr{L}_{t_j}(L_{t_j})$ preceding book L_{t_j}.

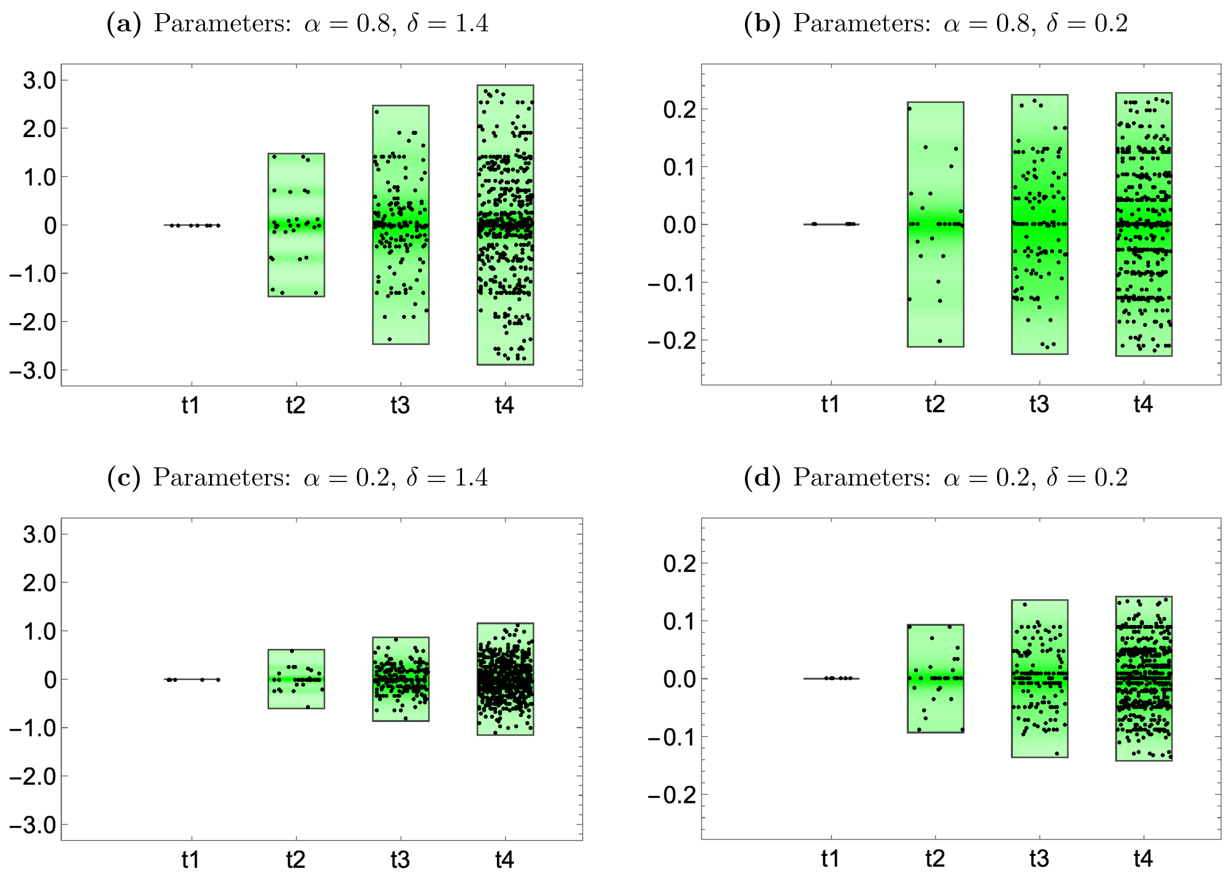
[26] Goettler et al. (2009) also includes the most recent transaction, in addition to the standing limit order book, in the investor state space.

[27] Time t_1 is included in the figure because books L_{t_1} at t_1 can potentially be produced by different sequences of active investor actions x_{t_1} and crowd responses at t_1.

scales).

Given that learning is non-Markovian, the next question is about what features of order-flow histories are informative. The next two sections consider several aspects of this question.

Figure 3: Informativeness of the Order History for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for Times t_1 through t_4. This figure shows the incremental information content of the past order history in excess of the information in the standing limit order book observed at the end of time t_j as measured by $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ where $\mathscr{L}_{t_j}(L_{t_j})$ is a history ending in the limit order book L_{t_j}. We only consider books L_{t_j} that occur in equilibrium in the different trading periods. The dots indicate values for particular books and paths, and the rectangles show the range of maximum and minimum values.



2.1.6 Price-impact of order flow

A standard empirical measure of price-discovery is the informational price-impact of order flow. The idea is that the price-impact of orders can be decomposed into two components: One measures the size of surprises in arriving orders relative to their expectations given the prior history, and the second measures the marginal (per-share) valuation impact of order-flow surprises. Fleming et al. (2017) and Brogaard et al. (2019) extend the Hasbrouck (1991) vector autoregression methodology — a standard empirical technique to estimate this decomposition — to allow for limit orders as well as market orders. Using our notation, their information innovation equation can be written as

$$E[v|x_t, \mathscr{L}_{t-1}] - E[v|\mathscr{L}_{t-1}] = \sum_k \lambda_k [Q_{k,t}^{x_t} - E[Q_{k,t}^{x_t}|\mathscr{L}_{t-1}]] \quad (14)$$

where $Q_{k,t}^{x_t} - E[Q_{k,t}^{x_t}|\mathscr{L}_{t-1}]$ in the innovation in the number of shares $Q_{k,t}^{x_t}$ associated with an order type k (e.g., a particular market or limit order) given the investor action x_t at time t, and λ_k is a constant marginal price-impact for order type k.

This section shows the potential need for a further extension of VAR estimation.[28]

Observation 7 The price-impacts of order flow are functions $\lambda_k(t, \mathscr{L}_{t-1})$ that are conditional on time t_j and the prior order history $\mathscr{L}_{t-1}$.

Simple empirical specifications might look at deterministic functions of time $\lambda_k(t)$ or conditioning (as in GPR 2009) on the standing limit order book L_{t-1} via a function $\lambda_k(t, L_{t-1})$. In its most general form, our analysis suggests using machine learning techniques to identify potentially high-dimensional relationships $\lambda_k(t, \mathscr{L}_{t-1})$ given prior order histories. Yueshen and Zhang (2019) find the price-impact of orders follows a stochastic process. Our results identify the standing limit order book and prior order-history paths as variables that may explain random time-varying price-impact.

Figure 4 shows that even our very simple model generates substantial variation in the conditional price-impact of orders. Consider an order sequence $\{\mathscr{L}_{t_{j-1}}, x_{t_j}\}$ where sequences $\{\mathscr{L}_{t_{j-1}}, x_{t_j}\}$ and $\{\mathscr{L}_{t_{j-1}}, NT\}$ both have positive equilibrium probabilities. As a metric for dispersion in the conditional price impact of order flow, we compute[29]

$$\max_{\mathscr{L}_{t_{j-1}}} \left[E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT]\right] - \min_{\mathscr{L}_{t_{j-1}}} \left[E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT]\right] \quad (15)$$

In words, $E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT]$ is the differential informational impact of a one-unit innovation in order type x_{t_j} relative to NT where differencing controls for expectations given the prior history $\mathscr{L}_{t_{j-1}}$. The diff-in-diff metric in (15) is the spread between the maximal and minimum differential informational innovation across all paths $\mathscr{L}_{t_{j-1}}$ such that order x_{t_j} and NT both occur with positive probability following the different paths $\mathscr{L}_{t_{j-1}}$. As can be seen, the amount of cross-path dispersion in the conditional price-impact of order flow can be substantial.[30]

2.1.7 Information and order-path characteristics

This section examines specific characteristics of order paths and their impact on price discovery. One natural conjecture is that the magnitude of valuation forecast errors is decreasing in the magnitude of prior valuation revisions caused by the preceding realized order history. In

[28] Note that path-dependence in the price-impact of orders $\lambda_k(t, \mathscr{L}_{t-1})$ is conceptually different from path-dependent order-flow expectations $E[Q_{k,t}^{x_t}|\mathscr{L}_{t-1}]$ used to construct order innovations $Q_{k,t}^{x_t} - E[Q_{k,t}^{x_t}|\mathscr{L}_{t-1}]$.

[29] The formulation here reflects the fact that our model only has one non-zero order size.

[30] Our order-impact statistic can only be computed once the book is sufficiently full such that NT becomes an option for arriving investors. In these parameterizations, that only happens at times t_3 and t_4.

other words, it may be easier for the trading process to reveal positive or negative valuation shocks than to reveal the absence of a valuation shock. Our results are consistent with this conjecture. Consider, for example, time t_4, which has the longest order histories for which both market and limit orders are potentially used. The valuation-error heteroscedasticity conjecture can be assessed using the correlation between the absolute cumulative Bayesian revision $|E[v|\mathscr{L}_{t_4}] - E[v]|$ up to time t_4 and the associated expected absolute forecast error $E[\,|v - E[v|\mathscr{L}_{t_4}]|\,]$ at time t_4. In Table 3, the first column in the four market parameterization cells shows that these correlations are negative (as conjectured) and strongly so for three of our four different parameterizations. In other words, the volatility of potential informational price changes later in the day are smaller when the path of realized orders earlier in the day has already revealed substantial information.

