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Are the Players in an Interactive Belief Model Meta-certain of the Model Itself?*

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Abstract

Are the players "commonly meta-certain" of an interactive belief model itself? The paper formalizes what it means by: "a player is (meta-)certain of her own belief-generating map" or "the players are (meta-)certain of the profile of belief-generating maps (i.e., the model)." The paper shows: a player is (meta-)certain of her own belief-generating map if and only if her beliefs are introspective. The players are commonly (meta-)certain of the model if and only if, for any event which some player i believes at some state, it is common belief at the state that player i believes the event. This paper then asks whether the "common meta-certainty" assumption is needed for epistemic characterizations of game-theoretic solution concepts. The paper shows: common belief in rationality leads to actions that survive iterated elimination of strictly dominated actions, as long as each player is logical and (meta-)certain only of her own strategy and belief-generating map.

JEL Classification: C70; D83
Keywords: Belief; Knowledge; Common Belief; Common Knowledge; Introspection; Epistemic Game Theory

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The common knowledge assumption underlies all of game theory and much of economic theory. Whatever be the model under discussion, whatever complete or incomplete information, consistent or inconsistent, repeated or one-shot, cooperative or non-cooperative, the model itself must be assumed common knowledge; otherwise the model is insufficiently specified, and the analysis incoherent.

—Aumann (1987b)

1 Introduction

In an economic or game-theoretic model in which the players make their interactive reasoning about their strategies or rationality, the analysts implicitly ("from outside of the model") assume that the players understand the model itself in a meta-sense. The above quotation from Aumann (1987b) suggests that the analysts should assume "the model is commonly known by the players" since otherwise "the model is insufficiently specified, and the analysis incoherent."

This paper has two objectives. The first is to explicitly formalize the "common knowledge" assumption of a model within the model itself. An interactive belief/knowledge model formally represents players' beliefs/knowledge about its ingredients, that is, events. The model itself does not tell whether the players (commonly) believe/know the model itself, although the analysts assume that the players (commonly) believe/know the model in a meta-sense. I refer to the knowledge/belief of the model as the "meta-knowledge/meta-belief" of the model.

The second objective is to examine the role that "meta-knowledge" of a model plays in game-theoretic analyses such as epistemic characterizations of solution concepts, robustness of solution concepts, or robustness of behaviors with respect to players' beliefs/knowledge. For a given epistemic characterization of a game-theoretic solution concept such as iterated elimination of strictly dominated actions, do the outside analysts need to formally assume that the players "meta-know" an epistemic model of a game (that describes their interactive beliefs about their strategies and rationality)?

Are the players (commonly) meta-certain of a model itself?[1] This first question has been puzzling theorists since the pioneering work of Aumann (1976, 1987a,b, 1999) on interactive knowledge models.[2] The main result regarding the first objective

[1]Since different epistemic models may feature different notions of qualitative or probabilistic beliefs or knowledge, I use the word the "(meta-)certainty" of a model to refer generically to the meta-knowledge or meta-belief of the model. In a probabilistic-belief model, by (meta-)certainty, it means that the players meta-believe the model with probability one. In a model of qualitative belief or knowledge whose degree of beliefs are stronger than probability-one belief, by (meta-)certainty, it means that the players meta-believe the model in the absolute sense or they meta-know the model.

[2]For this question, see also Bacharach (1985, 1990), Binmore and Brandenburger (1990), Brandenburger and Dekel (1989, 1993), Brandenburger, Dekel, and Geanakoplos (1992), Brandenburger and Keisler (2006), Dekel and Gul (1997), Fagin et al. (1999), Gilboa (1988), Myerson (1991), Pires

(Theorems 1A and 1B in the contexts of qualitative and probabilistic beliefs, respectively) in Section 4 characterizes the "implicit common meta-certainty" assumption as follows. According to the formal test to be discussed, the players are commonly (meta-)certain of a model if and only if, for any event which some player i believes at some state, it is common belief that player i believes the event at that state.

Moving on to the second question, Section 5 examines the role that the "common meta-certainty" assumption plays in epistemic characterizations of solution concepts of games. Especially, Section 5.1 studies the epistemic characterization of iterated elimination of strictly dominated actions (IESDA) in a strategic game. Informally, it states: if the players are "logical," if they are commonly meta-certain of a game, and if they commonly believe their rationality, then the resulting actions survive IESDA. Formally, it states: if the players commonly believe their rationality and if their common belief in their rationality is correct, then the resulting actions survive IESDA.[3] The main result regarding the second objective (Theorem 2) connects these two statements. If the players' beliefs are monotone (they believe any logical implication of what they believe), consistent (i.e., they do not simultaneously believe an event and its negation), and finitely conjunctive (if they believe E and F then they believe its conjunction $E \cap F$), and if each player is certain of her own strategy and the part of her own belief-generating process in the model (even if each player may not necessarily be certain of how the opponents' beliefs are generated in the model), then each player correctly believes her own rationality, and hence they have correct common belief in their rationality. Thus, if the players are "logical" and each of them is meta-certain of the part of the model that governs her *own* beliefs, then common belief in rationality leads to actions that survive IESDA. Section 5 also briefly studies epistemic characterizations of Nash equilibria and the role of meta-certainty assumption in communication protocols leading to agreement (e.g., Geanakoplos and Polemarchakis, 1982; Hart and Tauman, 2004; Sebenius and Geanakoplos, 1983). These results demonstrate the sense of robustness in which various epistemic characterizations of solution concepts of games hold even if players are not necessarily meta-certain of other players' belief-generating process in a belief model.

Now, I formally introduce a (belief) model described in Section 2. The model consists of the following three ingredients. The first is a measurable space of states of the world $(\Omega, \mathcal{D})$. Each state $\omega \in \Omega$ is a list of possible specifications of what the world is like, and the collection $\mathcal{D}$ of events (i.e., subsets of Ω) are the objects of the players' beliefs. The second is the players' belief operators $(B_i)_{i \in I}$. Player i's belief operator B_i associates, with each event E, the event that player i believes E. Iterative application of belief operators unpacks higher-order interactive beliefs. To focus on

(2021), Roy and Pacuit (2013), Samuelson (2004), Tan and Werlang (1988), Vassilakis and Zamir (1993), Werlang (1987), Tan and Werlang (1992), and Wilson (1987).

[3]The formal statement is taken from Fukuda (2020, Theorem 3), which holds irrespective of properties of beliefs. For seminal papers on implications of common belief in rationality, see, for example, Brandenburger and Dekel (1987), Stalnaker (1994), and Tan and Werlang (1988).

the (meta-)certainty of a model and to separate it from reasoning ability, this paper often assumes that the players' belief operators are monotone: if player i believes E at a state and if E implies (i.e., is included in) F, then she believes F at that state. That is, each player believes any logical implication of her own beliefs. Throughout the Introduction, I assume that the players' belief operators are monotone. The third is a common belief operator C, which associates, with each event E, the event that E is common belief among the players. Under certain assumptions on the players' beliefs, an event E is common belief if and only if everybody believes E, everybody believes that everybody believes E, and so on *ad infinitum*.

This framework nests various models of qualitative and probabilistic beliefs and knowledge. Broadly, the framework nests the following two standard models of belief or knowledge (or combinations thereof). First, the framework nests a possibility correspondence model of qualitative belief or knowledge when each player's belief or knowledge is induced by a possibility correspondence.[4] The possibility correspondence associates, with each state, the set of states that she considers possible. The player believes an event E at a state whenever the possibility set at ω implies (i.e., is included in) the event E. Second, the framework nests a Harsanyi (1967-1968) type space when each player's probabilistic beliefs are induced by her type mapping. The type mapping τ_i associates, with each state ω, her probability measure $\tau_i(\omega)$ on the underlying states at that state. The type mapping τ_i of player i induces her p-belief operator $B^p_{\tau_i}$ (Monderer and Samet, 1989): it associates, with each event E, the event that (i.e., the set of states at which) player i believes E with probability at least p (i.e., p-believes E). Certain properties of p-belief operators $(B^p_{\tau_i})_{p\in[0,1]}$ reproduce the underlying type mapping τ_i (Samet, 2000).

