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Reduced Strategies and Cognitive Hierarchies in the Extensive and Normal Form*

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Abstract

In a recent paper, Lin & Palfrey (2024) developed a theory of cognitive hierarchies (CH) in sequential games and observed that this solution concept is not reduced-normal-form invariant. In this paper I qualify and explain this observation. I show that *the CH model is normal-form invariant*, and that the differences arising from the application of the CH model to the *reduced* normal form depend only on how randomization by level-0 types is modeled. Indeed, while the uniform behavior strategy in the extensive form yields the uniform mixed strategy in the normal form, the latter does not correspond to the uniform randomization in the reduced normal form, because different reduced strategies may correspond to sets of equivalent strategies with different cardinalities. I also comment on (i) the invariance of the CH model to some transformations of the sequential game, and (ii) the independence of conditional beliefs about co-players' level-types.

KEYWORDS: Cognitive hierarchies, Sequential games, Extensive form, Normal form, Reduced normal form, Interchange of moves, Coalescing of moves, Independence, Observed deviators.

JEL codes: C72, C73, C92, D91.

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1 Introduction

This work is prompted by the paper on "Cognitive Hierarchies for Games in Extensive Form" by Lin & Palfrey (2022, 2024).[1] These authors observe that the Cognitive Hierarchies (CH) model is not reduced-normal-form invariant. This has important consequences for experimental design: insofar as experimental data are organized by means of the CH model, the commonly used strategy method, which makes subjects choose among reduced strategies, is not valid.[2] This is a new theoretical insight about the validity of the strategy method that complements extant considerations concerning dynamic consistency and hot/cold effects (see, e.g., the discussion in Section 7 of Battigalli & Dufwenberg 2022, and Aina *et al.* 2020).

I qualify and explain this observation by showing that *the CH model* gives the same predictions for all games with the same normal form, i.e., it *is normal-form invariant* and that the differences arising from the application of the CH model to the *reduced* normal form depend on how randomization by level-0 types is modeled.[3] Indeed, while the uniform behavior strategy in the extensive form yields the uniform mixed strategy in the normal form, the latter does not yield uniform randomization in the reduced normal form. The reason is that different (structurally) reduced strategies may correspond to sets of realization equivalent strategies with different cardinalities.[4] Specifically, let $\mathbf{r}_i \subseteq S_i$ denote **reduced strategies**, that is, elements of the equivalence partition (quotient set) of the set S_i of pure strategies, and let $|X|$ denote the number of elements of any finite set X. With this, it may well be the case that $|\mathbf{r}'_i| \neq |\mathbf{r}''_i|$ for different reduced strategies $\mathbf{r}'_i$ and $\mathbf{r}''_i$. Uniform randomization on S_i does not correspond to uniform randomization over reduced strategies, but rather to assigning probability $|\mathbf{r}_i| / |S_i|$ to each reduced strategy $\mathbf{r}_i$. This implies that the sequential CH model is *not invariant to*

[1] The 2024 article by Lin & Palfrey takes this note into account. On the cognitive hierarchies model in static games see Camerer *et al.* (2004) and the other relevant references in Lin & Palfrey (2024).

[2] This is documented by Lin & Palfrey (2024) by analyzing existing experimental results about the Centipede game. The strategy method was originally suggested by Selten (1967) to elicit subjects' off-path choices in experiments. See Brandts & Charness (2011).

[3] I systematically use the term "level-k type" (or "level-type"), because levels in the CH model (and also in the level-k thinking model) can be interpreted as types in games with incomplete information (cf. Liu 2024).

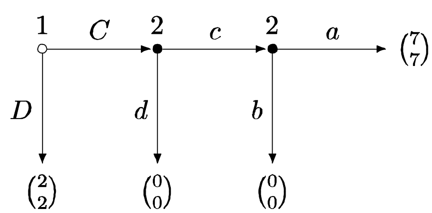
[4] Like Kuhn (1953), I consider the realization-equivalence relation between strategies, which is "structural" because it is independent of payoffs.

coalescing of moves and—of course—its inverse, sequential-agents splitting, because these operations change the cardinality of realization-equivalence classes, i.e., of reduced strategies.[5]

These observations are easily explained and illustrated by restricting attention to games with perfect information, as I do in most of this paper. Extending the analysis to games with imperfect information, I add two observations: (i) Assuming that the prior belief about level-types of different players is a product measure (as justified in standard experimental settings with random matching of subjects), conditional beliefs about co-players' level-types necessarily satisfy independence (only) in games with observed deviators. (ii) The sequential CH model satisfies invariance to the interchanging of essentially simultaneous moves.

1.1 Introductory examples

My theoretical analysis can be intuitively grasped by comparing two related common-interests (CI) games.



Game 1: A CI game with
sequential moves by player 2.

In Game 1, player 2 has two consecutive moves. The best reply of player 2 to any randomization that assigns positive probability to C is the backward-induction strategy $c.a$ (the strategy that selects c at the first node of player 2 and a at the second node). Hence, this is the prediction of the CH model for player 2 of level $k > 0$. The best reply by player 1 to the uniform behavior strategy of player 2 is D, because the sequence of actions (c, a) by player 2—that yields 7 utils—has probability $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; hence, C yields $\frac{7}{4} < 2$ in

[5]On the transformations that do not change the reduced normal form see Elmes & Reny (1994), Battigalli *et al.* (2020) and the relevant references therein.

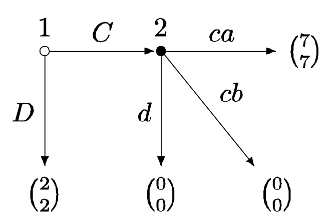
expectation. The CH model for sequential games of Lin & Palfrey (2024) assumes that the level-1 type of player 1 plays the best reply to the uniform behavior strategy of player 2, that is, D. The play of higher levels depends on the distribution of types: if the fraction of level-0 types of player 2 is high enough, some higher levels $k > 1$ of player 1 play D, otherwise the backward-induction pair $(C, c.a)$ obtains for levels $k > 1$ of both players.

$1\backslash 2$	$c.a$	$c.b$	$d.a$	$d.b$
C	7, 7	0, 0	0, 0	0, 0
D	2, 2	2, 2	2, 2	2, 2

Normal form of Game 1.

It can be checked that the same results obtain for the CH model applied to the normal form of Game 1, because the uniform mixed strategy of player 2 assigns probability $\frac{1}{4}$ to each one of the 4 strategies of player 2. This illustrates the normal-form invariance of the CH model.

By coalescing the sequential moves of player 2, one obtains the leader-follower Game 2. Now uniform randomization assigns probability $\frac{1}{3}$ to the action/strategy ca of player 2 that yields 7 utils; the best reply of player 1 is C, which yields $\frac{7}{3} > 2$ in expectation. If follows that the CH model yields the backward-induction strategy pair (C, ca) in Game 2 for all levels $k > 0$. This shows that the CH model for sequential games is not invariant to coalescing sequential moves of the same player.



Game 2: A leader-follower CI game.

It can be checked that, applying the CH model to the normal form of Game 2, the same strategies obtain for all levels of both players, illustrating again that the CH model is normal-form invariant.

1\2	ca	cb	d
C	7, 7	0, 0	0, 0
D	2, 2	2, 2	2, 2

Normal form of Game 2.

Note also that the normal form of Game 2 coincides, up to relabeling, with the (structurally) reduced normal form of Game 1.[6] Thus, the CH model is not reduced-normal-form invariant.

The rest of this paper is organized in several short sections. Section 2 introduces the basic formalism, restricting attention to games with perfect information. Section 3 briefly analyzes mixing and uniform randomization in the extensive and normal form. Section 4 introduces the key concept of (structurally) reduced strategy and related results. Section 5 relates predictive probabilities of co-players' actions to updated beliefs about co-players' level-types. Section 6 elaborates on sequential and *ex ante* best replies. Section 7 presents the core equivalence results. Section 8 extends the analysis to games with imperfect information. Section 9 offers some concluding remarks, including a discussion of the level-k thinking model in the extensive and normal form (cf. Schipper & Zhou 2024). The Appendix (Section 10) contains the more technical proofs.