This raises then the question of what specific path characteristics are informative. A natural set of candidate informative path characteristics are cumulative order imbalances. To explore this, the second two columns in Table 3 report correlations between expected absolute forecast errors $E[\,|v - E[v|\mathscr{L}_{t_j}]|\,]$ at time t_4 and the absolute value of different preceding order-flow imbalances. In particular, we consider imbalances for market and limit orders executed or posted at the inside quotes and also total imbalances at both the inside and outside quotes. The question is whether different types of market- and limit-order imbalances are informative. A negative (positive) correlation means larger imbalances for a particular type of order are more (less) informative. Table 3 shows that limit-order imbalances are consistently more informative than market-order imbalances. This is not surprising when the value volatility δ is small and informed investors do not use market orders, but it is also true when δ is high and informed investors do use market orders. As a result, larger market-order imbalances means more low- or non-informative market orders arrived over time than the more-informative limit orders. Once again, we also see that the information content of market orders executed at the inside quotes is different from all market orders pooled together.

2.1.8 Summary

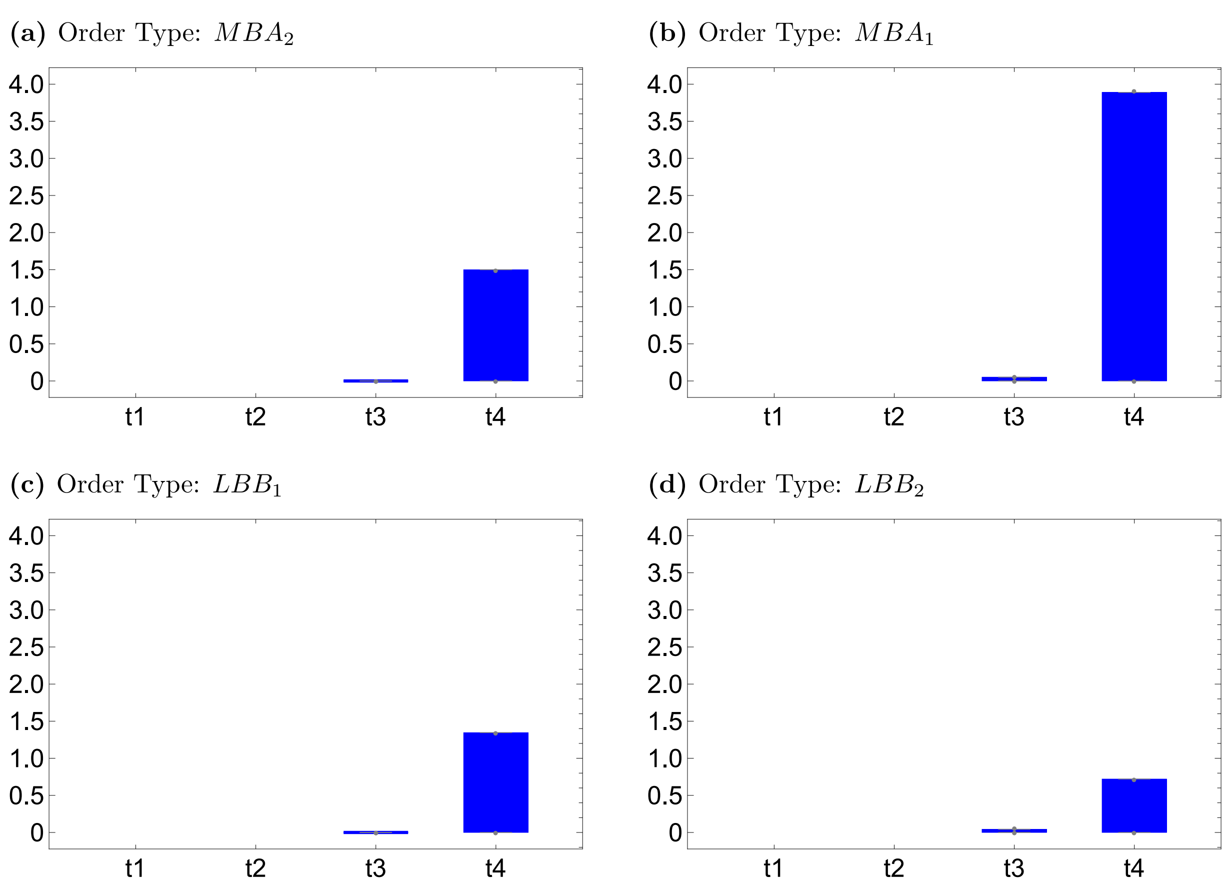
The analysis of our first model specification has identified a number of empirically testable predictions. First, liquidity and the relative information content of different orders differ in high-volatility markets (in which value shocks are large relative to the tick size) vs. in low-volatility markets. Second, it is possible for less-aggressive orders to be more informative than more aggressive orders and for the information content of some orders to be opposite the order sign. Third, price discovery is non-Markov. Fourth, the price-impact of orders

Table 3: Correlations between expected absolute pricing errors, prior absolute valuation revisions, and order imbalances. This table reports correlations between the uninformed traders' expected absolute pricing errors $E[|v - E[v|\mathscr{L}_{t_4}]|]$ at time t_4, the absolute past valuation revisions $|E[v|\mathscr{L}_{t_4}] - E[v]|$ (column 1), the market and limit order imbalances both for orders executed/submitted only at the inside quotes and for all orders at both quotes at t_1 (column 2) and foe $t-2$ through t_4 (column 3). Four combinations of value-shock volatilities and informed investor-arrival probabilities, δ and α, are considered. The private-value parameters are $\mu = 0$ and $\sigma = 15$, the tick size is $\kappa = 1$, and there are $N = 5$ trading dates.

			$\delta = 1.4$			$\delta = 0.2$		
				t_1	$t_2 - t_4$		t_1	$t_2 - t_4$
$\alpha = 0.8$		$\lvert E[v\lvert\mathscr{L}_{t_4}] - E[v]\rvert$	-0.027			-0.317		
	All	$\lvert\#MB\text{-}\#MS\rvert$		0.178	0.220		0.100	0.359
		$\lvert\#LB\text{-}\#LS\rvert$		-0.178	-0.052		-0.100	0.000
	Inside	$\lvert\#MB\text{-}\#MS\rvert$		N/A	0.129		N/A	0.051
		$\lvert\#LB\text{-}\#LS\rvert$		-0.099	0.033		0.429	-0.098
$\alpha = 0.2$		$\lvert E[v\lvert\mathscr{L}_{t_4}] - E[v]\rvert$	-0.352			-0.647		
	All	$\lvert\#MB\text{-}\#MS\rvert$		0.045	0.169		0.070	0.092
		$\lvert\#LB\text{-}\#LS\rvert$		-0.045	-0.196		-0.070	-0.017
	Inside	$\lvert\#MB\text{-}\#MS\rvert$		N/A	0.282		N/A	0.253
		$\lvert\#LB\text{-}\#LS\rvert$		0.355	-0.145		0.622	-0.305

varies conditional on the prior order history.

Figure 4: Dispersion in the price impact of order flow The plot reports $max_{\mathscr{L}_{t_{j-1}}}(E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT]) - min_{\mathscr{L}_{t_{j-1}}}(E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT])$ at different times, which shows how the prior order history affects the marginal price impact of the surprise in a given order. The parameterization is: $\alpha = 0.8$, $\delta = 1.4$



2.2 Informed and uninformed traders both have private-value motives

Our second model specification generalizes our earlier analysis. Now informed investors also have random private-valuation factors β_{t_j} with the same truncated-Normal distribution $\beta_{t_j} \sim Tr[\mathscr{N}(\mu, \sigma^2)]$ as the uninformed investors. Hence, informed traders not only speculate on their information, but they also have preference shocks, hedging needs and tax-induced private-value motives to trade. As a result, informed investors arriving at different times with the same valuation information potentially buy or sell from each other due to their different private values. This combination of trading motives has not been investigated in earlier models of dynamic limit order markets. Our second model specification lets us assess the robustness of the results in Section 2.1 and extend them.

2.2.1 Trading strategies

Tables 4 and 5 report order-submission probabilities and other statistics for our second model specification for time t_1 and for averages over times t_2 through t_4. There are a few differences relative to Tables 1 and 2 for the simpler model in Section 2.1. First, now all investors use all of the possible limit orders in both time windows, since now all investors have private-value motives to trade. Second, informed investors of all types use market orders at times t_2 through t_4 when their private-value trading motive is sufficiently strong. In particular, I_{v_0} investors with neutral news no longer just provide liquidity using limit orders. Third, trading against their private asset-value information is even stronger now for directionally-informed investors. In particular, $I_{\bar{v}}$ and $I_{\underline{v}}$ investors not only submit outside limit orders (which are always profitable given private good or bad news) as in our first model in Section 2.1, but now they also post inside limit orders against their private information (which are unprofitable when $\delta = 1.4$). Indeed, Tables 4 and 5 both show that outside limit orders are consistently used more by investors to trade opposite their information than to trade with their information. The same is also true on average at times t_2 through t_4 for inside limit orders in all of the parameterizations. These stronger results are due not just to deep-book effects (at outside prices), but also because their private-value motives can overwhelm their speculative motives (at inside as well as outside prices). The fact that informed investors frequently use limit orders to trade against their information will have important implications for the information content (considered below) of such limit orders.