With the framework in mind, I formalize the (meta-)certainty of a model in two steps. In the first step, Section 3.1 expands the objects of the players' beliefs from events to functions defined on the underlying states. Examples of such functions are random variables, strategies, and type mappings. Any such function x has to be defined on the state space Ω, but the co-domain X can be any set such as the set $\mathbb{R}$ of real numbers (a random variable), a set A_i of player i's actions (her strategy), and the set $\Delta(\Omega)$ of probability measures on $(\Omega, \mathcal{D})$ (a type mapping). I call the function $x : \Omega \to X$ a signal if its co-domain X has "observational" contents $\mathcal{X}$ (where "observation" is broadly construed as being an object of reasoning): it is a collection of subsets of X such that each $F \in \mathcal{X}$ is deemed an event $x^{-1}(F)$. Thus, a *signal (mapping)* is a function $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ such that each observational content $F \in \mathcal{X}$ is considered to be an event $x^{-1}(F) \in \mathcal{D}$. Player i is *certain* of the value of the signal x at a state ω if, for any observational content F that holds at ω (i.e., $\omega \in x^{-1}(F)$), player i believes the event $x^{-1}(F)$ at ω (i.e., $\omega \in B_i(x^{-1}(F))$). Player i is *certain* of x if she is certain of the value of x at every state. For example, let $x : \Omega \to X$ be the strategy of player i and let every singleton action $a \in X$ be

[4]See, for instance, Aumann (1976, 1999), Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), Geanakoplos (2021), and Morris (1996).

observable to her; then, player i is certain of her own strategy if, wherever she takes an action $a = x(\omega)$ at a state ω, she believes at ω that she takes action a. Thus, when my definition is applied to a player's strategy, it nests the previous formalization such as Aumann (1987a), Brandenburger, Dekel, and Geanakoplos (1992), Geanakoplos (2021), and McKelvey and Page (1986). Having defined individual players' (meta-)certainty, the players are *commonly certain* of the value of the signal x at a state ω if, for any observational content F that holds at ω, the event $x^{-1}(F)$ is common belief at ω (i.e., $\omega \in C(x^{-1}(F))$). The players are *commonly certain* of the signal x if they are commonly certain of its value at every state.

In the second step, Section 3.2 formulates a players' "belief-generating map" as a signal that associates, with each state, her beliefs at that state. By the second step, I can apply the formalization of certainty and common certainty in the first step to the ingredients of a given model (i.e., players' belief-generating maps). To that end, take player i's belief operator B_i from the model. I define a qualitative-type mapping t_{B_i}: it associates, with each state, whether player i believes each event or not at that state (formally, a binary set function from the collection of events to the binary values $\{0, 1\}$ where 1 indicates the belief of an event). The qualitative-type mapping is a binary "type" mapping analogous to a (probabilistic-)type mapping τ_i that represents player i's probabilistic beliefs at each state in the context of probabilistic beliefs. That is, the type mapping τ_i assigns, to each state ω, her probabilistic beliefs $\tau_i(\omega) : \mathcal{D} \to [0, 1]$ at ω. In a similar manner, the qualitative-type mapping t_{B_i} associates, with each state ω, her qualitative belief $t_{B_i}(\omega) : \mathcal{D} \to \{0, 1\}$ (where $t_{B_i}(\omega)(E) = 1$ if and only if $\omega \in B_i(E)$) on $(\Omega, \mathcal{D})$ at ω. The qualitative-type mapping t_{B_i} is player i's belief-generating mapping. Since the belief operator B_i and the qualitative-type mapping t_{B_i} are equivalent means of representing player i's beliefs, a model means the profile of qualitative-type mappings. Thus, the formal test for whether the players are commonly certain of a given belief model is whether the players are certain of the profile of their qualitative-type mappings.

Before asking when a player is certain of all the players' qualitative-type mappings (i.e., the model), Section 3.3 characterizes when a player is certain of her own qualitative-type mapping in terms of her introspective properties of beliefs. Roughly, Proposition 1A shows that a player is certain of her own qualitative-type mapping if and only if her belief is introspective. Proposition 1B also shows that a player is certain of her own (probabilistic-)type mapping if and only if her probabilistic belief is introspective. These results distinguish the fact that player i is certain of her own qualitative-type mapping and the fact that player i is (or the players are commonly) certain of the profile of the qualitative-type mappings.

Section 3.4 provides an alternative characterization of the fact that player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ in terms of her reasoning of the signal. If she is certain of the signal x, then she would be able to rank the underlying states based on the collection of observational contents that hold at each state. Call a state ω *at least as informative as* a state ω' according to the signal x if, for any observational content F that holds at ω' under x, it holds at ω under x. Section 3.4 then characterizes properties of beliefs and the certainty of a mapping in terms of the notion of informativeness. In the literature on type spaces such as Mertens and Zamir (1985), a player is "certain" of her own type if, at each state ω, she believes, with probability one, the set of states indistinguishable from (i.e., equally informative to) ω. I show that such "Harsanyi" property holds if and only if the player is (meta-)certain of her own type mapping in the strongest sense. Hence, my characterization also formally captures the original idea behind Harsanyi (1967-1968) that each player "is certain of" her own type mapping.

To the best of my knowledge, this is the first paper which systematically formalizes the statement that the players are (commonly) (meta-)certain of any given belief model within the model itself. The contribution of the paper also lies in presenting a formalization of the (meta-)certainty of a belief model within the model itself. The main result on this question, nevertheless, is related to Gilboa (1988). He constructs a particular syntactic model in which the statement that the model is common knowledge is incorporated within itself.[5] He formulates the sense in which the model is commonly known from Positive Introspection of common knowledge: if a statement is common knowledge then it is commonly known that the statement is common knowledge. In Theorems 1A and 1B, in contrast, the players are commonly certain of a given model if and only if, at each state and for any event which some player believes at that state, it is common belief that the player believes the event at that state. Thus, in this paper, the key criteria is the positive introspective property of common belief with respect to each player's beliefs. Whenever some individual player believes some event, it is common belief that she believes it. Bacharach (1985, 1990), in the context of partitional possibility correspondence models, formalizes the event that a player has an information partition by regarding it as a function.

One informal solution to the question of the meta-certainty of a model has been the use of a "universal" belief model in which each state encodes what the players believe at that state and in which the differences in the players' beliefs are all described within the underlying states themselves (see, for example, Brandenburger and Dekel, 1993, Heifetz and Samet, 1998, and Mertens and Zamir, 1985). Subsequently, the various strands of robustness literature relax the implicit "common meta-certainty" assumption of an environment among players by studying how equilibrium or non-equilibrium solutions (or allocations) would depend on specifications of players' interactive beliefs on a "universal" belief model.[6] This paper provides any belief model with a test under which the model is meta-certain by the players.

The paper is organized as follows. Section 2 defines the basic framework of the paper, i.e., a belief model. Section 3 characterizes the sense in which each player is

[5]This contrasts to the formalization of (meta-)certainty for *any* belief model in this paper.

[6]For robust mechanism design, see, for instance, Bergemann and Morris (2005), Heifetz and Neeman (2006), and Neeman (2004). For the robustness of solution concepts, see, for example, Weinstein and Yildiz (2007) in the context of rationalizability.

certain of how her belief is generated in a model. Section 4 examines the sense in which the players are commonly certain of a model itself (i.e., how the players' beliefs are generated in the model). Section 5 studies the role of the common certainty assumption of a model in game-theoretic analyses. Section 6 provides concluding remarks. The proofs are relegated to Appendix A.

2 Framework

Throughout the paper, let I denote a non-empty finite set of *players*. The framework represents players' interactive beliefs by belief operators on a state space so that it can capture various forms of qualitative and probabilistic beliefs and knowledge. Section 2.1 defines a belief model. Section 2.2 defines properties of beliefs.

2.1 A Belief Model

A *belief model (of I)* is a tuple $\overrightarrow{\Omega} := \langle(\Omega, \mathcal{D}), (B_i)_{i\in I}, C\rangle$, where: (i) $(\Omega, \mathcal{D})$ is a non-empty measurable space of states of the world (call Ω the *state space*); (ii) $B_i : \mathcal{D} \to \mathcal{D}$ is player i's *belief operator*; and (iii) $C : \mathcal{D} \to \mathcal{D}$ is a *common belief operator* to be defined in Expression (1) below.

While Ω constitutes a non-empty set of *states* of the world, each element E of $\mathcal{D}$ is an *event* about which the players reason. The assumption that $(\Omega, \mathcal{D})$ forms a measurable space accommodates players' qualitative and probabilistic beliefs in the same framework. Conceptually, the assumption means that: (i) if E is an object of beliefs, then so is its complement E^c (denote it also by $\neg E$); that (ii) if $(E_n)_{n\in\mathbb{N}}$ are objects of beliefs, then so are its union $\bigcup_{n\in\mathbb{N}} E_n$ and its intersection $\bigcap_{n\in\mathbb{N}} E_n$; and that (iii) any form of tautology Ω (e.g., $E \cup E^c$) is an object of beliefs.

For each event E, the set $B_i(E)$ denotes the event that (i.e., the set of states at which) player i believes E. Thus, player $i \in I$ believes an event $E \in \mathcal{D}$ at a state $\omega \in \Omega$ if $\omega \in B_i(E)$. I often assume that each player's belief operator satisfies *Monotonicity*: $E \subseteq F$ implies $B_i(E) \subseteq B_i(F)$. It means that if player i believes some event then she believes any of its logical consequences.