2 Representation of Sequential Games

For the sake of simplicity, within the class of sequential games represented in extensive form,[7] I mostly focus on *finite games with perfect information* and *without chance moves*. Small (Latin or Greek) letters will typically denote elements of sets, which are represented by the corresponding capital letters. Bold symbols will be used to denote equivalence classes.

[6] The (structurally) reduced strategies of player 2 in the Game 1 are the three realization-equivalence classes $\{c.a\}$, $\{c.b\}$ (both singletons), and $\{d.a, d.b\}$.

[7] I banned the terms "normal-form game" and "extensive-form game" from my vocabulary (despite their widespread use), because *the extensive and normal forms are kinds of representations of games, not kinds of games.* Game with simultaneous moves and games with sequential moves are kinds of games. Unfortunately, "normal-form game" is often confusingly used to mean "game with simultaneous moves."

I adopt the definition and representation of perfect-information games of Chapter 6 of the textbook of Osborne & Rubinstein (1994). The basic primitives of the game

$$\Gamma = \langle I, A, H, P, u \rangle$$

are as follows:

- A *finite* **set of actions** A.

- A *finite* **set of histories** H, that is, finite sequences of actions, including the **empty sequence** $\varnothing$. Thus,

  $$H \subset \{\varnothing\} \cup \left(\bigcup_{k \in \mathbb{N}} A^k \right);$$

  furthermore, H is *closed under the* canonical **prefix-of** relation $\preceq$, which makes it a tree with root $\varnothing$.[8] With this, $A(h) := \{a \in A : (h, a) \in H\}$ is the set of feasible actions given h, and $Z := \{h \in H : A(h) = \emptyset\}$ is the **set of terminal histories**. To avoid trivialities, I assume that *there are at least 2 feasible actions at each non terminal history*: $|A(h)| \geq 2$ for every $h \in H \backslash Z$.

- The **player set** is I and the **player function** is $P : H \backslash Z \to I$. With this, $H_i := P^{-1}(i)$ is the set of non-terminal histories where player $i \in I$ is active.[9]

- The description of the game is completed by the profile of payoff functions $u = (u_i : Z \to \mathbb{R})_{i \in I}$.

The analysis also requires the *exogenous specification of a probability measure* $p_i = (p_{i\ell})_{\ell \in \mathbb{N}_0} \in \Delta(\mathbb{N}_0)$ for each player $i \in I$, where each $\ell \in \mathbb{N}_0$ is interpreted as the level-type of player i, with level-0 being a uniformly randomizing type.

In what follows, I often refer to products of numbers and Cartesian products of sets defined over some finite index set, and I have to allow for the

[8] Sequence $x = (a^1, ..., a^k)$ is a (weak) **prefix** of sequence $y = (b^1, ..., b^\ell)$, written $x \preceq y$, if $k \leq \ell$ and $(a^1, ..., a^k) = (b^1, ..., b^k)$.

[9] In Osborne & Rubinstein (1994) and Lin & Palfrey (2022) the player set is instead denoted by N with the convention that $N = \{1, ..., n\}$. Since here such convention (a strict order on the player set) does not play any useful role, I stick to the notation of my textbook (Battigalli *et al.* 2024). With this, i is an element of I.

possibility that the index set is empty. This requires the use of convenient conventions. Thus, for any finite index set J and profile of numbers $(n_j)_{j\in J}$, $\prod_{j\in J} n_j = 1$ by convention if J is empty.[10] Similarly, if $(Y_j)_{j\in J}$ is a profile of sets, I let $\times_{j\in J} Y_j$ be a singleton if J is empty. From the aforementioned primitives and relying on such conventions one can *derive*:

- The **set of** (pure) **strategies** $S_i := \times_{h\in H_i} A(h)$ of each player i,[11] and the sets of profiles of strategies of all players and of i's co-players, $S := \times_{i\in I} S_i = \times_{h\in H\setminus Z} A(h)$, $S_{-i} := \times_{j\neq i} S_j$.
- The **path** or **outcome function** $O : S \to Z$, which specifies the terminal history induced by each strategy profile.
- The profile of **normal-form** payoff functions $(U_i = u_i \circ O : S \to \mathbb{R})_{i\in I}$.

Example 1 To illustrate, in Game 1 the sets of actions, histories, and terminal histories are[12]

$$\begin{aligned} A &= \{a, b, c, d, C, D\}, \\ H &= \{\varnothing, C, D, (C, c), (C, d), (C, c, a), (C, c, b)\}, \\ Z &= \{D, (C, d), (C, c, a), (C, c, b)\} \end{aligned}$$

All the other primitive elements can be easily understood from the picture representing Game 1. In all the *examples*, I adopt the convenient convention that *actions sets at distinct nodes have empty intersection* ($h' \neq h''$ implies $A(h') \cap A(h'') = \emptyset$). This allows to represent the strategies of a player as lists of action labels separated by dots, one for each node where this player is active. Thus, $S_1 = \{C, D\}$ and $S_2 = \{c.a, c.b, d.a, d.b\}$.

$O(\cdot,\cdot)$	$c.a$	$c.b$	$d.a$	$d.b$
C	(C, c, a)	(C, c, b)	(C, d)	(C, d)
D	D	D	D	D

Outcome function of Game 1.

The outcome function in the table yields the normal form of Game 1. ▲

[10] This is consistent with the standard convention about powers of numbers: $n^0 = 1$.

[11] I did not assume that the player function P is onto. Thus, H_i may be empty for some player i. In this case, by convention, S_i is a singleton, which may be interpreted as the set that only contains the "strategy of waiting." Similar considerations apply to continuation strategies in subgames.

[12] Redundant parentheses are omitted.

3 Randomizations

For any finite set X, let

$$\Delta(X) := \left\{\mu \in \mathbb{R}_{+}^{X} : \sum_{x\in X} \mu(x) = 1\right\},$$

$$\Delta^{\circ}(X) := \left\{\mu \in \mathbb{R}_{++}^{X} : \sum_{x\in X} \mu(x) = 1\right\}$$

respectively denote the simplex of probability mass functions on X, and its relative interior, i.e., the set of strictly positive probability mass functions on X.

For any player $i \in I$, the **set of mixed strategies** is $\Delta(S_i)$ and the **set of behavior strategies** is $\Sigma_i := \times_{h\in H_i}\Delta(A(h))$. The **uniform mixed strategy** of i, denoted μ_i^0, assigns the same probability to each pure strategy:

$$\forall s_i \in S_i,\ \mu_i^0(s_i) := \frac{1}{|S_i|}.$$

The **uniform behavior strategy** of i, denoted σ_i^0, assigns the same probability to all the actions at any given node where i is active:

$$\forall h \in H_i, \forall a \in A(h),\ \sigma_{ih}^0(a) := \frac{1}{|A(h)|}.$$

Clearly, uniform randomizations belong to the relevant relative interiors: $\mu_i^0(s_i) \in \Delta^{\circ}(S_i)$ and $\sigma^0 \in \Sigma_i{}^{\circ} := \times_{h\in H_i}\Delta^{\circ}(A(h))$. This implies that any mixtures giving strictly positive weights to uniform randomizations also belong to the relevant relative interiors, $\Delta^{\circ}(S_i)$ or $\Sigma_i{}^{\circ}$ (on mixtures of behavior strategies see Section 5). This is important in the CH models, where every level-type $k > 0$ assigns strictly positive weights to the uniform randomizations ascribed to co-players of level-type 0.