Consider next the impact of adverse selection on trading. The effect of higher δ and higher α on directionally-informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors differs when they are trading with or opposite their information. For investors trading with their information, we see the aggressiveness trading effect for limit orders again, similar to the results in Section 2.1. In particular,

Table 4: Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.4 and 0.2). The private-value parameters are $\mu = 0$ and $\sigma = 15$, the tick size is $\kappa = 1$, and there are $N = 5$ trading dates. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submisions at time t_1, and the expected welfare of investors arriving at t_1. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.4$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
$\alpha = 0.8$	LSA_2	0.107	0.053	0.032	0.062	0.064	0.054	0.048	0.042	0.048	0.048
	LSA_1	0.333	0.447	0.303	0.438	0.377	0.438	0.452	0.466	0.452	0.452
	LBB_1	0.303	0.447	0.333	0.438	0.377	0.466	0.452	0.438	0.452	0.452
	LBB_2	0.032	0.053	0.107	0.062	0.064	0.042	0.048	0.054	0.048	0.048
	MBA_2	0.224	0	0	0	0.060	0	0	0	0	0
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0.224	0	0.060	0	0	0	0	0
	NT	0	0	0	0	0	0	0	0	0	0
	E[Spread $\vert\cdot$]	2.363	2.106	2.363	2.123	2.247	2.096	2.096	2.096	2.096	2.096
	E[Depth A_2+A_1 $\vert\cdot$]	1.441	1.500	1.335	1.500	1.440	1.492	1.500	1.508	1.500	1.500
	E[Depth A_1 $\vert\cdot$]	0.333	0.447	0.303	0.438	0.377	0.438	0.452	0.466	0.452	0.452
	E[Depth B_1 $\vert\cdot$]	0.303	0.447	0.333	0.438	0.377	0.466	0.452	0.438	0.452	0.452
	E[Depth B_1+B_2 $\vert\cdot$]	1.335	1.500	1.441	1.500	1.440	1.508	1.500	1.492	1.500	1.500
	E[Welfare LO $\vert\cdot$]	2.776	4.454	2.776	4.295	3.527	4.462	4.465	4.462	4.461	4.462
	E[Welfare MO $\vert\cdot$]	1.671	0	1.671	0	0.891	0	0	0	0	0
	E[Welfare $\vert\cdot$]	4.447	4.454	4.447	4.295	4.419	4.462	4.465	4.462	4.461	4.462
$\alpha = 0.2$	LSA_2	0.061	0.050	0.043	0.050	0.050	0.049	0.048	0.046	0.048	0.048
	LSA_1	0.368	0.450	0.484	0.450	0.447	0.441	0.452	0.464	0.452	0.452
	LBB_1	0.484	0.450	0.368	0.450	0.447	0.464	0.452	0.441	0.452	0.452
	LBB_2	0.043	0.050	0.061	0.050	0.050	0.046	0.048	0.049	0.048	0.048
	MBA_2	0.045	0	0	0	0.003	0	0	0	0	0
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0.045	0	0.003	0	0	0	0	0
	NT	0	0	0	0	0	0	0	0	0	0
	E[Spread $\vert\cdot$]	2.148	2.101	2.148	2.101	2.107	2.096	2.096	2.096	2.096	2.096
	E[Depth A_2+A_1 $\vert\cdot$]	1.429	1.500	1.526	1.500	1.497	1.490	1.500	1.510	1.500	1.500
	E[Depth A_1 $\vert\cdot$]	0.368	0.450	0.484	0.450	0.447	0.441	0.452	0.464	0.452	0.452
	E[Depth B_1 $\vert\cdot$]	0.484	0.450	0.368	0.450	0.447	0.464	0.452	0.441	0.452	0.452
	E[Depth B_1+B_2 $\vert\cdot$]	1.526	1.500	1.429	1.500	1.497	1.510	1.500	1.490	1.500	1.500
	E[Welfare LO $\vert\cdot$]	4.093	4.452	4.093	4.433	4.389	4.466	4.465	4.466	4.465	4.465
	E[Welfare MO $\vert\cdot$]	0.422	0	0.422	0	0.056	0	0	0	0	0
	E[Welfare $\vert\cdot$]	4.516	4.452	4.516	4.433	4.445	4.466	4.465	4.466	4.465	4.465

Table 5: Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different value-shock volatilities δ (1.4 and 0.2) averaged over times t_2 through t_4. The private-value parameters are $\mu = 0$ and $\sigma = 15$, the tick size is $\kappa = 1$, and there are $N = 5$ trading dates. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at times t_2 through t_4, and the expected arriving investor welfare averaged over times t_2 through t_4. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.4$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
$\alpha = 0.8$	LSA_2	0.138	0.121	0.094	0.115	0.117	0.127	0.123	0.119	0.123	0.123
	LSA_1	0.103	0.057	0.050	0.065	0.069	0.057	0.053	0.048	0.053	0.053
	LBB_1	0.050	0.057	0.103	0.065	0.069	0.048	0.053	0.057	0.053	0.053
	LBB_2	0.094	0.121	0.138	0.115	0.117	0.119	0.123	0.127	0.123	0.123
	MBA_2	0.263	0.192	0.123	0.194	0.193	0.207	0.194	0.181	0.194	0.194
	MBA_1	0.158	0.128	0.069	0.124	0.119	0.133	0.128	0.124	0.129	0.128
	MSB_1	0.069	0.128	0.158	0.124	0.119	0.124	0.128	0.133	0.129	0.128
	MSB_2	0.123	0.192	0.263	0.194	0.193	0.181	0.194	0.207	0.194	0.194
	NT	0.003	0.003	0.003	0.004	0.004	0.004	0.003	0.004	0.004	0.004
	E[Spread $\vert\cdot$]	2.352	2.326	2.352	2.365	2.348	2.336	2.337	2.336	2.337	2.336
	E[Depth A_2+A_1 $\vert\cdot$]	1.602	1.599	1.550	1.570	1.581	1.590	1.593	1.596	1.593	1.593
	E[Depth A_1 $\vert\cdot$]	0.308	0.339	0.344	0.320	0.328	0.324	0.333	0.344	0.333	0.334
	E[Depth B_1 $\vert\cdot$]	0.344	0.339	0.308	0.320	0.328	0.344	0.333	0.324	0.333	0.334
	E[Depth B_1+B_2 $\vert\cdot$]	1.550	1.599	1.602	1.570	1.581	1.596	1.593	1.590	1.593	1.593
	E[Welfare LO $\vert\cdot$]	0.872	0.700	0.872	0.720	0.796	0.674	0.671	0.674	0.670	0.672
	E[Welfare MO $\vert\cdot$]	3.272	3.333	3.272	3.313	3.296	3.357	3.357	3.357	3.358	3.357
	E[Welfare $\vert\cdot$]	4.144	4.032	4.144	4.034	4.092	4.031	4.028	4.031	4.028	4.029
$\alpha = 0.2$	LSA_2	0.130	0.123	0.115	0.122	0.122	0.124	0.123	0.122	0.123	0.123
	LSA_1	0.058	0.054	0.049	0.053	0.053	0.053	0.053	0.052	0.053	0.053
	LBB_1	0.049	0.054	0.058	0.053	0.053	0.052	0.053	0.053	0.053	0.053
	LBB_2	0.115	0.123	0.130	0.122	0.122	0.122	0.123	0.124	0.123	0.123
	MBA_2	0.249	0.194	0.143	0.195	0.195	0.202	0.194	0.186	0.194	0.194
	MBA_1	0.156	0.127	0.095	0.127	0.127	0.133	0.128	0.124	0.128	0.128
	MSB_1	0.095	0.127	0.156	0.127	0.127	0.124	0.128	0.133	0.128	0.128
	MSB_2	0.143	0.194	0.249	0.195	0.195	0.186	0.194	0.202	0.194	0.194
	NT	0.004	0.003	0.004	0.004	0.004	0.004	0.003	0.004	0.004	0.004
	E[Spread $\vert\cdot$]	2.337	2.335	2.337	2.339	2.338	2.337	2.337	2.337	2.337	2.337
	E[Depth A_2+A_1 $\vert\cdot$]	1.552	1.595	1.632	1.591	1.592	3.066	3.026	3.066	2.442	1.592
	E[Depth A_1 $\vert\cdot$]	0.293	0.334	0.373	0.332	0.333	0.327	0.333	0.339	0.333	0.333
	E[Depth B_1 $\vert\cdot$]	0.373	0.334	0.293	0.332	0.333	0.339	0.333	0.327	0.333	0.333
	E[Depth B_1+B_2 $\vert\cdot$]	1.632	1.595	1.552	1.591	1.592	1.599	1.593	1.587	1.592	1.592
	E[Welfare LO $\vert\cdot$]	0.679	0.682	0.679	0.669	0.671	0.671	0.671	0.671	0.671	0.671
	E[Welfare MO $\vert\cdot$]	3.453	3.347	3.453	3.355	3.367	3.359	3.357	3.359	3.357	3.358
	E[Welfare $\vert\cdot$]	4.131	4.029	4.131	4.023	4.038	4.030	4.028	4.030	4.028	4.028

increased adverse selection (α or δ) leads to a reduced use of less-aggressive outside limit orders and an increased use of more aggressive orders when trading with directional news at t_1 and on average over t_2 through t_4. The net effect on aggressive inside limit orders is a priori ambiguous in these cases due to in-migration of probability from the reduced use of the outside limit orders but possible out-migration of probability to market orders. For example, at t_1 when δ increases, the probability of LBB_1 inside limit orders decreases when $\alpha = 0.8$ (from 0.466 to 0.303) but increases when $\alpha = 0.2$ (from 0.464 to 0.484).

The effect of adverse selection is different from above when investors trade opposite their directional information. Now greater adverse selection δ causes informed investors trading opposite their information to increase their use of less-aggressive outside limit orders (e.g., the LSA_2 submission probability at t_1 increases from 0.054 to 0.107 when δ increases from 0.2 to 1.4 with the high α). In particular, when δ increases, informed investors with good news $\bar{v}$ (bad news $\underline{v}$) know the security is worth more (less) and require a higher (lower) price when selling (buying). However, when α increases, there is a supply/demand effect: The demand for buying (selling) increases since now more investors know the good (bad) news, and, thus, informed investors willing to sell (buy) against their information can increase the price of the liquidity they provide.