Since I do not impose any assumption on the players' belief operators, I introduce the common belief operator $C : \mathcal{D} \to \mathcal{D}$ following Fukuda (2020). Call an event E a *common basis* if $E \subseteq B_I(F) := \bigcap_{i\in I} B_i(F)$ for any $F \in \mathcal{D}$ with $E \subseteq F$. That is, everybody believes any implication of E whenever E is true. Denote by $\mathcal{J}_{B_I}$ the collection of common bases. Note that when the players' belief operators are assumed to satisfy Monotonicity, an event E is a common basis if and only if (hereafter, iff) it is publicly evident: $E \subseteq B_I(E)$ (see, e.g., Milgrom, 1981 for public evidence). An event E is *common belief* at a state ω if there is a common basis that is true at ω and that implies the mutual belief in E: that is, $\omega \in F \subseteq B_I(E)$ for some $F \in \mathcal{J}_{B_I}$. Now, C is assumed to satisfy the property that the set of states at which E is common belief is an event for each $E \in \mathcal{D}$:

$$C(E) := \{\omega \in \Omega \mid \text{there is } F \in \mathcal{J}_{B_I} \text{ with } \omega \in F \subseteq B_I(E)\}. \tag{1}$$

This definition reduces to that of Monderer and Samet (1989) when the players' belief operators satisfy Monotonicity. Since $\mathcal{D}$ is closed under countable intersection, if E is common belief, then everybody believes E, everybody believes that everybody believes E, and so forth *ad infinitum*: $C(E) \subseteq \bigcap_{n \in \mathbb{N}} B_I^n(E)$. The converse (set inclusion) holds, for example, when the mutual belief operator B_I satisfies, in addition to Monotonicity, *Countable Conjunction*: $\bigcap_{n \in \mathbb{N}} B_I(E_n) \subseteq B_I(\bigcap_{n \in \mathbb{N}} E_n)$, meaning that everybody believes the countable conjunction of events whenever everybody believes each of them.[7] Hence, if, for example, the mutual belief operator satisfies Monotonicity and Countable Conjunction, then C is always a well-defined operator without assuming it.

While a possibility correspondence model often allows any subset of Ω to be an event, I represent the players' beliefs on a measurable space $(\Omega, \mathcal{D})$ instead of the power set algebra $(\Omega, \mathcal{P}(\Omega))$ so that I can analyze players' qualitative and probabilistic beliefs (such as the possibility correspondence and type space models) under the same framework. I will analyze the players' (countably-additive) probabilistic beliefs on a measurable space $(\Omega, \mathcal{D})$ by *p-belief operators* (Monderer and Samet, 1989).[8] For each $p \in [0, 1]$, player i's p-belief operator $B_i^p : \mathcal{D} \to \mathcal{D}$ associates, with each event E, the event that player i believes E with probability at least p (she *p-believes* E). I will also introduce the common p-belief operator C^p. Samet (2000) specifies conditions on p-belief operators under which a player's beliefs are equivalently represented by a type mapping $\tau_i : \Omega \to \Delta(\Omega)$ that associates, with each state of the world, the player's probabilistic beliefs at that state, where $\Delta(\Omega)$ denotes the set of countably-additive probability measures on $(\Omega, \mathcal{D})$. This framework also enables one to analyze both qualitative and probabilistic beliefs at the same time (e.g., Fukuda, 2024a): for example, in an extensive-form game with perfect information, each player has knowledge about players' past moves while she has beliefs about the future moves of the opponents.

2.2 Properties of Beliefs

Next, I introduce additional eight properties of beliefs. Various possibility correspondence models of qualitative beliefs and knowledge are represented as belief models that satisfy certain properties specified below. Fix a player i. I first introduce the following five logical properties of beliefs.

1. *Necessitation*: $B_i(\Omega) = \Omega$.

[7] Also, B_I satisfies Countable Conjunction if every player's belief operator B_i satisfies it.

[8] While one can analyze finitely-additive or non-additive beliefs, for ease of exposition I focus on countably-additive probabilistic beliefs when it comes to quantitative beliefs.

2. *Countable Conjunction*: $\bigcap_{n\in\mathbb{N}} B_i(E_n) \subseteq B_i(\bigcap_{n\in\mathbb{N}} E_n)$ (for any events $(E_n)_{n\in\mathbb{N}}$).
3. *Finite Conjunction*: $B_i(E) \cap B_i(F) \subseteq B_i(E \cap F)$.
4. The *Kripke property*: $B_i(E) = \{\omega \in \Omega \mid b_{B_i}(\omega) \subseteq E\}$, where $b_{B_i}(\omega) := \bigcap\{E \in \mathcal{D} \mid \omega \in B_i(E)\}$ is the set of states player i considers *possible* at ω.
5. *Consistency*: $B_i(E) \subseteq (\neg B_i)(E^c)$.

First, Necessitation means that the player believes a tautology such as $E \cup E^c$ at any state. Second, as discussed, Countable Conjunction means that if the player believes each of a countable collection of events, then she believes its conjunction. In the probabilistic environment, if the player believes E_n with probability one for each $n \in \mathbb{N}$, then she believes the intersection $\bigcap_{n\in\mathbb{N}} E_n$ with probability one. Third, Finite Conjunction is weaker than Countable Conjunction: if the player believes E and F then she believes its conjunction $E \cap F$. Fourth, to discuss the Kripke property, the player considers ω' possible at ω if, for any event E which she believes at ω, E is true at ω'. The Kripke property provides the condition under which i's belief is induced by her *possibility correspondence* $b_{B_i} : \Omega \to \mathcal{P}(\Omega)$: she believes E at ω iff her possibility set $b_{B_i}(\omega)$ at ω implies (i.e., is included in) E.[9] The Kripke property implies the previous three properties as well as Monotonicity. Fifth, Consistency means that the player cannot simultaneously believe an event E and its negation E^c.

Next, I move on to truth and introspective properties.

6. *Truth Axiom*: $B_i(E) \subseteq E$ (for all $E \in \mathcal{D}$).
7. *Positive Introspection*: $B_i(\cdot) \subseteq B_i B_i(\cdot)$ (i.e., $B_i(E) \subseteq B_i B_i(E)$ for all $E \in \mathcal{D}$).
8. *Negative Introspection*: $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.

Sixth, Truth Axiom says that the player can only "know" what is true. Truth Axiom turns belief into knowledge in that knowledge has to be true while belief can be false. Truth Axiom implies Consistency. While knowledge satisfies Truth Axiom, qualitative and probabilistic beliefs are often assumed to satisfy Consistency. Seventh, Positive Introspection states that if the player believes some event then she believes that she believes it. Eighth, Negative Introspection states that if the player does not believe some event then she believes that she does not believe it. Truth Axiom and Negative Introspection yield Positive Introspection (e.g., Aumann, 1999).

Three remarks are in order. First, the introspective properties will play important roles in whether a player is (meta-)certain of a belief model. They provide the senses in which the player believes her own belief-generating process.

[9] In fact, B_i satisfies the Kripke property iff B_i is induced by some possibility correspondence $b_i : \Omega \to \mathcal{P}(\Omega)$: $B_i(E) = B_{b_i}(E) := \{\omega \in \Omega \mid b_i(\omega) \subseteq E\}$ (Fukuda, 2019). Under the Kripke property, $B_i = B_{b_{B_i}}$ and $b_i = b_{B_{b_i}}$, i.e., the belief operator B_i and the possibility correspondence are equivalent representations of beliefs.

To see these points formally, an event E is a *basis* to i if $E \subseteq B_i(F)$ whenever $E \subseteq F$. That is, player i believes any implication of E whenever E is true. Especially, an event E is *self-evident* to i when $E \subseteq B_i(E)$. That is, i believes E whenever E is true. If B_i satisfies Monotonicity, then E is self-evident to i iff it is a basis to i. Positive Introspection means that i's belief in E is self-evident to i, and Negative Introspection means that i's lack of belief in E is self-evident to i. Denote by $\mathcal{J}_{B_i}$ the collection of bases to i.

Second, the last four properties can be restated in terms of b_{B_i} under the Kripke property: B_i satisfies Consistency iff b_{B_i} is serial (i.e., $b_{B_i}(\cdot) \neq \emptyset$); B_i satisfies Truth Axiom iff b_{B_i} is reflexive (i.e., $\omega \in b_{B_i}(\omega)$ for all $\omega \in \Omega$); B_i satisfies Positive Introspection iff b_{B_i} is transitive (i.e., $\omega' \in b_{B_i}(\omega)$ implies $b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$); and B_i satisfies Negative Introspection iff b_{B_i} is Euclidean (i.e., $\omega' \in b_{B_i}(\omega)$ implies $b_{B_i}(\omega) \subseteq b_{B_i}(\omega')$).

Third, various models of probabilistic and qualitative beliefs and knowledge take different sets of axioms. The framework accommodates possibility correspondence models of qualitative beliefs and knowledge when B_i satisfies the Kripke property. A partitional model of knowledge corresponds to the case when B_i satisfies Truth Axiom, Positive Introspection, and Negative Introspection.[10] A reflexive and transitive (non-partitional) possibility correspondence model is characterized by Truth Axiom and Positive Introspection.[11] When it comes to fully-introspective qualitative beliefs (e.g., Bonanno and Tsakas, 2018), b_{B_i} is serial, transitive, and Euclidean iff B_i satisfies Consistency, Positive Introspection, and Negative Introspection.