Kuhn's (1953) well-known Theorem 4 on the realization equivalence between mixed and behavior strategies relies on the following map from behavior to mixed strategies that preserves the probabilities of paths: behavior strategy $\sigma_i = (\sigma_{ih})_{h\in H_i} \in \Sigma_i$ yields the mixed strategy $\mu_i^{\sigma_i} \in \Delta(S_i)$ such that

$$\forall s_i \in S_i,\ \mu_i^{\sigma_i}(s_i) := \prod_{h\in H_i} \sigma_{ih}(s_{ih}), \tag{1}$$

where s_{ih} denotes the action selected by s_i at history h. In words, $\mu_i^{\sigma_i}$ is obtained under the assumption that the different "agents" ih ($h \in H_i$) of player i randomize at different histories/nodes independently of each other according to probability model σ_i. The following observation plays a key role in this note:

Remark 1 *For each player $i \in I$, the cardinality of i's strategy set is $|S_i| = \prod_{h \in H_i} |A(h)|$; therefore, the uniform behavior strategy σ_i^0 of i yields the uniform mixed strategy μ_i^0 under Kuhn's map (1).*

Proof. Using Kuhn's map (1), the mixed strategy obtained from the uniform behavior strategy σ_i^0 satisfies, for every $s_i \in S_i$,

$$\begin{aligned} \mu_i^{\sigma_i^0}(s_i) &= \prod_{h \in H_i} \sigma_{ih}^0(s_{ih}) = \prod_{h \in H_i} \frac{1}{|A(h)|} \\ &= \frac{1}{\prod_{h \in H_i} |A(h)|} = \frac{1}{|S_i|} = \mu_i^0(s_i). \ \blacksquare \end{aligned}$$

Example 2 Consider Game 1. The set strategies of player 2 is $S_2 = \{c, d\} \times \{a, b\}$. Thus, $|S_2| = 2 \times 2 = 4$ and $\mu_2^0(s_2) = \frac{1}{4}$ for every $s_2 \in S_2$; $\sigma_{2C}^0(c) = \sigma_{2C}^0(d) = \frac{1}{2}$, $\sigma_{2(C,c)}^0(a) = \sigma_{2(C,c)}^0(b) = \frac{1}{2}$, and $\mu_2^{\sigma_2^0}(s_2) = \frac{1}{4} = \mu_2^0(s_2)$ for every $s_2 \in S_2$. ▲

4 Reduced Strategies

The **set of** pure, structurally **reduced strategies** (sometimes called "plans of actions") is the quotient set $\mathbf{R}_i := S_i | \approx_i$, where $\approx_i$ is the behavioral/realization equivalence relation[13]

$$s_i' \approx_i s_i'' \iff (\forall s_{-i} \in S_{-i}, O(s_i', s_{-i}) = O(s_i'', s_{-i})).$$

In other words, $\mathbf{R}_i$ is the partition of S_i induced by equivalence relation $\approx_i$. I let $\bar{R}_i(s_i)$ denote the set of i's strategies realization equivalent to s_i, that

[13]Cf. Rubinstein (1991), Ch. 6.4 in Osborne & Rubinstein (1994), Battigalli *et al.* (2020), Chapter 9 in the textbook by Battigalli *et al.* (2024), and Theorem 1 in Kuhn (1953). I already explained why I call such strategies "structurally reduced." In Chapter 9 of my textbook I explain why I refrain from calling reduced strategies "plans of action." See also Battigalli & De Vito (2021).

is, the reduced strategy $\mathbf{r}_i \in \mathbf{R}_i$ such that $s_i \in \mathbf{r}_i$. I call the partitional map

$$\begin{array}{rccl} \bar{R}_i: & S_i & \to & \mathbf{R}_i \\ & s_i & \mapsto & \{s_i' \in S_i : s_i' \approx_i s_i\} \end{array}$$

"**reduction map**." Note that the right inverse $\bar{R}_i^{-1}$ is the identity map on partition $\mathbf{R}_i$. Also let $\mathbf{R} := \times_{i \in I} \mathbf{R}_i$ and $\mathbf{R}_{-i} := \times_{j \neq i} \mathbf{R}_j$. Since—by realization equivalence—$O(\cdot)$ is constant on each product of cells $\times_{i \in I} \mathbf{r}_i$, it makes sense to define the reduced-form outcome function $\mathbf{O} : \mathbf{R} \to Z$ as follows:

$$\forall (\mathbf{r}_i)_{i \in I} \in \mathbf{R}, \forall s \in \times_{i \in I} \mathbf{r}_i,\ \mathbf{O}\left((\mathbf{r}_i)_{i \in I}\right) := O(s).$$

The profile of (structurally) **reduced normal-form** payoff functions $\left(\bar{U}_i : \mathbf{R} \to \mathbb{R}\right)_{i \in I}$ is such that $\bar{U}_i\left(\left(\bar{R}_j(s_j)\right)_{j \in I}\right) = U_i(s)$ for all $s = (s_j)_{j \in I} \in S$, which is well posed by definition of $\mathbf{R}$ by means of realization equivalences.

For any $i \in I$ and $s_i \in S_i$, let

$$H_i(s_i) := \{h \in H_i : \exists s_{-i} \in S_{-i}, h \prec O(s_i, s_{-i})\}$$

denote the set of **histories** where i moves that are **allowed** (not prevented) **by strategy** s_i. For example, in Game 1, $H_2(d.x) = \{C\}$ and $H_2(c.x) = H_2$ for each action $x \in A(C, c)$. The following is Theorem 1 in Kuhn (1953):[14]

Lemma 1 *For any player $i \in I$, two strategies are realization equivalent if and only if they allow for the same set of non-terminal histories where i moves and prescribe the same actions at such histories, that is, for all $s_i', s_i'' \in S_i$,*

$$s_i' \approx_i s_i'' \iff \left(H_i(s_i') = H_i(s_i'') \wedge \left(\forall h \in H_i(s_i'), s_{ih}' = s_{ih}''\right)\right).$$

Lemma 1 implies the following:

Lemma 2 *For every player $i \in I$ and strategy s_i, the cardinality of the corresponding reduced strategy is*

$$\left|\bar{R}_i(s_i)\right| = \prod_{h \in H_i \setminus H_i(s_i)} |A(h)|.$$

[14] Adapted to perfect information games.

Proof. Fix a strategy s_i arbitrarily. By Lemma 1, all the strategies s_i' equivalent to s_i allow for the same set $H_i(s_i)$ of histories where i is active and select the same actions at those histories, that is, they can differ from s_i only at histories $h \in H_i \backslash H_i(s_i)$. Thus, the number of strategies equivalent to s_i, which is the cardinality $\left|\bar{R}_i(s_i)\right|$ of its reduction, is the number $\prod_{h \in H_i \backslash H_i(s_i)} |A(h)|$ of ways to select feasible actions at histories $h \in H_i \backslash H_i(s_i)$. ■

Definition 1 *Game Γ has the* ***one-move*** *property if no player moves more than once in any path of play, that is, for all $z \in Z$ and $i \in I$, $|\{h \prec z : P(h) = i\}| \leq 1$.*

As shown in the Appendix, Lemma 2 implies:

Remark 2 *Game Γ has the one-move property if and only if reduced and non reduced strategies coincide (that is, if and only if $\mathbf{R}_i$ is the finest partition of S_i for each $i \in I$).*

To illustrate, Game 2 is a one-move game, but Game 1 is not a one-move game, because player 2 has two consecutive moves. The normal form and reduced normal form coincide for Game 2, but they do not coincide for Game 1.

Given any mixed strategy $\mu_i \in \Delta(S_i)$, we obtain the corresponding image (pushforward) **reduced mixed strategy** $\bar{\mu}_i = \mu_i \circ \bar{R}_i^{-1} \in \Delta(\mathbf{R}_i)$ by means of the reduction map $\bar{R}_i : S_i \to \mathbf{R}_i$, that is,

$$\forall \mathbf{r}_i \in \mathbf{R}_i,\ \bar{\mu}_i(\mathbf{r}_i) = \left(\mu_i \circ \bar{R}_i^{-1}\right)(\mathbf{r}_i) = \sum_{s_i \in \mathbf{r}_i} \mu_i(s_i).$$

To ease notation, for any mixed strategy *profile* $\mu = (\mu_i)_{i \in I}$, I write $\mu \circ \bar{R}^{-1}$ for the image (pushforward) product measure induced by the collective reduction map

$$\begin{array}{rlll} \bar{R}: & S & \to & \mathbf{R}, \\ & (s_i)_{i \in I} & \mapsto & \left(\bar{R}_i(s_i)\right)_{i \in I}, \end{array}$$

that is,

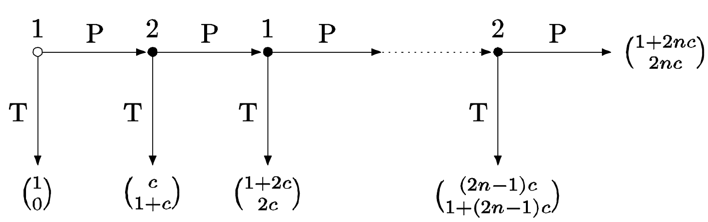
$$\forall \mathbf{r} = (\mathbf{r}_i)_{i \in I} \in \mathbf{R},\ \left(\mu \circ \bar{R}^{-1}\right)(\mathbf{r}) = \prod_{i \in I} \sum_{s_i \in \mathbf{r}_i} \mu_i(s_i).$$