The effects of higher volatility δ on uninformed U traders differs slightly at t_1 relative to times t_2 through t_4. At t_1, uninformed traders do not use market orders in these parametrizations, but they do tend to post slightly more patient outside limit orders when adverse selection increases (comparing the strategy probabilities for LBB_2 and LSA_2). This change in order-submission strategies is the consequence of uninformed traders offering liquidity at more profitable price levels to make up for the increased adverse selection costs. In later periods t_2 through t_4, as uninformed traders learn about the fundamental value of the asset, they still take liquidity at the outside quotes (using MBA_2 and MSB_2), but move to the inside quotes to supply liquidity (the LSA_1 and LBB_1 submission probabilities increase to 0.065 for times t_2 through t_4) when moving to the high-δ/high-α parameterization. As they learn about the future value of the asset, uninformed traders perceive less adverse selection costs and can afford to offer liquidity at more aggressive quotes. In contrast, the effect of increased value-shock volatility on the trading behavior of I_{v_0} investors with neutral news is relatively modest both at time t_1 and at times t_2 through t_4.

2.2.2 Market quality

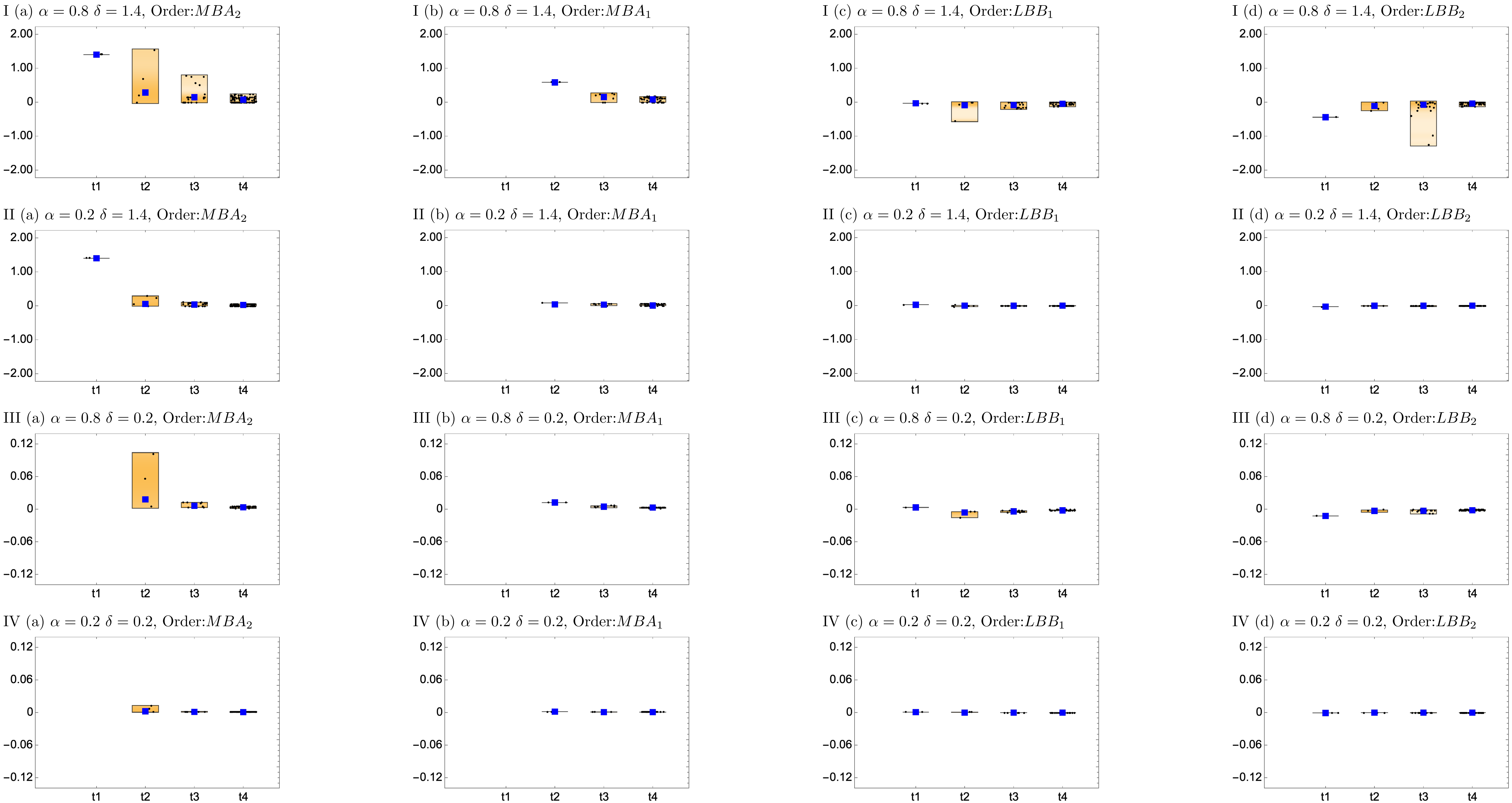
Market quality — as measured by both expected spreads and inside depth in Tables 3 and 4 — is almost always decreasing in adverse selection in this second model. This is a notable difference from our first model. However, this is not surprising given the generally greater use of market orders due to the potentially large range of private values. In particular, when the gains-from-trade are large, order execution is more important than price improvement.

2.2.3 Information content of orders

Figure 5 shows the distribution of Bayesian revisions for the different orders at different times and conditional on different prior order-history paths. The format is the same as in Figure 2. Once again, there is heterogeneity in the information content of orders over time and conditional on the preceding history. Not surprisingly, the magnitudes are smaller since there is substantially less price discovery in this second model specification given that informed-investor orders are now affected by noise from private values as well as information. Once again, we still see violations of the order-aggressiveness conjecture. Consider, for example, the high adverse-selection high-δ/high-α parameterization. As before, the small squares denote cross-path means. The most informative orders on average at t_1 and t_2 here are the market orders. However, the less-aggressive LBB_2 outside limit orders have a greater average informativeness than the aggressive LBB_1 inside limit orders at t_1 and also, less obvious visually, at t_2. The same is also true for limit orders at t_1 in the low-δ/high-α parameterization.

Violations of the order-sign conjecture are more common in our second model specification. For example, in the high-δ/high-α parameterization, LBB_2 limit buys at t_1 reveal bad news (rather than good news as one might expect given that they are buy orders). The same is true, but less obvious visually, of LBB_2 at dates t_2 through t_4 and also of LBB_1 limit buys at t_1 through t_4. This is because, as noted above, limit buys in our second model specification are used more frequently by directionally informed investor to trade opposite (rather than with) their information (i.e., due to their private-values β_{t_j}).

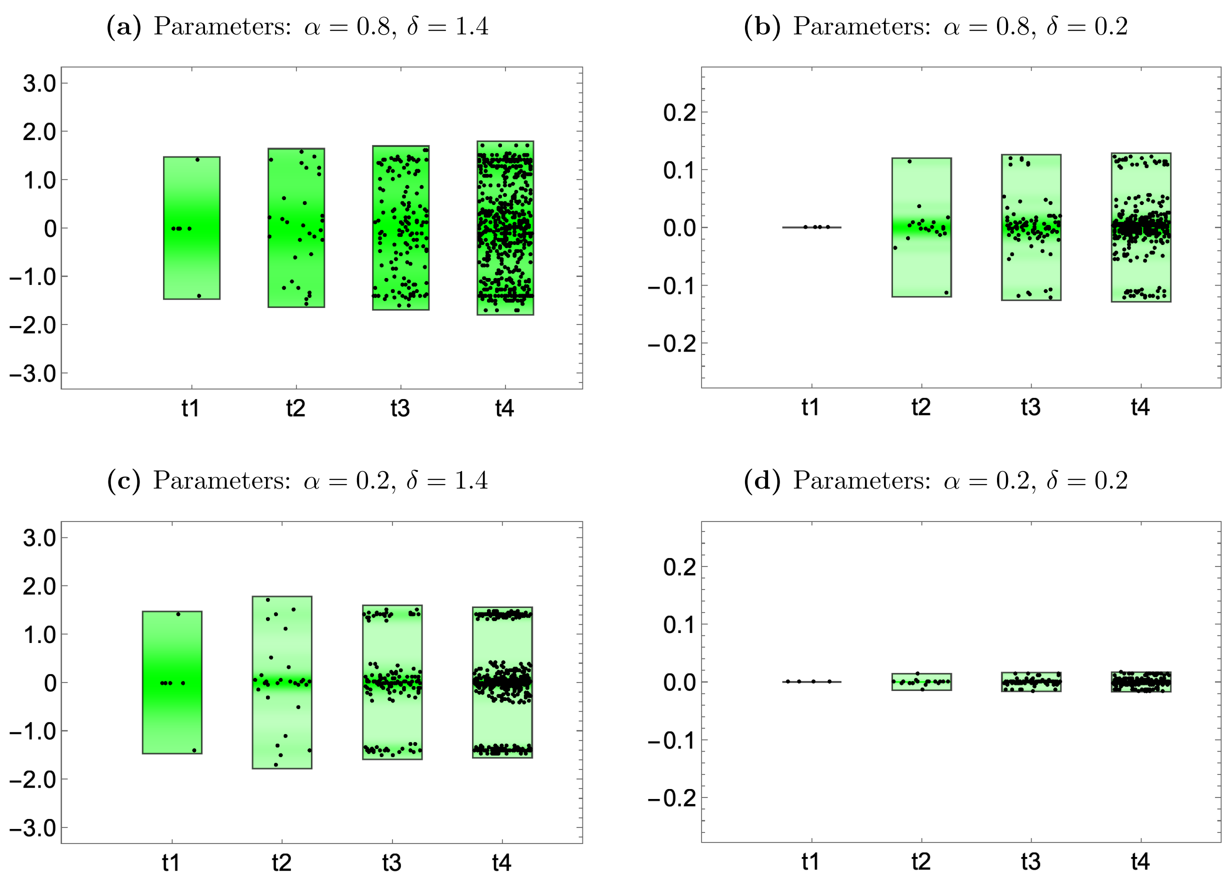
Figure 5: Order Informativeness for the Model with Informed Traders and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 to t_4. This figure shows the path-contingent Bayesian value-forecast revisions $E[v|\mathcal{L}_{t_j-1}, x_{t_j}] - E[v|\mathcal{L}_{t_j-1}]$, which shows the change in the uninformed traders's expected value of the fundamental conditional on different orders x_{t_j}. Plots a,c,e and g show graphs for the parametrization with $\alpha = 0.8$ and $\delta = 1.4$. Plots b,d,f and h show graphs for the parametrization with $\alpha = 0.8$ and $\delta = 0.2$. Plots i,k,m and o show graphs for the parametrization with $\alpha = 0.2$ and $\delta = 1.6$. Plots j,l,n and p show graphs for the parametrization with $\alpha = 0.2$ and $\delta = 0.2$. We only consider orders when they are equilibrium orders for the trading periods. Each dot indicates an equilibrium revision, the rectangles indicate the maximum and the minimum, and the squares denote probability-weighed cross-path means. The plots are grouped by their respective market parameterizations (δ, α).