The probability 1-belief operator B_i^1 (that maps each event E to the event that player i believes E with probability one) satisfies Necessitation, Countable Conjunction (thus Finite Conjunction), and Consistency. The next section shows that B_i^1 satisfies Positive Introspection and Negative Introspection if the player is certain of her type mapping. With the framework defined in this section, for a model of (qualitative or probabilistic) belief or knowledge, I will study the formal sense in which the players are certain of the model.

3 When Is a Player Certain of Her Belief-Generating Mapping?

The previous section has defined a belief model, in which the objects of beliefs are events. Here, Section 3.1 first extends an object of beliefs in a model from an event to a function ("signal") defined on the state space. That is, the subsection formu-

[10] In fact, Truth Axiom, Negative Introspection, and the Kripke property yield all the other properties defined in this section.

[11] The literature on non-partitional possibility correspondence models studies information processing errors that lead to the failure of Negative Introspection. See, for example, Bacharach (1985), Binmore and Brandenburger (1990), Dekel and Gul (1997), Geanakoplos (2021), Lipman (1995), Pires (2021), Samet (1990), and Shin (1993).

lates the statement that a player is certain of a function defined on the state space. Next, Section 3.2 represents a player's "belief-generating mapping" as a signal which associates, with each state, whether she believes each event or not. Then, Section 3.3 asks the sense in which she is certain of her own belief-generating mapping in terms of the introspective properties. Finally, Section 3.4 relates the introspective properties to a notion of "informativeness" derived from a signal. Section 4, based on this formalization, provides a test by which the outside analysts can determine whether the players are (commonly) certain of a belief model.

3.1 Functions as Objects of Players' Beliefs

I start with defining a notion of a signal mapping. A signal mapping is a function from an underlying state space Ω into a space of "observation" X endowed with "observational" contents $\mathcal{X} \subseteq \mathcal{P}(X)$. By observation, it means that each $F \in \mathcal{X}$ is deemed an object of reasoning. That is, we call a mapping $x : \Omega \to X$ a signal mapping if each "observational" content $F \in \mathcal{X}$ can be regarded as an event $x^{-1}(F) \in \mathcal{D}$ through inverting the mapping.

Formally, for a non-empty set X and a non-empty subset $\mathcal{X}$ of $\mathcal{P}(X)$, call $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ a *signal* (mapping) if $x^{-1}(\mathcal{X}) \subseteq \mathcal{D}$. Mathematically, $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ is a signal if $x : (\Omega, \mathcal{D}) \to (X, \sigma(\mathcal{X}))$ is measurable, where $\sigma(\mathcal{X})$ is the σ-algebra generated from $\mathcal{X}$. Examples include strategies, action/decision functions, random variables, state-contingent contracts, conditional expectations, and so on.

The main purpose of this subsection is to define the statement that a player is certain of a signal. A player i is *certain of the value of* a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω, if she believes any implication of any observational content F at ω whenever it is true: if $F \in \mathcal{X}$ satisfies $x(\omega) \in F$ (i.e., an observational content F is true at ω) and if $E \in \mathcal{D}$ satisfies $x^{-1}(F) \subseteq E$ (i.e., the observational content implies E), then $\omega \in B_i(E)$ (i.e., player i believes E). She is *certain of* the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ if she is certain of its value at every ω.

Likewise, the players are *commonly certain of the value of* the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω, if the players commonly believe any implication of any observational content F at ω whenever it is true: if $F \in \mathcal{X}$ satisfies $x(\omega) \in F$ and if $E \in \mathcal{D}$ satisfies $x^{-1}(F) \subseteq E$, then $\omega \in C(E)$. The players are *commonly certain of* the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ if they are certain of its value at every ω. Note that the word "certainty" is not necessarily related to probability-one belief. This terminology is used generically to refer to various probabilistic or non-probabilistic belief and knowledge (recall footnote 1). Formally:

Definition 1. Let $\overrightarrow{\Omega}$ be a belief model, and let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ be a signal mapping.

1. (a) Player i is *certain of the value of the signal* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω if the following holds: if there exist $F \in \mathcal{X}$ with $x(\omega) \in F$ and $E \in \mathcal{D}$ with

$x^{-1}(F) \subseteq E$, then $\omega \in B_i(E)$.

   (b) Player i is *certain of the signal* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ if she is certain of the value of the signal x at any state.

2. (a) The players are *commonly certain of the value of the signal* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω if the following holds: if there exist $F \in \mathcal{X}$ with $x(\omega) \in F$ and $E \in \mathcal{D}$ with $x^{-1}(F) \subseteq E$, then $\omega \in C(E)$.

   (b) The players *are commonly certain of* the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ if they are commonly certain of the value of the signal x at every state.

For ease of terminology, player i is *certain of (the value of) the signal* $x : \Omega \to X$ (at ω) *with respect to* $\mathcal{X}$ if she is certain of (the value of) the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ (at ω). Likewise, the players are *commonly certain of (the value of) the signal* $x : \Omega \to X$ (at ω) *with respect to* $\mathcal{X}$ if they are commonly certain of (the value of) the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ (at ω).

Also, with some abuse of terminology, I often omit the mention to the "value" of a signal at a state when it is clear from the context. That is, I often state that player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω instead of stating that player i is certain of the value of the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω. The same applies to the common certainty.

In Definition 1, whether player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ depends on $\mathcal{X}$. The larger $\mathcal{X}$ is (in the sense of set inclusion), the harder it is to satisfy the definition.

When the players' beliefs are monotone, Definition 1 reduces to a simple form. Player i is certain of a signal mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω iff she believes any observational content $x^{-1}(F)$ at ω whenever the observational content F is true at ω. Likewise, the players are commonly certain of x at ω iff they commonly believe any observational content $x^{-1}(F)$ at ω whenever F is true at ω. Formally:

Remark 1. Let $\overrightarrow{\Omega}$ be a belief model in which the players' belief operators satisfy Monotonicity, and let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ be a signal mapping.

1. Player i is certain of the value of the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω iff $\omega \in B_i(x^{-1}(F))$ for any $F \in \mathcal{X}$ with $x(\omega) \in F$.

2. The players are commonly certain of the value of the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω iff $\omega \in C(x^{-1}(F))$ for any $F \in \mathcal{X}$ with $x(\omega) \in F$.

With this in mind, we consider an example in which the players' beliefs are monotone. Suppose that $x : \Omega \to X$ is a decision function of a player which associates, with each state, the action taken at that state. Suppose that the set of actions X is endowed with the collection of singleton actions $\mathcal{X} = \{\{a\} \mid a \in X\}$. Each action a corresponds to an observational content to the player, and x is a signal mapping if the set of states at which the player takes action a is an event, $x^{-1}(\{a\}) = \{\omega \in \Omega \mid x(\omega) = a\} \in \mathcal{D}$, for each $a \in X$. More specifically, I consider the following setup:

Example 1. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $X = \{a, b\}$. For each $i \in I = \{1, 2\}$, let B_i be given by (i) $B_i(E) = E \setminus \{\omega_3\}$ for each $E \neq \Omega$; and (ii) $B_i(\Omega) = \Omega$. Suppose player 1's decision function $x_1 : (\Omega, \mathcal{P}(\Omega)) \to (X, \{\{a\}, \{b\}\})$ is given by $(x_1(\omega))_{\omega \in \Omega} = (a, a, a)$. Since $B_1(\Omega) = \Omega$ and $B_1(\emptyset) = \emptyset$, whenever player 1 takes a certain action, she believes that she takes that action. Thus, player 1 is certain of x_1. In contrast, suppose that player 2's decision function $x_2 : (\Omega, \mathcal{P}(\Omega)) \to (X, \{\{a\}, \{b\}\})$ is given by $(x_2(\omega))_{\omega \in \Omega} = (a, b, a)$. At ω_3 at which she takes action a, she does not believe that she takes action a, because $B_2(\{\omega_1, \omega_3\}) = \{\omega_1\}$. Thus, player 2 is not certain of the value of x_2 at ω_3. Since $B_i(\cdot) \in \mathcal{J}_{B_I}$, it follows that $C = B_i$. Thus, while the players are commonly certain of x_1, they are not commonly certain of x_2.

Next, I consider the information structure of Rubinstein (1989)'s well-known e-mail game. The example below shows that the idea that one player is certain of which game is played is fit into the framework of this paper.

Example 2. There are two players $I = \{1, 2\}$. With probability $p \in (0, 1)$, they face a game G_b. With probability $1 - p$, they face a game G_a. Initially, only player 1 is informed of which the true game is. The players are restricted to communicating through the following e-mail protocol. If the true game is G_b then player 1's computer automatically sends a message to player 2's computer. If the game is G_a then no message is sent. If a computer receives a message then it automatically sends a confirmation message, including a confirmation of a confirmation, and so on. There is a fixed probability $\varepsilon > 0$ that any given message is lost. If a message does not arrive then the communication ends.

Let the state space be $\Omega = \{0, 1, 2, \dots\}$, and let $\mathcal{D} = \mathcal{P}(\Omega)$. If the state is $n > 0$, then n messages have been sent and $n - 1$ of them received.