Remark 3 *For the purposes of expected-payoff calculations, the only measures that matter are the probability measures on reduced strategies induced by each mixed strategy, that is, for every* $\mu = \left(\mu_j\right)_{j\in I} \in \times_{j\in I}\Delta\left(S_j\right)$ *and* $i \in I$,

$$\begin{aligned}\mathbb{E}_{\mu}\left(U_i\right) &= \sum_{s\in S} u_i\left(O\left(s\right)\right)\prod_{j\in I}\mu_j\left(s_j\right)\\ &= \sum_{\mathbf{r}\in\mathbf{R}} u_i\left(\mathbf{O}\left(\mathbf{r}\right)\right)\sum_{s\in\bar{R}^{-1}(\mathbf{r})}\prod_{j\in I}\mu_j\left(s_j\right) = \mathbb{E}_{\mu\circ\bar{R}^{-1}}\left(\bar{U}_i\right).\end{aligned}$$

Remark 4 *For each player* $i \in I$*, the mixed reduced strategy induced both by the uniform behavior strategy* σ_i^0 *and by the uniform (non-reduced) mixed strategy* μ_i^0 *is* $\mu_i^0 \circ \bar{R}_i^{-1}$ *with* $\left(\mu_i^0 \circ \bar{R}_i^{-1}\right)\left(\mathbf{r}_i\right) = \left|\mathbf{r}_i\right| / \left|S_i\right|$ *for every* $\mathbf{r}_i \in R_i$*; thus,* $\mu_i^0 \circ \bar{R}_i^{-1}$ *is uniform in every one-move game, but there are games where* $\mu_i^0 \circ \bar{R}_i^{-1}$ *is not uniform.*

The last claim of Remark 4 is well illustrated by Centipede-like games (those where, for each h starting a subtree of height 2 or more, $A\left(h\right)$ contains a terminating action T and a continuation/pass action P).



Game 3: Centipede.

Example 3 Consider Game 3, where $\left|H_1\right| = \left|H_2\right| = n \geq 2$. The set of strategies has cardinality $\left|S_i\right| = 2^{\left|H_i\right|}$; the set of reduced strategies has cardinality $\left|\mathbf{R}_i\right| = \left|H_i\right| + 1$ (player i can either terminate at the k^{th} opportunity, with $k \in \{1, ..., \left|H_i\right|\}$, or always pass), the cardinality of reduced strategy $\mathbf{T}_{i,k} \subseteq S_i$ (terminating at the k^{th} opportunity for player i) is twice the cardinality of reduced strategy $\mathbf{T}_{i,k+1}$: $\left|\mathbf{T}_{i,k}\right| = 2\left|\mathbf{T}_{i,k+1}\right|$.[15] It follows that $\mu_i^0\left(s_i\right) = \left(\frac{1}{2}\right)^{\left|H_i\right|} = \left(\frac{1}{2}\right)^n$ for every s_i, whereas the measure on $\mathbf{R}_i$ induced by μ_i^0 satisfies $\left(\mu_i^0 \circ \bar{R}_i^{-1}\right)\left(\mathbf{T}_{i,1}\right) = \frac{1}{2}$ and $\left(\mu_i^0 \circ \bar{R}_i^{-1}\right)\left(\mathbf{T}_{i,k}\right) = 2 \times \left(\mu_i^0 \circ \bar{R}_i^{-1}\right)\left(\mathbf{T}_{i,k+1}\right)$. ▲

[15] Cf. Figure 5 in Lin & Palfrey (2024).

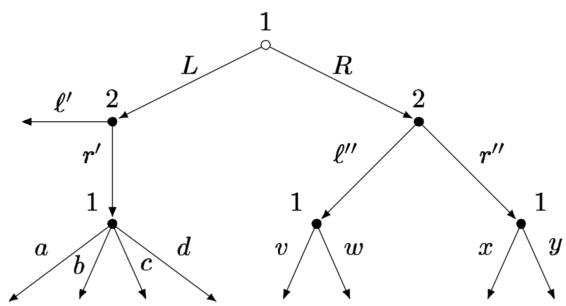
Next, I consider a generalization of one-move games.

Definition 2 *A game is* ***equi-reducible for player*** i *if all the realization-equivalence classes in* $\mathbf{R}_i = S_i| \approx_i$ *have the same cardinality. A game is* ***equi-reducible*** *if it is equi-reducible for every player.*[16]

Remark 4 implies:

Remark 5 *The uniform mixed strategies of the normal form induce the uniform mixed strategies of the reduced normal form* ($\mu_i^0 \circ \bar{R}_i^{-1}$ *is uniform for every player* $i \in I$) *if and only if the game is equi-reducible.*

By Remark 2, every one-move game is equi-reducible. Game 1 and Game 3 (Centipede) are not equi-reducible. The previous observations about the induced measures $\mu_i^0 \circ \bar{R}_i^{-1}$ in these games illustrate Remark 5.



Game 4: An equi-reducible game

Example 4 Game 4 does not satisfy the one-move property, but it is equi-reducible. Strategies and reduced strategies coincide for player 2, who moves only once on each path of play. Player 1 has $2^3 \times 4 = 32$ strategies. The set of reduced strategies of player 1 is

$$\mathbf{R}_1 = \{\mathbf{L.a}, \mathbf{L.b}, \mathbf{L.c}, \mathbf{L.d}, \mathbf{R.v.x}, \mathbf{R.v.y}, \mathbf{R.w.x}, \mathbf{R.w.y}\},$$

where $\mathbf{L.a} := \{L.a.v.x, L.a.v.y, L.a.w.x, L.a.v.y\}$ and so on. By Lemma 2, each reduced strategy is an equivalence class with four elements: if $s_{1\varnothing} = L$,

[16] In an earlier draft of this paper I called "balancedness" a related property of the game tree.

then $\bar{R}_1(s_1) = |A(R,\ell'')| \times |A(R,r'')| = 4$; if $s_{1\varnothing} = R$ then $\bar{R}_1(s_1) = |A(L,r')| = 4$. Thus, the uniform mixed strategy of the normal form assigns probability $\mu_1^0(s_1) = \frac{1}{32}$ to each pure strategy $s_1 \in S_1$, and the induced mixed strategy in the reduced normal form coincides with the uniform mixed strategy of the reduced normal form: $\left(\mu_1^0 \circ \bar{R}_1^{-1}\right)(\mathbf{r}_1) = \frac{1}{8} = \bar{\mu}_1^0(\mathbf{r}_1)$ for every $\mathbf{r}_1 \in \mathbf{R}_1$. ▲

5 Behavior strategy mixtures and independence

Given that each level-type k of player j uses behavior strategy σ_{jk} and under uncertainty about the level-type k of player j, the conditional predictive probabilities[17] of j's actions assigned by *player* i *of level* $\ell+1$ are obtained from the behavior strategy mixture (Selten, 1975) $\widetilde{\sigma}_j^\ell$ with *ex ante* subjective weights p_{jk}^ℓ ($p_{j\cdot}^\ell$ is the normalized truncation of p_j with support $\{0,...,\ell\}$). First, for each $\bar{h} \in H_j$ and level k, obtain the updated probability of k conditional on $\bar{h}$ by Bayes rule:

$$\nu_j^\ell\left(k|\bar{h}\right) = \frac{p_{jk}^\ell \prod_{h\in H_j\cap\left\{h':h'\prec\bar{h}\right\}} \sigma_{jk,h}\left(\alpha\left(h,\bar{h}\right)\right)}{\sum_{k'=0}^{\ell} p_{jk'} \prod_{h\in H_j\cap\left\{h':h'\prec\bar{h}\right\}} \sigma_{jk',h}\left(\alpha\left(h,\bar{h}\right)\right)},$$

where $\alpha\left(h,\bar{h}\right)$ is the action $\bar{a} \in A(h)$ such that $(h,\bar{a}) \preceq \bar{h}$. Next, for each $a \in A\left(\bar{h}\right)$, let

$$\widetilde{\sigma}_{j\bar{h}}^\ell(a) = \sum_{k=0}^{\ell} \nu_j^\ell\left(k|\bar{h}\right)\sigma_{jk,\bar{h}}(a).$$

The profile of behavior strategy mixtures describing the predictive probabilities assigned by player i of level $\ell+1$ to the co-players' actions is denoted $\widetilde{\sigma}_{-i}^\ell = \left(\widetilde{\sigma}_j^\ell\right)_{j\neq i}$.