2.2.4 Non-Markovian learning

Figure 6 shows once again the variation in the incremental information $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ in the prior order histories $\mathscr{L}_{t_j}(L_{t_j})$ preceding different books L_{t_j}. The plots here confirm qualitatively our earlier results about non-Markovian learning. In particular, there is cross-path heterogeneity in the information content of arriving orders. Not surprisingly, the heterogeneity is quantitatively less here since there is less information revelation in general due to the additional trading noise. This reduction is especially apparent when valuation volatility δ is low. One qualitative difference relative to the model in Section 2.1.5 is that the range of cross-path heterogeneity in Figure 6 is not growing over time as in Figure 3.

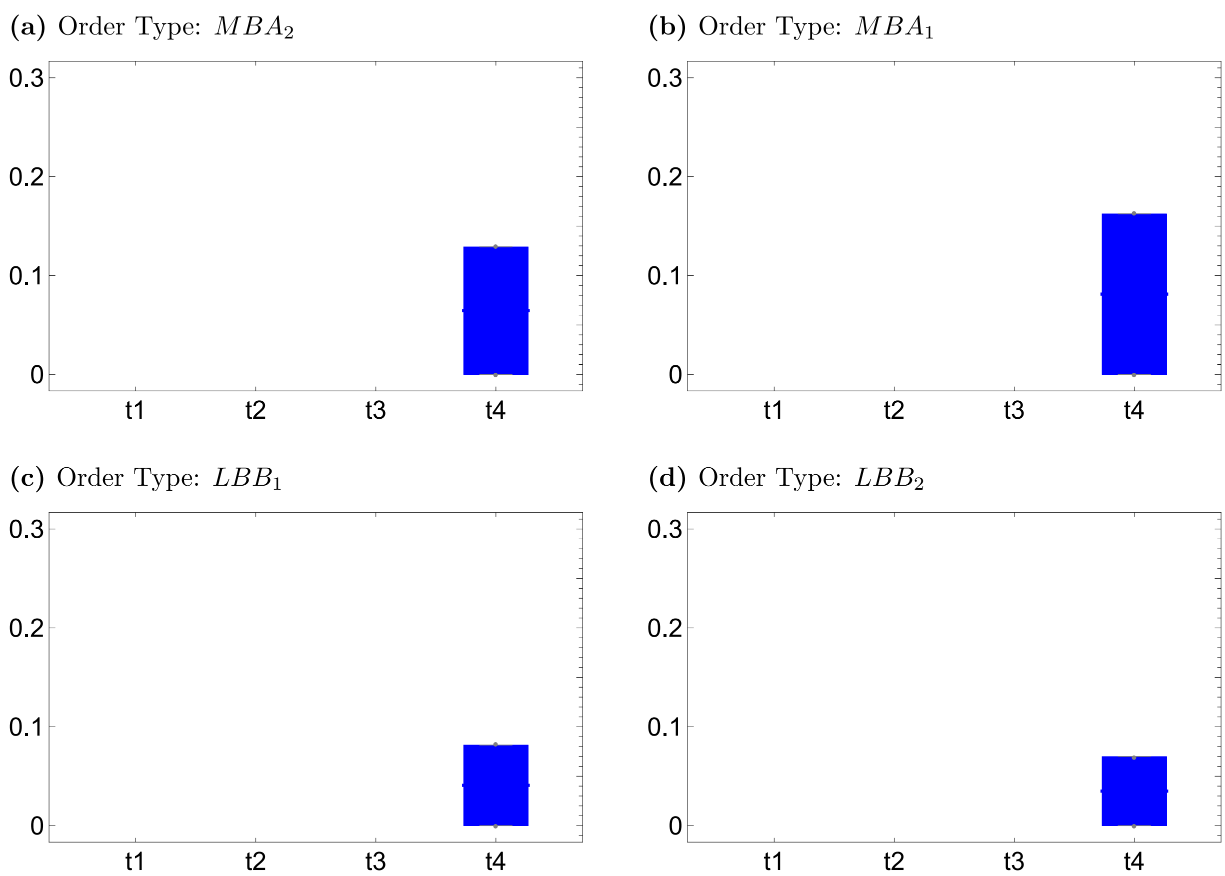
Figure 6: History Informativeness for Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 through t_4. This Figure shows the incremental information content of the past order history in excess of the information in the standing limit order book observed at the end of time t_j as measured by $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ where $\mathscr{L}_{t_j}(L_{t_j})$ is a history ending in the limit order book L_{t_j}. We only consider books L_{t_j} that occur in equilibrium in the different trading periods. The candlesticks indicate for each of these two metrics the maximum, the minimum, the median and the 75^{th} (and 25^{th}) percentile respectively as the top (bottom) of the bar.



2.2.5 Price-impact of order flow

Figure 7 confirms qualitatively our earlier results about intraday time-contingency and path-dependence in the price-impact of order flow. The main difference is that, as expected, the additional noise due to the informed-investor private values makes the magnitudes of the effects here smaller than in Figure 4 (i.e., note difference in vertical scaling). However, they are still present and material relative to the overall amount of price discovery.

Figure 7: Dispersion in the price impact of order flow The plot reports $max_{\mathscr{L}_{t_{j-1}}}(E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT]) - min_{\mathscr{L}_{t_{j-1}}}(E[v|\mathscr{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathscr{L}_{t_{j-1}}, NT])$ at different times, which shows how the prior order history affects the marginal price impact of the surprise in a given order. The parameterization is: $\alpha = 0.8$, $\delta = 1.6$



2.2.6 Summary

The results for our second model specification — with its richer specification of informed investor trading motives — confirm the robustness of the results from our first model specification and extend them. First, increased adverse selection affects informed-investor trading differently when directionally informed investors trade with their information than (because of private-value shocks) against their information. Second, the informativeness of orders can again be opposite both order aggressiveness and order direction. Third, information content of arriving orders is again history-dependent. Not surprisingly, the quantitative magnitudes of these effects are smaller due to the additional trading noise, but they are material relative to the overall amount of price discovery.

3 Robustness

Our analysis makes a number of simplifying assumptions for tractability, but we conjecture that our qualitative results are robust to relaxing these assumptions. We consider two of these assumptions here. First, our model of the trading day only has five periods. Relatedly, our analysis abstracts from limit orders being carried over from one day to the next. It is true that, with more trading rounds, as information is revealed, different paths should all eventually converge to the correct valuation. However, along the way at early trading times, paths should still differ depending on difference in how quickly and accurately information is revealed over time. Second, arriving investors are only allowed to submit single orders that cannot be cancelled or modified subsequently. However, it seems likely that order-flow histories will still be informative if orders at different points in time are correlated due to correlated actions of returning investors.

4 Conclusions

This paper has identified a number of notable theoretical properties about information aggregation and liquidity provision in dynamic limit order markets. First, informed investors switch between endogenously demanding liquidity via market orders and supplying liquidity via limit orders. Second, the information content/price impact of orders can differ from the order direction and order aggressiveness. Third, the information aggregation process is non-Markovian. In particular, the prior order history has information content beyond that in the standing limit order book, and the price-impact of order flow is also history dependent. These findings have implications for conditional estimation of the empirical price-impact of different types of orders. Moreover, we have shown the robustness of our results in two different specification of private trading motives, and have argued they should also be robust with dynamic trading strategies and more sophisticated market makers.

Our model suggests several directions for future research. Most importantly, our analysis provides a framework for empirical research about the changing price-impacts of orders conditional on the order history and time of day. There are also promising directions for future theory. First, the model can be enriched by allowing investors to trade dynamically over time and to face quantity decisions and to use multiple orders. Second, the model could be extended to allow for trading in multiple fragmented limit order markets and with Dark Pools. Third, the model could be used to study high frequency trading in limit order markets and the effect of different investors processing and trading on different types of information at different latencies.