For player 1, let her belief operator B_1 be induced by the possibility correspondence b_1 which is given by

$$b_1(\omega) = \begin{cases} \{0\} & \text{if } \omega = 0 \\ \{2n-1, 2n\} & \text{if } \omega = 2n-1 \text{ for some } n \in \mathbb{N} \\ \{2n-1, 2n\} & \text{if } \omega = 2n \text{ for some } n \in \mathbb{N} \end{cases}.$$

For player 2, let her belief operator B_2 be induced by the possibility correspondence b_2 which is given by

$$b_2(\omega) = \begin{cases} \{0, 1\} & \text{if } \omega = 0 \\ \{2n-2, 2n-1\} & \text{if } \omega = 2n-1 \text{ for some } n \in \mathbb{N} \\ \{2n, 2n+1\} & \text{if } \omega = 2n \text{ for some } n \in \mathbb{N} \end{cases}.$$

Define a signal mapping $G : (\Omega, \mathcal{D}) \to (\{G_a, G_b\}, \{\{G_a\}, \{G_b\}\})$ that determines which game the true game is:

$$G(\omega) = \begin{cases} G_a & \text{if } \omega = 0 \\ G_b & \text{if } \omega \neq 0 \end{cases}.$$

Thus, $G^{-1}(\{G_a\}) = \{0\}$ and $G^{-1}(\{G_b\}) = \Omega \setminus \{0\}$.

Now, I show that player 1 is certain of $G : (\Omega, \mathcal{D}) \to (\{G_a, G_b\}, \{\{G_a\}, \{G_b\}\})$. For any E with $\{0\} \subseteq E$, $0 \in B_1(\{0\}) \subseteq B_1(E)$. For any E with $\Omega \setminus \{0\} \subseteq E$ and for any $\omega \in E$, $\omega \in B_1(\Omega \setminus \{0\}) \subseteq B_1(E)$. In contrast, I show that player 2 is not certain of $G : (\Omega, \mathcal{D}) \to (\{G_a, G_b\}, \{\{G_a\}, \{G_b\}\})$ (at $\omega = 0$). Take $E = \{0\}$. Then, $0 \notin \emptyset = B_2(E)$. Consequently, the players are not commonly certain of $G : (\Omega, \mathcal{D}) \to (\{G_a, G_b\}, \{\{G_a\}, \{G_b\}\})$ (at $\omega = 0$).

For the rest of this subsection, six remarks on Definition 1 are in order. First, Remark 2 below restates the fact that a player is certain of a signal in terms of bases (recall Section 2.2 for the definition of bases).

Remark 2. Let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ be a signal.

1. Player i is certain of the signal x iff $x^{-1}(\mathcal{X}) \subseteq \mathcal{J}_{B_i}$.
2. The players are commonly certain of the signal x iff $x^{-1}(\mathcal{X}) \subseteq \mathcal{J}_{B_I}$.
3. The players are commonly certain of the signal x iff every player i is certain of the signal x.

Part (1) states that player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff any observational content $F \in \mathcal{X}$ (i.e., any event $x^{-1}(F) \in \mathcal{D}$) is a basis to i. Part (2) states that the players are commonly certain of the signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff any observational content $F \in \mathcal{X}$ is a common basis. Consequently, Part (3) says that the players are commonly certain of a signal at every state iff every player is certain of it at every state. Hence, for the outside analysts to assert that the players are commonly certain of some signal, it suffices to show that each player is certain of it.[12] Remark 2 follows from straightforwardly from Definition 1 and the definitions of bases and common bases.

Second, Remark 3 below shows that, when $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ is a player's strategy, Definition 1 formalizes the statement that the player is certain of the strategy in the literature on characterizations of solution concepts of games in state space models such as Brandenburger, Dekel, and Geanakoplos (1992) and Geanakoplos (2021). To see this, assume that $\mathcal{X}$ contains a singleton $\{x(\omega)\}$ to reason about the action taken at ω for all $\omega \in \Omega$. That is, the set of states $[x(\omega)] := x^{-1}(\{x(\omega)\}) = \{\omega' \in \Omega \mid x(\omega') = x(\omega)\}$ at which player i takes the same action as she does at ω is an event. Also, assume that B_i satisfies Monotonicity. Then, player i is certain of her action $x(\omega)$ (i.e., the value of the signal x) at ω iff $\omega \in B_i([x(\omega)])$, that is, player i believes that her action is $x(\omega)$ at ω. More generally:

[12] In contrast, Appendix A.2 shows that it is not necessarily the case that each player is certain of a signal at a state ω iff the players are commonly certain of the signal at ω. In light of Example 2, at state $n > 0$, the players are mutually certain of the value of $G : (\Omega, \mathcal{D}) \to (\{G_a, G_b\}, \{\{G_a\}, \{G_b\}\})$, they are mutually certain that they are mutually certain of the value of G, and so on $n - 1$ times but not n times.

Remark 3. Let X be a set of actions available to player i, and let $x : \Omega \to X$ be a strategy of player i with respect to $\mathcal{X} = \{\{x(\omega)\} \mid \omega \in \Omega\}$: the set of actions that could have been taken at each state. Then, Definition 1 states that player i is certain of her strategy x iff $[x(\omega)]$ is a basis to i at every $\omega \in \Omega$.

The proof of Remark 3 is similar to that of Remark 2 (1), and thus it is omitted.

Definition 1 also subsumes the formulation of the certainty of the strategy by Aumann (1987a) in the (countable) partitional state space model of knowledge. Let $\{b_{B_i}(\omega)\}_{\omega\in\Omega}$ be a countable partition on Ω. In Aumann (1987a), the player "knows" her own strategy x iff the strategy x is measurable with respect to the partition (which turns out to be equivalent to $b_{B_i}(\cdot) \subseteq [x(\cdot)]$). Since the partition is countable, the σ-algebra generated by the partition turns out to be equal to the collection of bases: $\mathcal{J}_{B_i} = \sigma(\{b_{B_i}(\omega) \in \mathcal{D} \mid \omega \in \Omega\})$. Hence, player i is certain of her strategy $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff $x : (\Omega, \mathcal{J}_{B_i}) \to (X, \sigma(\mathcal{X}))$ is measurable.

Third, Remark 4 below states that if player i is certain of a signal $x : \Omega \to X$ with respect to the collection of singletons $\{\{a\} \mid a \in X\}$, then she is certain of x, in the strongest sense, with respect to $\mathcal{P}(X)$.

Remark 4. Player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{P}(X))$ iff she is certain of $x : \Omega \to X$ with respect to the collection of values of x: $\{\{x(\omega)\} \mid \omega \in \Omega\}$. Thus, player i is certain of her strategy $x : (\Omega, \mathcal{D}) \to (X, \mathcal{P}(X))$ in the strongest sense iff she is certain of her strategy with respect to the actions $\{\{x(\omega)\} \mid \omega \in \Omega\}$ that she could have taken.

To see this, it can be shown that player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff she is certain of $x : \Omega \to X$ with respect to the collection of unions of $\mathcal{X}$: $\{F \in \mathcal{P}(X) \mid F = \bigcup_{\lambda\in\Lambda} F_\lambda \text{ for some } \{F_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{X}\}$. Remark 4 follows from this observation.

Fourth, Remark 5 provides conditions on beliefs under which a player is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff she is certain of $x : (\Omega, \mathcal{D}) \to (X, \sigma(\mathcal{X}))$ when x is her own "belief-generating map" in Section 3.3.

Remark 5. Under the following conditions on player i's belief operator B_i, the collection of i's beliefs $\mathcal{B}_i := \{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ forms a sub-σ-algebra of $\mathcal{D}$.

1. B_i satisfies Monotonicity, Consistency, Countable Conjunction, Positive Introspection, and Negative Introspection.

2. B_i satisfies Monotonicity, Truth Axiom and Negative Introspection.

For Part (2), one can show that $\mathcal{B}_i$ coincides with the collection of bases $\mathcal{J}_{B_i}$, which forms a sub-σ-algebra of $\mathcal{D}$. In fact, it can be seen that the conditions in (2) imply those in (1). Fully-introspective qualitative or probability-one beliefs satisfy the conditions in Part (1), and fully-introspective knowledge satisfies the conditions in Part (2).

Fifth, Remark 6 below shows that B_i satisfies Necessitation iff player i is certain of any constant signal. Likewise, the common belief operator C satisfies Necessitation (equivalently, every B_i satisfies Necessitation) iff the players are commonly certain of any constant signal.

Remark 6. 1. Player i's belief operator B_i satisfies Necessitation iff she is certain of any constant signal.

2. The common belief operator C satisfies Necessitation iff the players are commonly certain of any constant signal iff she is certain of any constant signal.

The proof of Remark 6 is in Appendix A.1.1.