Since the payoff of player i of level-type $\ell+1$ does not directly depend on the level-types of co-players, for the purpose of expected-payoff calculations only the conditional predictive probabilities $\widetilde{\sigma}_{-i}^\ell$ matter. Yet, it is interesting to keep track of updated beliefs about co-players' level-types. Starting from

[17] In compliance with the language of Bayesian statistics, a **predictive probability** is the probability of an observable event, possibly conditional on another observable event: unlike levels/types, histories and actions are observable.

the product measure $p_{-i}^{\ell} = \times_{j\neq i} p_j^{\ell}$, the updated beliefs of player i of level-type $\ell+1$ on the levels/types of the co-players conditional on $\bar{h}$ is the product measure $\times_{j\neq i}\nu_j^{\ell}\left(\cdot|\bar{h}\right)$. See Proposition 1 in Lin & Palfrey (2024) and the more general argument provided in Section 8 below about games with observed deviators.

6 Best Replies

Using Kuhn's map (1) and the assumption that every pure strategy profile of the co-players has strictly positive probability due to the presence of a positive fraction of level-0 types for each co-player (role) $j \neq i$, we can equivalently express i's conjectures about co-players as products of (I) totally mixed strategies $\mu_j \in \Delta^{\circ}(S_j)$, (II) totally mixed reduced strategies $\bar{\mu}_j \in \Delta^{\circ}(\mathbf{R}_j)$, or (III) totally randomized (predictive) behavior strategies $\hat{\sigma}_j \in \times_{h\in H_j}\Delta^{\circ}(A(h))$.

For all $h \in H_i$ and $\sigma_i \in \times_{h'\in H_i}\Delta(A(h'))$, let $\sigma_i^{\succeq h} \in \times_{h'\in H_i\cap\left\{\bar{h}:\bar{h}\succeq h\right\}}\Delta(A(h'))$ denote the **restriction of σ_i to the subgame with root** h; symbol $s_i^{\succeq h} \in \times_{h'\in H_i\cap\left\{\bar{h}:\bar{h}\succeq h\right\}} A(h')$ has the analogous meaning for pure strategies. With this,

$$\operatorname{supp}\left(\sigma_i^{\succeq h}\right) := \left\{s_i^{\succeq h} : \forall h' \in H_i \cap \left\{\bar{h} : \bar{h} \succeq h\right\}, \sigma_{ih'}^{\succeq h}\left(s_{ih'}^{\succeq h}\right) > 0\right\}$$

denotes the **support of** $\sigma_i^{\succeq h}$, that is, the support of the h-subgame mixed strategy obtained from $\sigma_i^{\succeq h}$ by means of the restriction of Kuhn's map (1) to the subgame:

$$s_i^{\succeq h} \in \operatorname{supp}\left(\sigma_i^{\succeq h}\right) \iff \prod_{h'\in H_i\cap\left\{\bar{h}:\bar{h}\succeq h\right\}} \sigma_{ih'}(s_{ih'}) > 0.$$

To simplify the statement of results, I define randomized best replies by assuming, in the spirit of the cognitive-hierarchies literature, that ties at the top are broken by randomizing uniformly on top actions. For any terminal history z, nonterminal history h, h-subgame strategy $s_i^{\succeq h}$, and conjecture (co-players' behavior strategies profile) $\hat{\sigma}_{-i} = (\hat{\sigma}_j)_{j\neq i}$, I let $\mathbb{P}\left(z|h; s_i^{\succeq h}, \hat{\sigma}_{-i}\right)$ denote the probability of reaching z from h given $s_i^{\succeq h}$, and $\hat{\sigma}_{-i} \in \Sigma_{-i}$, where $\Sigma_{-i} := \times_{j\in I\setminus\{i\}}\Sigma_j$ is the set of behavior strategy profiles of i's co-players.

Definition 3 *The **sequential best reply** of i to (predictive) conjecture $\hat{\sigma}_{-i} \in \Sigma_{-i}$ is the (possibly degenerate) behavior strategy $\sigma_i = \mathrm{BR}_i(\hat{\sigma}_{-i})$ that maximizes expected payoff given $\hat{\sigma}_{-i}$ in every subgame and such that each local randomization is uniform on its support, that is, for every $h \in H_i$,*

$$\begin{aligned} supp\left(\sigma_i^{\succeq h}\right) &= \arg\max_{s_i^{\succeq h}} \sum_{z \succeq h} \mathbb{P}\left(z|h; s_i^{\succeq h}, \hat{\sigma}_{-i}\right) u_i(z), \\ \forall a &\in supp(\sigma_{ih}), \ \sigma_{ih}(a) = \frac{1}{|supp(\sigma_{ih})|}. \end{aligned}$$

Well-known results about dynamic programming yield the following:[18]

Remark 6 (One-Deviation Principle) *For any pair $(\sigma_i, \hat{\sigma}_{-i}) \in \Sigma_i \times \Sigma_{-i}$, σ_i is the sequential best reply of i to conjecture $\hat{\sigma}_{-i}$—that is, $\sigma_i = \mathrm{BR}_i(\hat{\sigma}_{-i})$—if and only if, for every $h \in H_i$,*

$$\begin{aligned} supp(\sigma_{ih}) &= \arg\max_{a \in A(h)} \sum_{z \succeq (h,a)} \mathbb{P}(z|(h,a); \sigma_i, \hat{\sigma}_{-i}) u_i(z), \\ \forall a &\in supp(\sigma_{ih}), \ \sigma_{ih}(a) = \frac{1}{|supp(\sigma_{ih})|}, \end{aligned}$$

where $\mathbb{P}(z|(h,a); \sigma_i, \hat{\sigma}_{-i})$ is the probability of z conditional on (h,a) when behavior complies with σ_i and $\hat{\sigma}_{-i}$ in the subgame with root (h,a).

Let $H_i(\mu_i) := \bigcup_{s_i \in \mathrm{supp}(\mu_i)} H_i(s_i)$ denote the the set of non terminal histories allowed (not precluded) by mixed strategy μ_i. For any behavior strategy σ_i that is realization equivalent to μ_i, write $H_i(\sigma_i) = H_i(\mu_i)$.

Definition 4 *The **weakly sequential best reply** of i to $\hat{\sigma}_{-i} \in \Sigma_{-i}$ is the (possibly degenerate) behavior strategy $\bar{\sigma}_i = \overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})$ such that, for every $h \in H_i(\sigma_i)$,*

$$\begin{aligned} supp\left(\bar{\sigma}_i^{\succeq h}\right) &= \arg\max_{s_i^{\succeq h}} \sum_{z \succeq h} \mathbb{P}\left(z|h; s_i^{\succeq h}, \hat{\sigma}_{-i}\right) u_i(z), \\ \forall a &\in supp(\bar{\sigma}_{ih}), \ \bar{\sigma}_{ih}(a) = \frac{1}{|supp(\bar{\sigma}_{ih})|}, \end{aligned}$$

[18] See, e.g., Chapter 10 of Battigalli *et al.* (2024).

and furthermore

$$\forall h \in H_i \backslash H_i(\bar{\sigma}_i), \forall a \in A(h), \bar{\sigma}_{ih}(a) = \frac{1}{|A(h)|}.$$

Note that the specification of $\bar{\sigma}_i$ outside $H_i(\bar{\sigma}_i)$ is immaterial, but the uniform distribution is in the spirit of the CH model and allows a simplified statement of results. The following observations follow from well-known results about expected payoff maximization in sequential games:[19]