5 Appendix A: Illustration of order paths and Bayesian updating

This appendix uses an excerpt of the extensive form of the trading game in our model to illustrate order-submission and trading dynamics and the associated Bayesian updating. The particular realized order path in Figure 8 is from the equilibrium for a model specification in which informed and uninformed investors both have random private-values (β_{t_j}). The model is considered in detail in Section 2.2. There are $N = 5$ rounds of trade, and the market parameters are $\kappa = 1$, $\mu = 0$, $\sigma = 15$, $\alpha = 0.8$, and $\delta = 1.4$. This is a relatively high informed-investor arrival probability and large value shocks. In this example, Nature has chosen an underlying economic state with good value news ($\overline{v}$), a realized sequence of arriving traders $\{I, U, U, I, I\}$, and a sequence of realized private values $\{\beta_{t1}, \ldots, \beta_{t_5}\}$ (not shown for brevity). For simplicity, we only consider a few possible nodes of the trading game and show the possible outgoing total books (from both the crowd and the arriving investors) for the different possible equilibrium order choices given the realized arriving trader types.[31] The realized books along this particular path are indicated in bold with double bars ("||"). Trading starts at t_1 with a book $[1, 0, 0, 1]$ consisting of limit orders at the outside prices A_2 and B_2 from the trading crowd.

Along the particular equilibrium path in this example, the optimal orders are unique. However, orders are random given the arriving investors' informational types (I_v or U) due to their random private values β_{t_j}. Below each possible order type at each time, Figure 8 shows the order's equilibrium submission probability for the realized arriving trader. For example, the informed investor $I_{\overline{v}}$ arriving here at t_1 chooses a limit order LSA_2 to sell at A_2 with an ex ante probability 0.107. Figure 1 in the main body of the paper shows an example of how order-submission probabilities are determined for different ranges of β_{t_j} values. In this example, the initial trader has a β_{t_1} such that she posts a limit sell LSA_2 and has rational-expectation beliefs that its execution probability is 0.625.[32] This equilibrium execution probability depends on all of the possible future trading paths proceeding from her submission at time t_1 up through time t_5. Continuing along this realized path, an uninformed trader arrives at t_2 and posts a limit sell LSA_1 at A_1, thereby undercutting the earlier LSA_2 order, so that the book at the end of t_2 is $[2, 1, 0, 1]$). In this scenario, the initial LSA_2 order from t_1 can only be executed if the LSA_1 order at t_2 is executed first. For example, the probability of a market order MBA_1 hitting the limit order at A_1 at t_3 is 0.352, and then

[31] For example, NT is not included at t_1, since Section 2.2 shows NT is not an equilibrium action here.

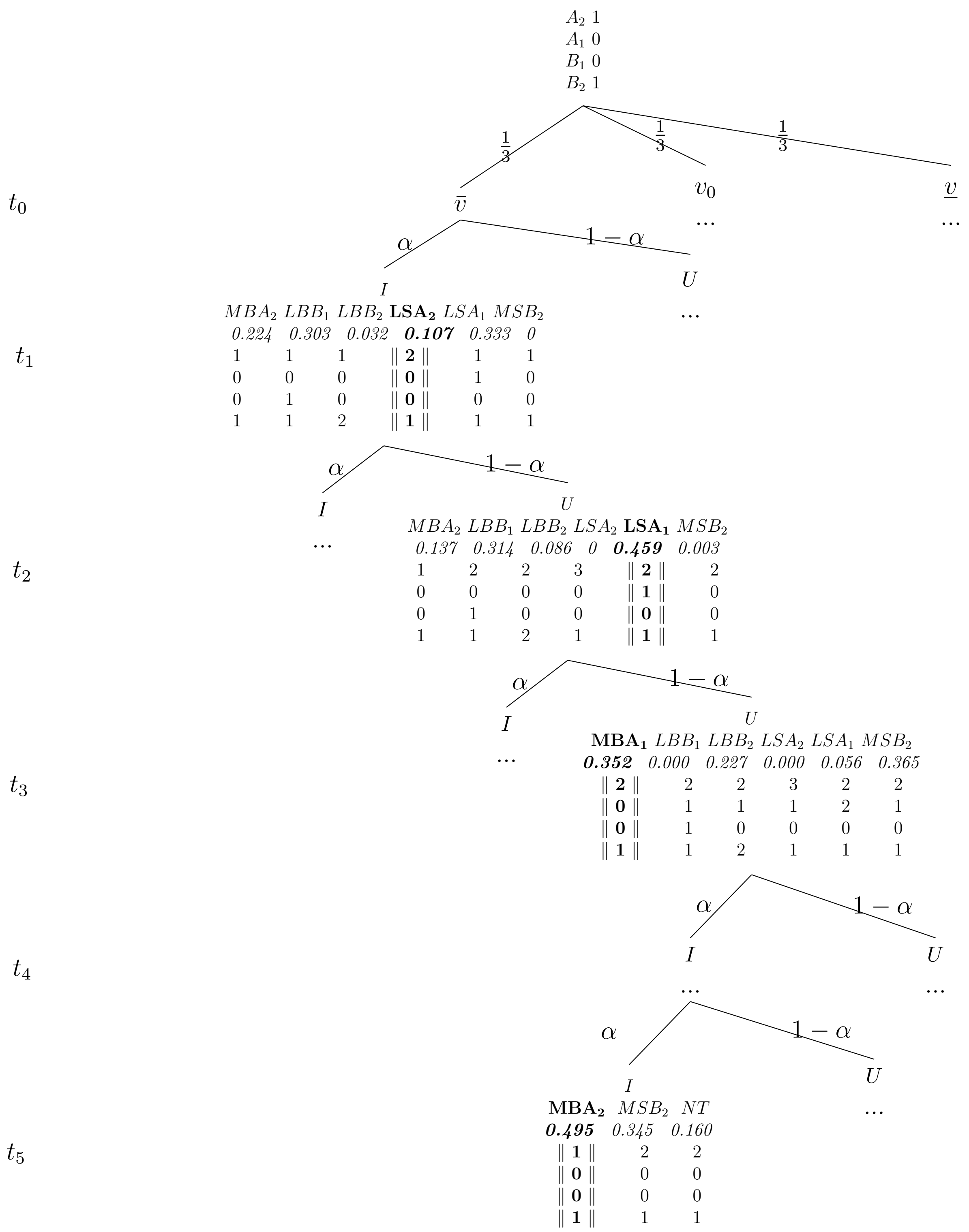
[32] Some of the numerical values discussed here are from equilibrium calculations reported in more detail in Tables 4 and 5 and in Table C2 in Appendix C. Others are unreported calculations available from the authors upon request.

the probability of another market order hitting the initial limit sell at A_2 is 0.407 at t_4 and 0.495 at t_5.[33] Therefore, there is a chance that the LSA_2 order from t_1 will still be executed even after it is undercut by the order LSA_1 at t_2.

The path in Figure 8 also illustrates Bayesian updating. After the investor at t_1 submits the LSA_2 limit sell, the uninformed trader who arrives at time t_2 in this example — who just knows the submitted order at time t_1 but not the identity or information of the trader at time t_1 — updates his equilibrium conditional valuation to $E[\tilde{v}|LSA_2] = 10.442$ and his execution-contingent expectation given his limit order LSA_1 at time t_2 is $E[\tilde{v}|LSA_2, \theta_{t_2}^{LSA_1}] = 10.724$. In subsequent periods, later investors observe additional realized orders and further update their beliefs.

[33]Due to space constraints, we do not include the t_4 node in Figure 8.

Figure 8: Excerpt of the Extensive Form of the Trading Game. This figure shows one possible realized order path of the trading game with parameters $\alpha = 0.8$, $\delta = 1.4$, $\mu = 10$, $\sigma = 15$, $\kappa = 1$, and $N = 5$ periods. When trading starts at time t_1, the incoming book $[1, 0, 0, 1]$ consists of just the initial limit orders from the crowd at A_2 and B_2. Arriving traders choose optimal orders at each period which may be limit orders LSA_i (LBB_i) $i = 1, 2$, market orders at the best standing ask, MBA_{i_t}, or bid, MSB_{i_t}, or no trade, NT. Below each possible optimal order at the nodes shown here, we report in italics the order's equilibrium submission probability given randomness in the investor private values β. Boldfaced orders and the associated states of the book (within double vertical bars) indicate realized equilibrium outcomes along this particular realized order path.



6 Appendix B: Algorithm for computing equilibrium

The computational problem to solve for a Perfect Bayesian Nash equilibrium in our model (as defined in Section 1.1) has three parts. First, the optimal order-submission problems in (6) and (7) require computing limit-order execution probabilities $Pr(\theta^x_{t_j}|v, \mathscr{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})$ for informed and uninformed investors conditional on past order histories $\mathscr{L}_{t_{j-1}}$ and asset-value expectations $E[v|\mathscr{L}_{t_{j-1}}]$ for uninformed investors that are conditional on both the order history $\mathscr{L}_{t_{j-1}}$ and on future state-contingent limit-order execution $\theta^x_{t_j}$ for each time t_j at each node of the trading game. Thus, the learning problem is both backward- and forward-looking. Second, optimal strategies are needed for each possible investor given their different information and different private values. Third, rational expectations (RE) involves a fixed point so that the beliefs underlying the optimal order-submission strategies are consistent with the execution probabilities and value expectations that the endogenous optimal strategies produce in equilibrium.