In light of the certainty of a signal, Necessitation allows the players to be certain of any constant "random variable" that does not depend on the realization of a state. For example, consider whether player i is certain that an event $B_j(E)$ is equal to an event F in a belief model. The outside analysts determine whether player i believes that player j believes an event E at a state ω by examining whether $\omega \in B_i B_j(E)$ since player j's belief $B_j(E)$ itself is an event. The (implicit) assumption in any (semantic) belief model is that $E = F$ implies $B_i(E) = B_i(F)$. Thus, if two events are extensionally the same (e.g., E is the set of 1 and -1, and F is the set of real solutions of $x^2 = 1$) then each player's belief in the two events are the same.[13] To assess player i's belief about player j's belief about E, how can the outside analysts justify the fact that player i is able to equate $B_j(E)$ with another event (say, F)? Since either $B_j(E) = F$ or $B_j(E) \neq F$, player i is *certain that* $B_j(E)$ *is an event* F if player i is certain of the indicator function $\mathbb{I}_{B_j \leftrightarrow F}$, where $(B_j(E) \leftrightarrow F) := ((\neg B_j)(E) \cup F) \cap ((\neg F) \cup B_j(E))$. If player i's belief operator B_i satisfies Necessitation, then player i is certain of a constant function $\mathbb{I}_{B_j \leftrightarrow F}$. Thus, under Necessitation, player i is certain that $B_j(E) = F$ if it is indeed the case. This argument justifies that, under Necessitation, the outside analysts can say that the players are certain of equating two extensionally equivalent events (say, $B_j(E)$ and F) if they are indeed extensionally equivalent.

Necessitation also follows if player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ where $X = \bigcup_{F \in \mathcal{X}} F$. Thus, for example, if player i is certain of her strategy $x : (\Omega, \mathcal{D}) \to (X, \{\{a\} \mid a \in X\})$, then B_i satisfies Necessitation.

Sixth, Remark 7 below shows that player i is certain of a profile of signals (e.g., a strategy profile) iff she is certain of each of them.

Remark 7. Let A be a non-empty set, and let $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$ be a signal for each $\alpha \in A$. Let $X := \prod_{\alpha \in A} X_\alpha$, and let $\pi_\alpha : X \to X_\alpha$ be the projection. By

[13] Although such identification of events are implicitly assumed for any (semantic) belief model, one can construct a canonical ("universal") semantic model from a syntactic language which maximally distinguishes the denotations of events. In the canonical model, such identification of events can be minimized in a way such that two events are equated only when they are explicitly assumed to be equivalent by the outside analysts (see Fukuda (2024b) for a formal assertion).

definition, every $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$ is a signal iff $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ is a signal, where $\mathcal{X} := \bigcup_{\alpha \in A} \{(\pi_\alpha)^{-1}(F_\alpha) \in \mathcal{P}(X) \mid F_\alpha \in \mathcal{X}_\alpha\}$. Then, player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff she is certain of every $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$.

The proof of Remark 7 is in Appendix A.1.1. Under either condition in Remark 5, player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \sigma(\mathcal{X}))$ iff she is certain of every $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$. Observe that $\sigma(\mathcal{X})$ is the product σ-algebra if each $\mathcal{X}_\alpha$ is a σ-algebra.

Remarks 2 and 7 imply that the players are commonly certain of a profile of signals iff every player is certain of every signal.

3.2 A Qualitative-Type Mapping that Represents a Player's Beliefs

In order to formulate a test under which the outside analysts can examine whether the players are commonly certain of a belief model, I define the "belief-generating map," which I call the qualitative-type mapping, of a player. Given the belief operator of the player, the qualitative-type mapping associates, with each state, a set function that maps each event to a binary value indicating whether the player believes the event in an analogous manner to the type mapping in the type-space literature.[14]

To that end, recall the notion of probabilistic types. A (probabilistic-)type is a σ-additive probability measure $\nu \in \Delta(\Omega)$. A (probabilistic-)type mapping is a measurable mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, where $\mathcal{D}_\Delta$ is the σ-algebra generated by $\beta^p_E := \{\nu \in \Delta(\Omega) \mid \nu(E) \geq p\}$ for all $(E, p) \in \mathcal{D} \times [0, 1]$ (Heifetz and Samet, 1998). It associates, with each state ω, the player's probabilistic beliefs $\tau_i(\omega) \in \Delta(\Omega)$ at that state. Given the type mapping τ_i, define player i's p-belief operator $B^p_{\tau_i} : \mathcal{D} \to \mathcal{D}$ as $B^p_{\tau_i}(E) := \tau_i^{-1}(\beta^p_E)$. Thus, $\omega \in B^p_{\tau_i}(E)$ iff $\tau_i(\omega)(E) \geq p$. As in Samet (2000), the type mapping τ_i and the collection of p-belief operators $(B^p_{\tau_i})_{p \in [0,1]}$ are equivalent, that is, a type space of the form $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I} \rangle$ is equivalent to $\langle (\Omega, \mathcal{D}), (B^p_{\tau_i})_{(i,p) \in I \times [0,1]} \rangle$.

With this in mind, let $M(\Omega)$ be the set of binary set functions $\mu : \mathcal{D} \to \{0, 1\}$ (i.e., $M(\Omega) \subseteq \{0, 1\}^{\mathcal{D}}$) that satisfy a given set of logical properties of beliefs defined in Section 2.2 (these properties will be shortly expressed in terms of μ). Call each $\mu \in M(\Omega)$ a *qualitative-type*. Interpret $\mu(E) = 1$ as the belief in an event $E \in \mathcal{D}$. Once $M(\Omega) \subseteq \{0, 1\}^{\mathcal{D}}$ is defined as the set of qualitative-types that satisfy the given set of logical properties of beliefs, I represent player i's beliefs by a *qualitative-type mapping* $t_i : \Omega \to M(\Omega)$ satisfying a certain measurability condition specified below. It is a measurable mapping which associates, with each state $\omega \in \Omega$, player i's qualitative-type $t_i(\omega) \in M(\Omega)$ at ω. Thus, player i believes an event E at ω if $t_i(\omega)(E) = 1$.

Now, I define the logical properties of μ in an analogous way to the corresponding logical properties of belief operators. Fix $\mu \in \{0, 1\}^{\mathcal{D}}$.

0. *Monotonicity*: $E \subseteq F$ implies $\mu(E) \leq \mu(F)$.

[14] Fukuda (2017, Section 6) constructs a universal knowledge space when each player's knowledge is represented by her qualitative-type mapping.

1. *Necessitation*: $\mu(\Omega) = 1$.
2. *Countable Conjunction*: $\min_{n \in \mathbb{N}} \mu(E_n) \leq \mu(\bigcap_{n \in \mathbb{N}} E_n)$.
3. *Finite Conjunction*: $\min(\mu(E), \mu(F)) \leq \mu(E \cap F)$.
4. The *Kripke property*: $\mu(E) = 1$ iff $\bigcap\{F \in \mathcal{D} \mid \mu(F) = 1\} \subseteq E$.
5. *Consistency*: $\mu(E) \leq 1 - \mu(E^c)$.

The interpretations of the above properties are similar to those in Section 2.2. For Countable Conjunction, if E_n is believed (i.e., $\mu(E_n) = 1$) for every $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} E_n$ is believed (i.e., $\mu(\bigcap_{n \in \mathbb{N}} E_n) = 1$). The Kripke property characterizes the condition under which player i's beliefs are induced by a possibility correspondence when every qualitative-type $t_i(\omega)$ satisfies it. Whether all of these properties are assumed or not depend on the model that the outside analysts study. For example, if the outside analysts examine a partitional possibility correspondence, then since the corresponding belief operator satisfies all the logical properties, $M(\Omega)$ is the set of qualitative-types that satisfy all the logical properties. In contrast, if the outside analysts study a belief model in which only Monotonicity is assumed, then $M(\Omega)$ is the set of qualitative-types that satisfy Monotonicity.

I formally define the measurability condition of a qualitative-type mapping. A *qualitative-type mapping* is a measurable mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ which satisfies given (logical and) introspective properties of beliefs, where $\mathcal{D}_M$ is the σ-algebra generated by the sets of the form $\beta_E := \{\mu \in M(\Omega) \mid \mu(E) = 1\}$ for all $E \in \mathcal{D}$. The set β_E is the set of types under which E is believed. Thus, β_E is an informational content indicating that event E is believed. Note that $t_i : \Omega \to M(\Omega)$, by construction, satisfies given logical properties because any element in $M(\Omega)$ satisfies them. For example, if every $\mu \in M(\Omega)$ satisfies the Kripke property, then every $t_i(\omega)$ satisfies it. Denote $b_{t_i}(\omega) := \bigcap\{E \in \mathcal{D} \mid t_i(\omega)(E) = 1\}$ for each $\omega \in \Omega$.

The measurablity condition of t_i requires each $t_i^{-1}(\beta_E) = \{\omega \in \Omega \mid t_i(\omega)(E) = 1\}$ to be the event that player i believes E. Next, I define Truth Axiom and the introspective properties of t_i.