Remark 7 *Weakly sequential best replies are invariant to realization equivalences, that is, for all players $i \in I$, conjectures $\hat{\sigma}_{-i}$, and pure strategies $s_i \in$supp$\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right)$, $\bar{R}_i(s_i) \subseteq$supp$\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right)$.*

Remark 8 *Sequential best replies and weakly sequential best replies coincide and yield the same expected payoffs on realizable histories, that is, for all $i \in I$ and $\hat{\sigma}_{-i} \in \Sigma_{-i}$,*

$$H_i(\mathrm{BR}_i(\hat{\sigma}_{-i})) = H_i\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right),$$

$$\forall h \in H_i(\mathrm{BR}_i(\hat{\sigma}_{-i})), \mathrm{BR}_{ih}(\hat{\sigma}_{-i}) = \overline{\mathrm{BR}}_{ih}(\hat{\sigma}_{-i}),$$

and

$$\sum_{z \succeq h} \mathbb{P}\left(z|h; \mathrm{BR}_i^{\succeq h}(\hat{\sigma}_{-i}), \hat{\sigma}_{-i}\right) u_i(z) = \sum_{z \succeq h} \mathbb{P}\left(z|h; \overline{\mathrm{BR}}_i^{\succeq h}(\hat{\sigma}_{-i}), \hat{\sigma}_{-i}\right) u_i(z).$$

Example 5 In Game 1, sequential and weakly sequential best replies coincide for player 2, because the optimal course of action in the subgame with root C is to play $c.a$, and $H_2(c.a) = H_2$. Now consider the following *modification of Game 1*: path (C, d) yields 8 utils to player 2. According to Definition 4, the weakly sequential best reply to any σ_1 is the behavior strategy $\bar{\sigma}_2$ such that $\bar{\sigma}_{2C}(d) = 1$ (to grab 8 utils) and $\bar{\sigma}_{2(C,c)}(a) = \frac{1}{2}$, because $(C, c) \notin H_2(\bar{\sigma}_2)$. The sequential best reply is $d.a$, that is, the degenerate behavior strategy σ_2 with $\sigma_{2C}(d) = 1$ and $\sigma_{2(C,c)}(a) = 1$. Thus, $H_2(\bar{\sigma}_2) = \{C\} = H_2(\sigma_2)$ and the two strategies select action d probability 1 at history C. ▲

[19] See, e.g., Chapter 10 in Battigalli *et al.* (2024).

7 Equivalence Results

In this section, I build on previous observations and results to explain the normal-form invariance of the CH model.

Definition 5 *The **ex ante best reply** of i to $\mu_{-i} \in \Delta(S_{-i})$ is the (possibly degenerate) mixed strategy μ_i^* such that*

$$\begin{aligned} \mathrm{supp}\left(\mu_i^*\right) &= \arg\max_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} U_i\left(s_i, s_{-i}\right) \mu_{-i}\left(s_{-i}\right), \\ \forall s_i &\in \mathrm{supp}\left(\mu_i^*\right), \, \mu_i^*\left(s_i\right) = \frac{1}{\left|\mathrm{supp}\left(\mu_i^*\right)\right|}. \end{aligned}$$

In what follows, for every behavioral strategy profile $\sigma_{-i} \in \Sigma_{-i}$ of the co-players, I let $\mu_{-i}^{\sigma_{-i}} \in \Delta(S_{-i})$ denote the product measure resulting from any realization-equivalent profile of mixed strategies; for definiteness, consider the one obtained by means of Kuhn's map (1), i.e., the product measure $\mu_{-i}^{\sigma_{-i}} \in \Delta(S_{-i})$ such that

$$\forall s_{-i} \in S_{-i}, \, \mu_{-i}^{\sigma_{-i}}\left(s_{-i}\right) = \prod_{j \neq i} \prod_{h \in H_j} \sigma_{jh}\left(s_{jh}\right).$$

Lemma 3 *For all strictly positive conjectures, ex ante best replies coincide with weakly sequential best replies: specifically, for all $i \in I$ and $\hat{\sigma}_{-i} \in \Sigma^{\circ}{}_{-i}$, the ex ante best reply to $\mu_{-i}^{\hat{\sigma}_{-i}}$ is the Kuhn's transformation of $\overline{\mathrm{BR}}_i\left(\hat{\sigma}_{-i}\right)$.*

Recall that p_j^{ℓ} denotes the ℓ-truncation of p_j, that is, for every level-type $k \in \{0, ..., \ell\}$, $p_{jk}^{\ell} = \left(\sum_{\kappa=0}^{\ell} p_{j\kappa}\right)^{-1} p_{jk}$ and $p_{jm}^{\ell} = 0$ for $\ell < m$. With this, p_j^{ℓ} is the initial belief of player i of level-type $\ell+1$ about the level-types of player j. Recall that $\widetilde{\sigma}_{-i}^{\ell}$ is the profile of behavior strategy mixtures representing the predictive probabilities assigned by player i of level-type $\ell+1$ to the co-players' actions. Similarly, in a game with simultaneous moves (such as the normal form of the given sequential game), we let μ_j^k denote the mixed strategy of level-type k of player j, so that the conjecture of player i of level-type $\ell+1$ about the co-players' strategies is $\widetilde{\mu}_{-i}^{\ell} = \times_{j \neq i}\left(\sum_{k=0}^{\ell} p_{jk}^{\ell} \mu_j^k\right)$.

Proposition 1 *Consider the CH models applied to the normal-form and extensive-form representations of a finite game (with perfect information). For every player* $i \in I$ *and every level* $\ell \geq 0$*, the level-*$(\ell+1)$ *mixed best reply* $\mu_i^{\ell+1}$ *to conjecture* $\widetilde{\mu}_{-i}^{\ell} = \times_{j \neq i} \left(\sum_{k=0}^{\ell} p_{jk}^{\ell} \mu_j^k \right)$ *in the normal form is the Kuhn's transformation of the weakly sequential best reply* $\bar{\sigma}_i^{\ell+1} = \overline{\mathrm{BR}}_i \left(\widetilde{\sigma}_{-i}^{\ell} \right)$ *to behavior strategy mixture* $\widetilde{\sigma}_{-i}^{\ell}$ *in the extensive form, which is realization equivalent to the sequential best reply* $\sigma_i^{\ell+1} = \mathrm{BR}_i \left(\widetilde{\sigma}_{-i}^{\ell} \right)$.

Proof The proof is by induction on ℓ. The basis step $\ell = 0$ follows from Remark 1 and Lemma 3, because $\widetilde{\mu}_{-i}^{0} = \times_{j \neq i} \mu_j^0$, where μ_j^0 is the uniform (hence, strictly positive) probability measure induced by the uniform behavior strategy σ_j^0. For $\ell > 0$, suppose by way of induction that the result holds for each $k \in \{0, ..., \ell\}$ and fix any $i \in I$. One can show that the strictly positive conjecture $\widetilde{\sigma}_{-i}^{\ell}$ is realization-equivalent to $\widetilde{\mu}_{-i}^{\ell}$. Thus, Lemma 3 yields the result. ■

Corollary 1 *Consider the* modified *CH model applied to the reduced strategic form where the level-*0 *type of each player* i *strictly randomizes with the (possibly non-uniform) reduced mixed strategy* $\bar{\mu}_i^0 = \mu_i^0 \circ \bar{R}_i^{-1} \in \Delta(\mathbf{R}_i)$ *obtained from the uniform mixed strategy* $\mu_i^0 \in \Delta(S_i)$*. For every player* $i \in I$ *and every level* $\ell \geq 0$*, if the reduced mixed strategy of level* $\ell + 1$ *of* i *is pure (degenerate,* $\bar{\mu}_i^{\ell+1} = \delta_{\mathbf{r}_i^{\ell+1}}$*), then the corresponding pure reduced strategy* $\mathbf{r}_i^{\ell+1}$ *satisfies* $\mathbf{r}_i^{\ell+1} = supp\left(\mu_i^{\ell+1}\right)$*, where* $\mu_i^{\ell+1} \in \Delta(S_i)$ *is the mixed strategy of level* $\ell + 1$ *of* i *in the normal form.*