Our algorithm uses backward induction to solve for optimal order strategies given a set of asset-value beliefs for all dates and nodes in the trading game and uses an iterative recursion to solve for the RE equilibrium asset-value and order-execution beliefs fixed point. The backward induction makes order-execution probabilities consistent with optimal order-submission. It also takes future state-contingent execution into account in the uninformed-investor value expectations. We then embed the optimal order strategy calculation in an iterative recursion to solve for a fixed point for the RE asset-value beliefs. For a generic round r in this recursion, asset-value probabilities $\pi^{v,r-1}_{t_j}$ from round $r-1$ are used iteratively as incoming asset-value beliefs in round r. Thus, the recursion for a generic round r involves solving by backward induction for optimal strategies for each t_j

$$\max_{x \in X_{t_j}} w^{I,r}(x\,|\,v, \mathscr{L}_{t_{j-1}}) = [v + \beta_{t_j} - p(x)]\, f(x)\, Pr^r(\theta^x_{t_j}|\,v, \mathscr{L}_{t_{j-1}}) \quad (16)$$

$$\max_{x \in X_{t_j}} w^{U,r}(x\,|\mathscr{L}_{t_{j-1}}) = [v_0 + E^r[\Delta\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] + \beta_{t_j} - p(x)]\, f(x)\, Pr^r(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}}) \quad (17)$$

where $f(x)$ is the fill function (equal to 1 for buy orders, 0 for NT, and -1 for sell orders) and

$$E^r[\Delta|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] = (\hat{\pi}^{\bar{v},r}_{t_j}\,\bar{v} + \hat{\pi}^{v_0,r}_{t_j}\,v_0 + \hat{\pi}^{\underline{v},r}_{t_j}\,\underline{v}) - v_0 \quad (18)$$

$$\hat{\pi}^{v,r}_{t_j} = \frac{\Pr^r(\theta^x_{t_j}|v, \mathscr{L}_{t_j})}{Pr^r(\theta^x_{t_j}|\mathscr{L}_{t_j})}\pi^{v,r-1}_{t_j}. \quad (19)$$

At each time t_j the backward induction in round r has already determined limit-order execution contingencies $\theta^x_{t_j}$ at subsequent times $t > t_j$. Thus, the order-execution probabilities $Pr^r(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr^r(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})$, and the history-and execution-contingent probabilities $\hat{\pi}^{v,r}_{t_{j-1}}$ and associated asset-value expectations $E^r[\Delta|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$ are hybrid moments in that history-contingent asset-value beliefs $\pi^{v,r-1}_{t_{j-1}}$ from round $r-1$ for a path $\mathscr{L}_{t_{j-1}}$ for dates t_1 to t_{j-1} are updated using the order-execution contingencies already computed for subsequent dates t_{j+1} to t_N in the backward induction for round r. At the end of round r, the updated outgoing asset-value beliefs $\pi^{v,r}_{t_j}$ for round r are then used as incoming beliefs for the next round $r+1$. The fixed-point recursion starts in round $r = 1$ by setting the initial asset-value beliefs $\pi^{v,0}_{t_j}$ of uninformed traders at each time t_j in the backward induction to be the unconditional priors $Pr(v)$ in (1). In other words, the algorithm starts in round $r = 1$ by ignoring history. The recursion is then iterated to find a RE fixed point $\pi^v_{t_j}$ in the uninformed investor asset-value beliefs given history.

Order-execution in generic round r is modeled as follows: Starting at t_N (the last order-decision time), the execution-probability for new limit orders is zero, and so optimal strategies only involve market orders and NT. Given linearity of the expected payoffs in the private value $\beta \in [-10, 10]$ in (16) and (17), the optimal orders for an informed trader $I_v \in \{I_{\underline{v}}, I_{v_0}, I_{\overline{v}}\}$ at t_N are[34]

$$x^{I,r}_{t_N}(\beta|\mathscr{L}_{t_4}, v) = \begin{cases} MSB_{i_{t_N}} & if\ \beta \in [-10, \beta^{r,I_v,MSB_{i_{t_N}},NT}_{t_N}) \\ NT & if\ \beta \in [\beta^{r,I_v,MOB_{i_{t_N}},NT}_{t_N}, \beta^{r,I_v,NT,MBA_{i_{t_N}}}_{t_N}] \\ MBA_{i_{t_N}} & if\ \beta \in [\beta^{r,I_v,NT,MBA_{t_N}}_{t_N}, 10] \end{cases} \tag{20}$$

where for each possible combination of $MSB_{i_{t_N}} = \{MSB_1, MSB_2\}$ and $MBA_{i_{t_N}} = \{MBA_1, MBA_2\}$

$$\begin{aligned} \beta^{r,I_v,MSB_{i_{t_N}},NT}_{t_N} &= B_{i_{t_N}} - v \\ \beta^{r,I_v,NT,MBA_{i_{t_N}}}_{t_N} &= A_{i_{t_N}} - v \end{aligned} \tag{21}$$

are the critical thresholds that solve $w^{I,r}(MSB_{i_{t_N}}|v, \mathscr{L}_{t_{N-1}}) = w^{I,r}(NT|v, \mathscr{L}_{t_{N-1}})$ and $w^{I,r}(NT|v, \mathscr{L}_{t_{N-1}}) = w^{I,r}(MBA_{i_{t_N}}|v, \mathscr{L}_{t_{N-1}})$, respectively. The optimal trading strategies and β thresholds for an uninformed traders are similar but the conditioning set does not include the asset value v:

$$x^{U,r}_{t_N}(\beta|\mathscr{L}_{t_{N-1}}) = \begin{cases} MSB_{i_{t_N}} & if\ \beta \in [-10, \beta^{r,U,MSB_{i_{t_N}},NT}_{t_N}) \\ NT & if\ \beta \in [\beta^{r,U,MSB_{i_{t_N}},NT}_{t_N}, \beta^{r,U,NT,MBA_{i_{t_N}}}_{t_N}] \\ MBA_{i_{t_N}} & if\ \beta \in (\beta^{r,U,NT,MBA_{i_{t_N}}}_{t_N}, 10] \end{cases} \tag{22}$$

[34] For instance, an informed trader would post a market buy MBA_1 at t_N only if the payoff is positive and thus outperforms the NT payoff of zero, i.e, if $v + \beta_{t_N} - A_1 > 0$.

where the critical thresholds are

$$\begin{aligned}\beta_{t_N}^{r,U,MSB_{i_{t_N}},NT} &= B_{i_{t_N}} - (v_0 + E^{r-1}[\Delta|\mathscr{L}_{t_{N-1}}]) \\ \beta_{t_N}^{r,U,NT,MBA_{i_{t_N}}} &= A_{i_{t_N}} - (v_0 + E^{r-1}[\Delta|\mathscr{L}_{t_{N-1}}]).\end{aligned} \tag{23}$$

Given the β_{t_N} ranges for each possible action at t_N, we compute submission probabilities associated with each optimal order at t_N using the truncated-Normal density $\mathfrak{n}(\cdot)$ for the private values β_{t_N}.[35] At time t_{N-1} these are the execution probabilities for new limit orders by an informed trader $I_v \in \{I_{\underline{v}}, I_{v_0}, I_{\overline{v}}\}$ at the different possible best bids and asks, $B_{i,t_{N-1}}$ and $A_{i,t_{N-1}}$ respectively at time t_N:

$$Pr^r(\theta_{t_{N-1}}^{LBB_i}|\mathscr{L}_{t_{N-2}}, v) = \alpha\Big[\int_{-10}^{\beta_{t_N}^{r,I_v,MSB_{i_{t_N}},NT}} \mathfrak{n}(\beta)\,d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_{-10}^{\beta_{t_N}^{r,U,MSB_{i_{t_N}},NT}} \mathfrak{n}(\beta)d\beta\Big] \tag{24}$$

$$Pr^r(\theta_{t_{N-1}}^{LSA_i}|\mathscr{L}_{t_{N-2}}, v) = \alpha\Big[\int_{\beta_{t_N}^{r,I_v,NT,MBA_{i_{t_N}}}}^{10} \mathfrak{n}(\beta)\,d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_{\beta_{t_N}^{r,U,NT,MBA_{i_{t_N}}}}^{10} \mathfrak{n}(\beta)\,d\beta\Big] \tag{25}$$

which may be zero if particular orders are never submitted. The analogous execution probabilities for an uninformed U investor arriving at time t_{N-1} and who does not know v a priori are:

$$Pr^r(\theta_{t_{N-1}}^{LBB_i}|\mathscr{L}_{t_{N-2}}) = \alpha\Big[\sum_{v\in\{\overline{v},v_0,\underline{v}\}} \hat{\pi}_{t_{N-1}}^{v,r} \int_{-10}^{\beta_{t_N}^{r,I_v,MSB_{i_{t_N}},NT}} \mathfrak{n}(\beta)\,d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_{-10}^{\beta_{t_N}^{r,U,MSB_{i_{t_N}},NT}} \mathfrak{n}(\beta)\,d\beta\Big] \tag{26}$$

$$Pr^r(\theta_{t_{N-1}}^{LSA_i}|\mathscr{L}_{t_{N-2}}) = \alpha\Big[\sum_{v\in\{\overline{v},v_0,\underline{v}\}} \hat{\pi}_{t_{N-1}}^{v,r} \int_{\beta_{t_N}^{r,I_v,NT,MBA_{i_{t_N}}}}^{10} \mathfrak{n}(\beta)\,d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_{\beta_{t_N}^{r,U,NT,MBA_{i_{t_N}}}}^{10} \mathfrak{n}(\beta)\,d\beta\Big] \tag{27}$$

At t_{N-1} there is only one period before the end of the trading game. Thus, the execution probability for a limit order is positive if and only if the order is posted at the best price on its own side of the market ($A_{i_{t_{N-1}}}$ or $B_{i_{t_{N-1}}}$), and if there are no non-crowd limit orders already standing in the limit order book at that price at the time the new limit order is posted.

The backwards induction continues at generic times $t_j < t_N$ in round r as follows: Investors now potentially use limit orders as well as market orders and NT. From the backwards induction for dates $t > t_j$, we have order-execution probabilities (for informed and uninformed investors) and conditional valuation expectations (for uninformed investors) for each potential order an investor might submit at time t_j, and so we can compute expec-

[35] The discussion here is for the case where both informed and uninformed investors have random private factors β.

ted payoffs for each order for each investor given their information and given each possible private value β_{t_j}. Since expected payoffs at each t_j in (16) and (17) are linear in an investor's β, optimal strategies given investor beliefs are characterized by the upper envelope of their expected payoffs for each order with respect to possible values of β. The critical thresholds for optimal orders at t_j are the β values where expected payoffs for different optimal orders are equal (i.e., potential orders dominated by other orders are not used). The probability of the truncated normal distribution for β in between the different critical thresholds gives the order-submission probabilities associated with investors' optimal orders. Figure 1 illustrates this construction.[36] The order-execution probabilities together with the model parameters and the uninformed-investor beliefs then give order-execution probabilities at time t_j for potential orders at time t_{j-1}. This backwards induction for round r continues back to time t_1.