6. *Truth Axiom*: $t_i(\omega)(E) = 1$ implies $\omega \in E$.
7. *Positive Introspection*: $t_i(\omega)(E) = 1$ implies $t_i(\omega)(\{\omega' \in \Omega \mid t_i(\omega')(E) = 1\}) = 1$ (i.e., $t_i(\omega)(t_i^{-1}(\beta_E)) = 1$).
8. *Negative Introspection*: $t_i(\omega)(E) = 0$ implies $t_i(\omega)(\{\omega' \in \Omega \mid t_i(\omega')(E) = 0\}) = 1$ (i.e., $t_i(\omega)(\neg t_i^{-1}(\beta_E)) = 1$).

So far, a qualitative-type mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ is introduced. Finally, I demonstrate that a belief operator and a qualitative-type mapping are equivalent. A given belief operator B_i induces the qualitative-type mapping t_{B_i} by

$$t_{B_i}(\omega)(E) := \begin{cases} 1 & \text{if } \omega \in B_i(E) \\ 0 & \text{otherwise} \end{cases}.$$

Conversely, a given qualitative-type mapping t_i induces the belief operator B_{t_i} defined as $B_{t_i}(E) := t_i^{-1}(\beta_E)$. It can be seen that $B_{t_{B_i}} = B_i$ and $t_i = t_{B_{t_i}}$.

3.3 Certainty of Own Type Mapping

I apply the certainty of a signal to a qualitative- and probabilistic-type mappings. The main results of this subsection are Propositions 1A and 1B. Roughly, they state that a player is certain of her own qualitative- and probabilistic-type mappings iff her qualitative and probabilistic beliefs are introspective, respectively. Hereafter, "A" and "B" in Proposition, Theorem, and Remark refer to the case with qualitative and probabilistic beliefs, respectively.

3.3.1 Certainty of Own Qualitative-Type Mapping

I start with the certainty of a qualitative-type mapping.

Proposition 1A. *Let $\overrightarrow{\Omega}$ be a belief model in which B_i satisfies Monotonicity, and let $t_{B_i} : \Omega \to M(\Omega)$ be player i's qualitative-type mapping.*

1. *(a) Player i is certain of t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ iff B_i satisfies Positive Introspection: $B_i(\cdot) \subseteq B_i B_i(\cdot)$.*

   (b) Player i is certain of t_{B_i} with respect to $\{\neg\beta_E \mid E \in \mathcal{D}\}$ iff B_i satisfies Negative Introspection: $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$.

   (c) If player i is certain of $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, then B_i satisfies Positive Introspection and Negative Introspection.

2. *(a) Let B_i satisfy Truth Axiom. Player i is certain of $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff B_i satisfies (Positive Introspection and) Negative Introspection.*

   (b) Let B_i satisfy Consistency and Countable Conjunction. Player i is certain of $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff B_i satisfies Positive Introspection and Negative Introspection.

While Part (1) characterizes the certainty of the qualitative-type mapping t_{B_i} with respect to the possession or lack of beliefs, Part (2) examines the sense in which player i is certain of her qualitative-type mapping $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ in a model of knowledge (i.e., Part (2a)) and belief (i.e., Part (2b)).

Part (1a) states that player i is certain of her qualitative-type mapping t_{B_i} with respect to the possession of beliefs iff her belief operator B_i satisfies Positive Introspection. Parts (1a) and (1b) jointly state that B_i satisfies Positive Introspection and

Negative Introspection iff player i is certain of her qualitative-type mapping t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\} \cup \{\neg\beta_E \mid E \in \mathcal{D}\}$.

Without Monotonicity, Part (1) is stated as follows: (i) player i is certain of t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ iff player i believes any logical implication of her own beliefs: for any $E, F \in \mathcal{D}$, $B_i(E) \subseteq F$ implies $B_i(E) \subseteq B_i(F)$; and (ii) player i is certain of t_{B_i} with respect to $\{\neg\beta_E \mid E \in \mathcal{D}\}$ iff she believes any logical implication of her own dis-beliefs: for any $E, F \in \mathcal{D}$, $(\neg B_i)(E) \subseteq F$ implies $(\neg B_i)(E) \subseteq B_i(F)$.

I discuss three additional implications of Proposition 1A. First, the proposition sheds light on the literature of non-partitional knowledge models in which Negative Introspection fails. The question is, when a player commits an information-processing error leading to the failure of Negative Introspection, is she certain of her own possibility correspondence?[15] The dichotomous answer leads to the following issue. If the player is certain of her own possibility correspondence, then she is "certain" that she commits the information-processing error and yet she fails to overcome the lack of Negative Introspection. If she is not certain of her own possibility correspondence, where do her beliefs come from?

Part (1) provides the following eclectic answer: player i is not fully certain of her qualitative-type mapping. That is, without imposing Negative Introspection, player i is not certain of her own qualitative-type mapping with respect to $\mathcal{D}_M$ (or $\{\beta_E \mid E \in \mathcal{D}\} \cup \{\neg\beta_E \mid E \in \mathcal{D}\}$). Rather, she takes her own information at face value in the sense that she is only certain of her qualitative-type mapping with respect to her own beliefs $\{\beta_E \mid E \in \mathcal{D}\}$. Proposition 1A formalizes the very sense in which "she takes her own information at face value."

In contrast, Proposition 1A (2a) shows that, in a partitional possibility correspondence model of knowledge, the axioms of Truth Axiom, (Positive Introspection) and Negative Introspection characterize the sense in which a player is fully certain of her possibility correspondence. While the proposition does not necessarily require B_i to satisfy the Kripke property, consider a model of knowledge in which B_i satisfies Truth Axiom and the Kripke property, i.e., B_i is induced by the reflexive possibility correspondence b_{B_i}. Then, player i is certain of her "knowledge-generating" mapping iff B_i satisfies (Positive Introspection and) Negative Introspection.

Likewise, Proposition 1A (2b) demonstrates that, in a serial possibility correspondence model, the axioms of Consistency, Positive Introspection and Negative Introspection characterize the sense in which a player is fully certain of her possibility correspondence (note that the Kripke property implies Countable Conjunction).

Second, Proposition 1A also sheds light on the identification of events discussed in Section 3.1. The belief operator B_i of player i associates, with each event E, the event $B_i(E)$ that she believes E. Since the players' beliefs themselves are events, player i can reason about player j's belief in E: $B_iB_j(E)$. However, the question is how does player i evaluate another player j's belief in E? The implicit assumption is

[15] See, for instance, Brandenburger, Dekel, and Geanakoplos (1992), Dekel and Gul (1997), and Geanakoplos (2021)

again that "the belief model is commonly certain among the players."

In Proposition 1A, Positive Introspection and Negative Introspection pertain to every event E including $E = B_j(F)$ for some $F \in \mathcal{D}$. This means that if player i is certain of her qualitative-type mapping t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ then she is also certain of such qualitative-type mapping as $t_{B_i B_j}$ with respect to $\{\beta_E \mid E \in \mathcal{D}\}$, where $t_{B_i B_j}$ is the qualitative-type mapping associated with the operator $B_i B_j$ (i.e., $t_{B_i B_j}(\omega)(E) = 1$ iff $\omega \in B_i B_j(E)$). That is, if player i is certain of her own belief-generating mapping, then she is also certain of the mapping that generates i's belief about j's belief about events. Although it is implicitly assumed in the belief model that player i figures out what B_j is,[16] the fact that each i is certain of $t_{B_i B_j}$ could possibly be a justification for why the outside analysts can assume that "i is certain of j's belief operator in i's mind." Section 4 studies the question whether the outside analysts can assume that each player i is certain of each other's qualitative-type mapping t_{B_j}.[17]

Third, I can extend Proposition 1A to the case where a player has qualitative belief and knowledge. Consider a belief model $\langle(\Omega, \mathcal{D}), (K_i)_{i \in I}\rangle$ where $K_i : \mathcal{D} \to \mathcal{D}$ is player i's monotone knowledge operator (for simplicity, I omit the common-knowledge operator). Now, for each player i, let $B_i : \mathcal{D} \to \mathcal{D}$ be her monotone qualitative-belief operator. Let t_{B_i} be player i's qualitative-type mapping that represents B_i, and ask whether player i is certain of her qualitative-type mapping $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$. Proposition 1A (2a) implies that player i is certain of $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff K_i satisfies *Positive Certainty* (with respect to B_i): $B_i(\cdot) \subseteq K_i B_i(\cdot)$ and *Negative Certainty* (with respect to B_i): $(\neg B_i)(\cdot) \subseteq K_i(\neg B_i)(\cdot)$. Whenever player i believes an event, she knows that she believes it. Whenever player i does not believe an event, she knows that she does not believe it. In fact, these two properties are often assumed in a model of belief and knowledge, and Proposition 1A (2a) justifies the assumptions in terms of the certainty of one's knowledge about her own beliefs.

3.3.2 Certainty of Own Probabilistic-Type Mapping

Next, I study when a player is certain of her own (probabilistic-)type mapping. Thus, I add to a belief model (in which $(B_i)_{i \in I}$ is a primitive) each player i's (probabilistic-)type mapping τ_i, which induces her p-belief operator $B^p_{\tau_i}$. Especially, the case with $B_i = B^1_{\tau_i}$ studies whether player i is certain of her type mapping within the type space $\langle(\Omega, \mathcal{D}), (\tau_i)_{i \in I}\rangle$ itself. In the case in which B_i is either a knowledge or qualitative belief operator, the outside analysts consider players' knowledge or qualitative beliefs about their probabilistic beliefs.