The introductory examples are simple illustrations of Proposition 1. Corollary 1 is illustrated by the following:

Example 6 Consider Game 1. The set of reduced strategies of player 2 is $\mathbf{R}_2 = \{\mathbf{c.a}, \mathbf{c.b}, \mathbf{d}\}$, where $\mathbf{c.x} = \{c.x\}$ (singleton) for each $x \in A(C, c) = \{a, b\}$ and $\mathbf{d} = \{d.a, d.b\}$. Then $\bar{\mu}_2^0(\mathbf{c.a}) = \bar{\mu}_2^0(\mathbf{c.b}) = \frac{1}{4}$, and $\bar{\mu}_2^0(\mathbf{d}) = \mu_2^0(d.a) + \mu_2^0(d.b) = \frac{1}{2}$. The best reply to $\bar{\mu}_2^0$ for the first level-type of player 1 in the reduced normal form is the same as the best reply to μ_2^0 for the first level-type of player 1 in the normal form. It follows that the modified CH solution in the reduced normal form is equivalent to the CH solution in the normal form, which is equivalent (by Proposition 1) to the CH solution in the extensive form. ▲

8 Games with Imperfect Information

The foregoing analysis extends seamlessly to games with **observed actions**, where players may choose simultaneously at some stage and previous moves are perfectly observed.[20] As for games with *imperfectly* observed actions, the main complication is due to the presence of information sets. The main change in this case is that the conditional belief about co-players' level-types at an information set in games with three or more players need not be a product measure (see Figure 10, Section 7.2 in Lin & Palfrey 2024). It is, however, a product measure in all games with **observed deviators**,[21] that is, games where, for every player i and information set $\mathbf{h}_i \in \mathbf{H}_i$, the set

$$S(\mathbf{h}_i) := \{s \in S : \exists h \in \mathbf{h}_i, h \prec O(s)\}$$

of pure strategy profiles inducing a path through $\mathbf{h}_i$ is a Cartesian product of its projections:

$$S(\mathbf{h}_i) = \times_{j \in I} \mathrm{proj}_{S_j} S(\mathbf{h}_i).$$

Note that perfect recall implies that, for every player $i \in I$ and information set $\mathbf{h}_i \in \mathbf{H}_i$, set $S(\mathbf{h}_i)$ can be factorized as the product of its projections onto S_i and S_{-i}:

$$S(\mathbf{h}_i) = \mathrm{proj}_{S_i} S(\mathbf{h}_i) \times \mathrm{proj}_{S_{-i}} S(\mathbf{h}_i).$$

Therefore, under perfect recall, every two-person game has observed deviators.

Proposition 2 *Suppose that the information structure satisfies the* observed deviators *property, that is,* $S(\mathbf{h}_i) = \times_{j \in I} S_j(\mathbf{h}_i)$ *for all players* $i \in I$ *and information sets* $\mathbf{h}_i \in \mathbf{H}_i$. *Then players' updated beliefs about co-players' level-types conditional on their information sets are product measures.*

The main results stated for perfect information games also hold for all sequential games (assuming perfect recall): essentially, in Remark 1 one has to replace, for each player $i \in I$, histories $h \in H_i$ with information sets

[20] See, e.g., Chapter 9 in Battigalli *et al.* (2024); cf. Chapter 6.3.2 in Osborne & Rubinstein (1994): deviating from standard terminology, Osborne & Rubinstein stipulate that also such games have "perfect information."

[21] On observed deviators see Fudenberg & Levine (1993) and Battigalli (1996, 1997). The notation used here for information sets is the same as Battigalli *et al.* (2020).

$\mathbf{h}_i \in \mathbf{H}_i$ (corresponding to personal histories of signals received and actions taken by i, see Battigalli & Generoso 2024); the cardinality of the strategy set S_i is $|S_i| = \times_{\mathbf{h}_i \in \mathbf{H}_i} |A(\mathbf{h}_i)|$, and the same counting argument as in the proof of Remark 1 implies that the uniform behavior strategy σ_i^0 yields the uniform mixed strategy μ_i^0 by means of Kuhn's map. Results on sequential, weakly sequential and ex ante best replies extend seamlessly to all sequential games as long as players have perfect recall (which makes conditional expected utility maximization dynamically consistent). The extension of Proposition 1 to sequential games with imperfect information follows.

With this, known results on transformations of extensive form structures yield the following:[22]

Observation *The CH model is invariant to interchanging essentially simultaneous moves, but it is not invariant to coalescing sequential moves by the same player and its inverse, sequential-agents splitting.*

To see why this is true, note that Battigalli *et al.* (2020) prove that two extensive-form structures have the "same" map $\mathbf{O} : \mathbf{S} \rightarrow Z$ (up to isomorphisms) from structurally reduced strategy profiles to induced terminal histories if and only if it is possible to transform one into the other by means of a sequence of interchanging and coalescing/splitting transformations. One can also show that two extensive-form structures have the same map $O : S \rightarrow Z$ (up to isomorphisms) if and only if one can transform one into the other by means of a sequence of interchanging transformations (see Bonanno 1992). On the one hand, the latter result and the extension of Proposition 1 to games with imperfect information imply that the CH model is invariant to interchanging essentially simultaneous moves. On the other hand, the result by Battigalli *et al.* (2020), Remarks 2 and 5 imply that the CH model is not invariant to sequential-agents splitting (the inverse of coalescing), a transformation that destroys the equi-reducibility of a game; see the comparison of Game 2 with Game 1.

9 Discussion

The CH model is mostly used to organize data of experimental games. Sequential games are often played in experiments with the strategy method by

[22] Since I did not introduce the mathematical definitions of the involved transformations, the result can only be stated informally.

making subjects irreversibly choose among *reduced* strategies, which are easy for subjects to understand and conceptualize. A natural question is whether subjects who are presented with a sequential game and then have to choose between reduced strategies are better modeled by assuming that they think of uniform randomization as equalizing the probabilities of possible actions at any given node of the sequential game, or equalizing the probabilities of reduced strategies. Lin & Palfrey (2024) report interesting evidence supporting the latter hypothesis.

Be that as it may, we have to recognize that the normal form $\mathcal{N}(\Gamma)$ of a sequential game Γ and the reduced normal form $\mathcal{RN}(\Gamma)$—*interpreted as games where players irreversibly and covertly choose strategies (reduced strategies) in advance*—are different from each other (except when Γ is a one-move game), and that they are very different from the sequential game Γ. Whether we should expect players to behave "in the same way" in Γ, $\mathcal{N}(\Gamma)$, and $\mathcal{RN}(\Gamma)$—or whether behavior should be expected to be invariant to some specific transformations of the game—cannot but depend on the adopted theory of strategic interaction and the corresponding solution concept. It is well known that some solution concepts like Nash equilibrium and iterated admissibility are essentially reduced-normal-form invariant,[23] while others like trembling-hand perfect equilibrium, sequential equilibrium, and notions of rationalizability for sequential games are not reduced normal form invariant. Similarly, some solution concepts are invariant to transformations like interchanging essentially simultaneous moves and coalescing/sequential-agents splitting: these transformations do not change the structurally reduced normal form (see Battigalli *et al.* 2020); thus, all the reduced-normal-form invariant solution concepts, like Nash equilibrium and iterated admissibility, are necessarily invariant to these transformations; but also other *non*-reduced-normal-form invariant solution concepts, like initial and strong rationalizability, are invariant to these two transformations.[24] I proved that the CH model with uniform randomization by level-0 types is normal-form invariant, although it is *not* reduced-normal-form invariant. It follows that the model is invariant to interchanging essentially simultaneous moves, but not invariant to coalescing/sequential-agents splitting. Are these lacks of in-

[23] It is less well known that also selfconfirming equilibrium is essentially reduced-normal form invariant. See Battigalli et al. (2019).

[24] On initial (also called "weak") rationalizability and strong rationalizability (a.k.a. "extensive-form rationalizability") see, e.g., the textbook of Battigalli *et al.* (2024) and the relevant references therein.

variances mere "representation effects"? My position is that, even if different games can be obtained from each other by some transformations preserving some basic structures, *they remain different and should not be presumed to be played in the same way unless one explicitly spells out and adopts a theory entailing this.* Some solution concepts have clear and explicit foundations in theories of strategic reasoning, or learning, or adaptive play. If we like those theories, we must accept the equivalences they entail and no more.