Off-equilibrium beliefs: Round r of the backwards-induction recursion requires history-contingent asset-value beliefs $\pi_{t_j}^{v,r-1} = Pr^{r-1}(v|\mathscr{L}_{t_j})$ from round $r-1$ for all feasible paths that traders may use. These beliefs can be computed using Bayes' Rule for all paths $\mathscr{L}_{t_j}$ that occur with positive probability in round $r-1$. However, Bayes' Rule cannot be used to update beliefs for paths that involve orders that are not used with positive probability in round $r-1$. Thus, similar to the discussion in Section 1.1, the model needs candidate off-equilibrium beliefs for such paths. There are many ways to search over candidate off-equilibrium beliefs, such as grid search, simulate annealing, and introducing trembles. However, we found that a relatively simple procedure worked well: If a path $\mathscr{L}_{t_j}$ does not occur in round $r-1$, we set the candidate off-equilibrium belief $Pr^{r-1}(v|\mathscr{L}_{t_j})$ to be the probability $Pr^{r-k}(v|\mathscr{L}_t)$ of the most recent recursion $r-k$ in which the longest subpath starting at t_1 and continuing up to the largest $t < t_j$ consistent with path $\mathscr{L}_{t_j}$ occurred. If no such recursion exists, then $Pr^{r-1}(v|\mathscr{L}_{t_j})$ is set to be the unconditional probability $Pr(v)$.

Mass points: In our first model specification in Section 2.1, all informed investors have neutral private values $\beta = 0$. Thus, rather than there being a probability density function over different informed investors with different private values β, there are mass points. In this specification, the integrals associated with the upper-envelope construction giving the cumulative densities between critical thresholds must be replaced with probability masses for the different informed investors.

Mixed strategies: We allow for both pure and mixed strategies in our Perfect Bayesian

[36]One reason for the model's tractability, despite having a continuum of investor β types, is that all investors with private values β in between two critical thresholds, by construction, have the same unique optimal order.

Nash equilibrium. Our algorithm starts with a conjecture of pure strategies. However, if a pair of orders (x, x') for a particular investor repeatedly oscillates between order x being optimal when it is conjectured that the investor uses order x' and x' being optimal when it is conjectured that the investor uses x, then we search over mixed strategies such that the investor is indifferent between using the two orders given other investors' behavior given the conjectured mixing probabilities. For example, in Section 2.1, neutrally informed investors mix equally between a limit buy LSA_2 and a limit sell LSB_1 at time t_1. Another example is that, in the high-δ/high-α market, an informed investor with good news mixes unequally between LBB_1 and LBB_2 limit orders.

Mixed strategies and mass points in the private-value distribution are connected.

Proposition 2 If the private-value distributions for all investors have continuous densities, then the market equilibrium will have pure strategies. However, if the private-value distributions have mass points (by themselves or in mixtures with continuous densities), then is is possible that the equilibrium will involve mixed strategies.

Proof: The proof follows from the fact that investor expected payoffs on different orders change linearly in investor private values β_{t_j}. Thus, if an investor with a given β_{t_j} value is indifferent between two orders (a set of measure zero if there are no private-value mass points), investors with adjacent β_{t_j} values nearby will strictly prefer one order over the other, and, thus, cannot optimally mix over those orders. □

Convergence: A Perfect Bayesian Nash equilibrium is obtained by solving the model recursively for multiple rounds until the updating process converges to a fixed point (i.e, the RE beliefs) in that uninformed traders no longer revise their asset-value beliefs and when mixed strategies yield the same payoffs. Numerically, convergence deemed to have been reached when i) the probabilities $\pi_{t_j}^{\overline{v},r}$, $\pi_{t_j}^{v_0,r}$ and $\pi_{t_j}^{\underline{v},r}$ in round r are "close enough" to the probabilities from round $r-1$ in that the absolute values of the differences between the two are all less than 10^{-7} and ii) the absolute differences in expected payoffs for any mixed strategies are also within 10^{-7}.

7 Appendix C: Additional results

The tables in this section provide additional information on the execution probabilities of limit orders for informed investor with positive, neutral and negative signals, ($I_{\overline{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders. The tables also report the asset-value expectations of the uninformed investor at time t_2 after observing all the possible buy orders submissions at time t_1. The expectations for sell orders are symmetric. Table C1 reports results for our first model specification in which only uninformed traders have random private values. Table C2 reports results for our second model in which both the informed and uniformed traders have private-value motives.

Table C1: Order Execution Probabilities and Asset-Value Expectation for Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different values of the informed-investor arrival probability α (0.8 and 0.2) and for two different values of the value-shock volatility δ (1.4 and 0.2). $\sigma = 15$. For each set of parameters, the first four columns report the equilibrium limit order probabilities of executions for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports the unconditional order-execution probabilities in the market. Next, the columns report conditional and unconditional future order-execution probabilities and the asset-value expectations of an uniformed investor at time t_2 after observing different order submissions at time t_1.

		$\delta = 1.4$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
$\alpha = 0.8$	$P^{EX}(LSA_2\|\cdot)$	0.269	0.205	0.059	0.177	0.177	0.180	0.229	0.170	0.193	0.193
	$P^{EX}(LSA_1\|\cdot)$	0.999	0.154	0.090	0.414	0.414	0.323	0.323	0.323	0.323	0.323
	$P^{EX}(LBB_1\|\cdot)$	0.090	0.154	0.999	0.414	0.414	0.323	0.323	0.323	0.323	0.323
	$P^{EX}(LBB_2\|\cdot)$	0.059	0.205	0.269	0.177	0.177	0.170	0.229	0.180	0.193	0.193
	E[v$\|LBB_1\ \|\cdot$]					11.331					10.000
	E[v$\|LBB_2\ \|\cdot$]					10.701					10.130
	E[v$\|MBA_1\ \|\cdot$]										
	E[v$\|MBA_2\ \|\cdot$]					10.000					10.000
$\alpha = 0.2$	$P^{EX}(LSA_2\|\cdot)$	0.563	0.490	0.402	0.485	0.485	0.514	0.499	0.476	0.496	0.496
	$P^{EX}(LSA_1\|\cdot)$	0.872	0.772	0.735	0.793	0.793	0.792	0.792	0.790	0.791	0.791
	$P^{EX}(LBB_1\|\cdot)$	0.735	0.772	0.872	0.793	0.793	0.790	0.792	0.792	0.791	0.791
	$P^{EX}(LBB_2\|\cdot)$	0.402	0.490	0.563	0.485	0.485	0.476	0.499	0.514	0.496	0.496
	E[v$\|LBB_1\ \|\cdot$]					10.245					10.000
	E[v$\|LBB_2\ \|\cdot$]					10.000					10.089
	E[v$\|MBA_1\ \|\cdot$]										
	E[v$\|MBA_2\ \|\cdot$]					10.000					10.000

Table C2: Order Execution Probabilities and Asset-Value Expectation for Informed and Uninformed Traders both with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu,\, \sigma^2)]}$. This table reports results for two different values of the informed-investor arrival probability α (0.8 and 0.2) and for two different values of the value-shock volatility δ (1.4 and 0.2). $\sigma = 15$. For each set of parameters, the first four columns report the equilibrium limit order probabilities of executions for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports the unconditional order-execution probabilities in the market. Next, the columns report conditional and unconditional future order-execution probabilities and the asset-value expectations of an uniformed investor at time t_2 after observing different order submissions at time t_1.

		$\delta = 1.4$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	$P^{EX}(LSA_2\|\cdot)$	0.625	0.498	0.419	0.514	0.514	0.502	0.487	0.472	0.487	0.487
	$P^{EX}(LSA_1\|\cdot)$	0.906	0.834	0.720	0.820	0.820	0.849	0.837	0.824	0.836	0.836
	$P^{EX}(LBB_1\|\cdot)$	0.720	0.834	0.906	0.820	0.820	0.824	0.837	0.849	0.836	0.836
	$P^{EX}(LBB_2\|\cdot)$	0.419	0.498	0.625	0.514	0.514	0.472	0.487	0.502	0.487	0.487
$\alpha = 0.8$											
	E[v|LBB_1 |·]					9.970					10.003
	E[v|LBB_2 |·]					9.558					9.988
	E[v|MBA_1 |·]										
	E[v|MBA_2 |·]					11.400					
	$P^{EX}(LSA_2\|\cdot)$	0.519	0.492	0.471	0.494	0.494	0.490	0.487	0.483	0.487	0.487
	$P^{EX}(LSA_1\|\cdot)$	0.851	0.834	0.816	0.834	0.834	0.839	0.837	0.834	0.837	0.837
	$P^{EX}(LBB_1\|\cdot)$	0.816	0.834	0.851	0.834	0.834	0.834	0.837	0.839	0.837	0.837
	$P^{EX}(LBB_2\|\cdot)$	0.471	0.492	0.519	0.494	0.494	0.483	0.487	0.490	0.487	0.487
$\alpha = 0.2$											
	E[v|LBB_1 |·]					10.024					10.001
	E[v|LBB_2 |·]					9.967					9.999
	E[v|MBA_1 |·]										
	E[v|MBA_2 |·]					11.400					

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