Finally, let me point out that—as anticipated in the Introduction—part of my observations about the CH model also apply to the level-k thinking model, whereby a level-k type plays the best reply to the strategy profiles of level-$(k-1)$ types of the co-players. Indeed, in both models level-0 types are assumed to play the uniform randomization and level-1 types play the best reply to the profile of uniformly randomized strategies. Thus, what I observed about differences of predictions between the normal form and the reduced normal form, as well as the lack of invariance to coalescing/sequential-agents splitting, applies to the level-k model as well. The two models differ for levels $k \geq 2$. The key conceptual issue of the level-k model applied to sequential games is that types of level $k \geq 2$ may be completely surprised by some observed moves of the level-$(k-1)$ co-players; this implies that *ex ante* best replies need not be weakly sequential best replies. Thus, there is no equivalence between reasonable extensions of the level-k model to sequential games and the level-k model on the normal form of the game. Furthermore, as observed by Schipper & Zhou (2024), extensions of the level-k model to sequential games require the addition of a theory of how players revise their beliefs when they are surprised.[25] This is not necessary for the CH model, because every type of level $k > 0$ assigns positive probability to level-0 types of the co-players, making every node reachable with positive probability.

10 Appendix

Proof of Remark 2 If Γ has the one-move property, then $H_i(s_i) = H_i$ for every $s_i \in S_i$, because no strategy of i can prevent the realization of any history $h \in H_i$. Therefore, Lemma 2 implies that $\mathbf{R}_i$ contains only singletons: recalling the convention about products (Section 2), for every $s_i \in S_i$, since $H_i \backslash H_i(s_i) = \emptyset$, $\left|\bar{R}_i(s_i)\right| = \prod_{h \in H_i \backslash H_i(s_i)} |A(h)| = 1$.

[25] On the level-k thinking model and its extensions to sequential games see Schipper & Zhou (2024) and the relevant references therein.

Now suppose that Γ does *not* have the one-move property. Then there are a player $i \in I$ and a pair of histories $h, \bar{h} \in H_i$ such that $h \prec \bar{h}$. Recall that, $\alpha\left(h, \bar{h}\right)$ denotes the action $\bar{a} \in A(h)$ such that $(h, \bar{a}) \preceq \bar{h}$. Also recall that every feasible action set $A(h')$ ($h' \in H \backslash Z$) has at least two elements. Thus, there is a strategy s_i with $s_{ih} \neq \alpha\left(h, \bar{h}\right)$, so that $\bar{h} \in H_i \backslash H_i(s_i)$, and Lemma 2 implies that $\left|\bar{R}_i(s_i)\right| \geq \left|A\left(\bar{h}\right)\right| \geq 2$. ■

Proof of Lemma 3 It is well known that, if every strategy profile of the co-players is deemed possible ex ante, then ex ante expected payoff maximization is equivalent to expected payoff maximization conditional on each history allowed by the optimizing strategy.[26] As for the probabilities assigned by the mixed best reply, observe that—since, by Remark 8, all the actions in the support of sequential best reply $\mathrm{BR}_i(\hat{\sigma}_{-i})$ realization-equivalent to weakly sequential best reply $\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})$ yield the same, maximal conditional expected payoff and $\mathrm{BR}_{ih}(\hat{\sigma}_{-i}) = \overline{\mathrm{BR}}_{ih}(\hat{\sigma}_{-i})$ for all $h \in H_i(\mathrm{BR}_i(\hat{\sigma}_{-i})) = H_i\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right)$—then a kind of exchangeability property holds:

$$\begin{aligned}
\mathrm{supp}(\mu_i^*) &= \left(\times_{h \in H_i(\mathrm{BR}_i(\hat{\sigma}_{-i}))} \mathrm{supp}(\mathrm{BR}_{ih}(\hat{\sigma}_{-i}))\right) \times \left(\times_{h' \in H_i \backslash H_i(\mathrm{BR}_i(\hat{\sigma}_{-i}))} A(h')\right) \\
&= \mathrm{supp}\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right),
\end{aligned}$$

where μ_i^* is the ex ante best reply. Therefore, for every $s_i \in$supp(μ_i^*),

$$\begin{aligned}
\mu_i^*(s_i) &= \frac{1}{|\mathrm{supp}(\mu_i^*)|} \\
&= \frac{1}{\prod_{h \in H_i(\mathrm{BR}_i(\hat{\sigma}_{-i}))} |\mathrm{supp}(\mathrm{BR}_{ih}(\hat{\sigma}_{-i}))| \cdot \prod_{h' \in H_i \backslash H_i(\mathrm{BR}_i(\hat{\sigma}_{-i}))} |A(h')|} \\
&= \frac{1}{\left|\mathrm{supp}\left(\overline{\mathrm{BR}}_i(\hat{\sigma}_{-i})\right)\right|} \\
&= \prod_{h \in H_i} \frac{1}{\left|\mathrm{supp}\left(\overline{\mathrm{BR}}_{ih}(\hat{\sigma}_{-i})\right)\right|} = \prod_{h \in H_i} \overline{\mathrm{BR}}_{ih}(\hat{\sigma}_{-i})(s_{ih}). \quad ■
\end{aligned}$$

Proof of Proposition 2 Following the hint in footnote 8 of Battigalli (1996), model i's uncertainty about co-players $j \neq i$ as distributional strategies $\delta_j \in \Delta(\Theta_j \times S_j)$, where $\Theta_j \cong \mathbb{N}_0$ is the set of level-types of player j. (Of course, the ex ante belief on $\times_{j \neq i} \Theta_j$ of level-type $k+1$ of player i is the

[26] See, e.g., Chapter 10 in Battigalli *et al.* (2024).

product of the normalized truncations on $\{0, ..., k\}$.) The initial belief about co-players is the product measure $\delta_{-i} = \times_{j \neq i} \delta_j$. Observed deviators implies that the updated probability of profile $(\theta_{-i}, s_{-i}) = (\theta_j, s_j)_{j \neq i}$ conditional on $\mathbf{h}_i \in \mathbf{H}_i$ is

$$\begin{aligned}
\delta_{-i}\left(\theta_{-i}, s_{-i} | \Theta_{-i} \times S_{-i}\left(\mathbf{h}_i\right)\right) &= \frac{\delta_{-i}\left(\theta_{-i}, s_{-i}\right)}{\delta_{-i}\left(\Theta_{-i} \times S_{-i}\left(\mathbf{h}_i\right)\right)} = \\
\frac{\delta_{-i}\left(\theta_{-i}, s_{-i}\right)}{\delta_{-i}\left(\times_{j \neq i} \Theta_j \times S_j\left(\mathbf{h}_i\right)\right)} &= \frac{\prod_{j \neq i} \delta_j\left(\theta_j, s_j\right)}{\prod_{j \neq i} \delta_j\left(\Theta_j \times S_j\left(\mathbf{h}_i\right)\right)} = \\
\prod_{j \neq i} \frac{\delta_j\left(\theta_j, s_j\right)}{\delta_j\left(\Theta_j \times S_j\left(\mathbf{h}_i\right)\right)} &= \prod_{j \neq i} \delta_j\left(\theta_j, s_j | \Theta_j \times S_j\left(\mathbf{h}_i\right)\right),
\end{aligned}$$

where the denominators are strictly positive because there is a strictly positive fraction of level-0 types, who play every action with strictly positive probability. The conditional probability of each profile of co-players level-types $\theta_{-i} = (\theta_j)_{j \neq i}$ is the (product) marginal of $\delta_{-i}\left(\cdot | \Theta_{-i} \times S_{-i}\left(\mathbf{h}_i\right)\right)$:

$$\nu_i\left(\theta_{-i} | \mathbf{h}_i\right) = \prod_{j \neq i} \delta_j\left(\{\theta_j\} \times S_j\left(\mathbf{h}_i\right) | \Theta_j \times S_j\left(\mathbf{h}_i\right)\right).$$

■

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