[16] Recalling footnote 13, this pertains to the assumption in any (semantic) belief model that if $E = F$ then $B_i(E) = B_i(F)$.

[17] Note that here I ask whether each player i can be certain of how she herself can evaluate an opponent's belief-generating process through studying whether player i is certain of the mapping $t_{B_i B_j}$ that generates the beliefs of player i about player j's beliefs.

As in the previous discussion, a belief operator B_i satisfies *Positive Certainty* (with respect to $B^p_{\tau_i}$) if $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$. Likewise, B_i satisfies *Negative Certainty* (with respect to $B^p_{\tau_i}$) if $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$. With these in mind:

Proposition 1B. *Let $\overrightarrow{\Omega}$ be a belief model in which B_i satisfies Monotonicity, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. *(a) Player i is certain of her type mapping τ_i with respect to $\{\beta^p_E \mid (p, E) \in [0, 1] \times \mathcal{D}\}$ iff B_i satisfies Positive Certainty: $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$.*

   (b) Player i is certain of her type mapping τ_i with respect to $\{\neg\beta^p_E \mid (p, E) \in [0, 1] \times \mathcal{D}\}$ iff B_i satisfies Negative Certainty: $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$.

   (c) If player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then B_i satisfies Positive Certainty and Negative Certainty.

2. *(a) Let B_i satisfy Truth Axiom and Negative Introspection. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty iff B_i satisfies Negative Certainty.*

   (b) Let B_i satisfy Consistency, Countable Conjunction, Positive Introspection, and Negative Introspection. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty and Negative Certainty.

   (c) Let B_i satisfy Entailment: $B_i(\cdot) \subseteq B^1_{\tau_i}$. Player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff B_i satisfies Positive Certainty and Negative Certainty.

Part (1) characterizes the statement that player i is certain of her type mapping τ_i with respect to the possession of p-beliefs $\{\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\}$ or the lack of p-beliefs $\{\neg\beta^p_E \mid (E, p) \in \mathcal{D} \times [0, 1]\}$. It also states that if player i is certain of the type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then the belief operator B_i satisfies Positive Certainty and Negative Certainty: $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$. Without Monotonicity, (i) player i is certain of her type mapping τ_i with respect to $\{\beta^p_E \mid (p, E) \in [0, 1] \times \mathcal{D}\}$ iff $B^p_{\tau_i}(E) \subseteq F$ implies $B^p_{\tau_i}(E) \subseteq B_i(F)$; and (ii) player i is certain of her type mapping τ_i with respect to $\{\neg\beta^p_E \mid (p, E) \in [0, 1] \times \mathcal{D}\}$ iff $(\neg B^p_{\tau_i})(E) \subseteq F$ implies $(\neg B^p_{\tau_i})(E) \subseteq B_i(F)$.

Part (2a) corresponds to the case when B_i is a fully-introspective knowledge operator in addition to her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. Part (2b) corresponds to the case in which B_i is a fully-introspective qualitative belief operator in addition to her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. When probabilistic beliefs and knowledge (or qualitative belief) are present, the introspective properties of Positive Certainty $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ and Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ are the standard assumptions (e.g., Aumann, 1999). Whenever player i believes an event E with probability at least p, she knows that she believes E with probability at least p. Whenever player i does not believe an event E with probability at least p, she knows that she does not believe E with probability at least p. In this environment, Parts (2a) and (2b) formalize the sense in which player i is certain of her probabilistic beliefs (her type mapping).

Part (2c) sheds light on the certainty of a type mapping in the type space (i.e., purely probabilistic model) in the case in which B_i is taken as the probability 1-belief operator $B^1_{\tau_i}$. The introspective properties of probabilistic beliefs are now formulated in terms of probability-one belief about own beliefs: $B^p_{\tau_i}(\cdot) \subseteq B^1_{\tau_i} B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(\cdot)$. That is, if player i p-believes an event E, then she believes with probability one that she p-believes E; and if player i does not p-believe an event E, then she believes with probability one that she does not p-believe E. These two introspective properties are essential in the syntactic formulation of type spaces such as Heifetz and Mongin (2001) and Meier (2012). Part (2c) justifies the statement that player i is certain of her own type mapping in a type space.

Part (2c) also justifies the structural assumption in a product type space. In the product type space, an underlying state space Ω is the product space of an underlying set of nature states S (e.g., the set of action profiles of a given strategic game) and the players' type sets $(T_j)_{j \in I}$. The structural assumption refers to the assumption that player i's type is a probability measure on the product of the nature states S and the types of the opponents $(T_j)_{j \in I \setminus \{i\}}$. Appendix A.3 formally shows that, under such specification, player i is certain of her type mapping.

I remark on two additional implications of Proposition 1B. First, Proposition 1B and Remark 6 allow one to formalize the sense in which each player is certain of her "prior." Consider a model $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I}, (\mu_i)_{i \in I} \rangle$ with the following properties: $(\Omega, \mathcal{D})$ is a measurable space, $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ is player i's (probabilistic-)type mapping, and $\mu_i \in \Delta(\Omega)$ is a *prior* satisfying

$$\mu_i(E) = \int_\Omega \tau_i(\omega)(E) \mu_i(d\omega) \text{ for all } E \in \mathcal{D}. \tag{2}$$

That is, the prior belief $\mu_i(E)$ is equal to the expectation of the posterior beliefs $t_i(\cdot)(E)$ with respect to the prior μ_i (e.g., Mertens and Zamir, 1985). The model admits a *common prior* if $\mu_i = \mu_j$ for all $i, j \in I$. If each μ_i would be identified as a constant mapping $\mu_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then

$$\mu_i^{-1}(\beta^p_E) = \begin{cases} \emptyset & \text{if } \mu_i(E) < p \\ \Omega & \text{if } \mu_i(E) \geq p \end{cases}.$$

Since $B^1_{\tau_i}$ satisfies Necessitation, each player i is certain of every player j's prior. In fact, the players are commonly certain of the priors.

Second, if the players are certain of their own type mappings, then the common p-belief operator reduces to the iteration of mutual p-beliefs.

Remark 8. Let $(\Omega, \mathcal{D})$ be a measurable space, and let $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ be player i's type mapping for each $i \in I$. Let $B^1_{\tau_i}$ be player i's probability-one belief operator. If each player i is certain of her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ (according to her probability-one belief), then the common p-belief operator reduces to the iteration of mutual p-beliefs and is well-defined: $C^p(\cdot) = \bigcap_{n \in \mathbb{N}} (B_I^p)^n(\cdot) \in \mathcal{D}$.

3.4 Informativeness, Possibility, and Certainty

If a player is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then, for each state ω, she would be able to conceive the collection of observational contents $F \in \mathcal{X}$ which hold at ω. Comparing such collections among all states, she would be able to rank the states according to "informativeness." As the player is certain of the signal, the information derived from the signal must have already been incorporated in her beliefs.

Section 3.4.1 defines the informativeness of a signal (Definition 2), and applies the informativeness criteria to qualitative- and probabilistic-type mappings to characterize the sense in which a player is certain of her type mapping in terms of informativeness (Propositions 2A and 2B).

Section 3.4.2 studies the assumption in a Harsanyi (1967-1968) type space in terms of the informativeness (Proposition 3A and 3B): at each state, a player assigns probability-one to the set of states equally informative to the state.

3.4.1 Informativeness, Possibility, and Certainty

I start with defining the informativeness of a signal:

Definition 2. For states ω and ω' in Ω, ω is *at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$* if

$$\{F \in \mathcal{X} \mid \omega' \in x^{-1}(F)\} \subseteq \{F \in \mathcal{X} \mid \omega \in x^{-1}(F)\}. \tag{3}$$

States ω and ω' are *equally informative according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$* if

$$\{F \in \mathcal{X} \mid \omega' \in x^{-1}(F)\} = \{F \in \mathcal{X} \mid \omega \in x^{-1}(F)\}. \tag{4}$$

The ideas behind Definition 2 are (i) that the informational content of a signal mapping $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω is expressed as the collection of observational contents $\{F \in \mathcal{X} \mid x(\omega) \in F\}$ true at ω and (ii) that informational contents are ranked by the implication in the form of set inclusion.[18] While the notion of informativeness (i.e., the relation induced by Expression (3)) is reflexive and transitive, the notion of equal informativeness (i.e., the relation induced by Expression (4)) is an equivalence relation.

The following remark characterizes the certainty of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ from informativeness. Namely, player i is certain of the signal x iff the notion of

[18] The notion of informativeness is closely related to that of information studied by Bonanno (2002). Ghirardato (2001), Lipman (1995), and Mukerji (1997) also study information processing in which informational contents are ranked by the implication in the form of set inclusion.

possibility derived from her beliefs is incorporated in the notion of informativeness derived from the signal in the following sense: whenever player i considers state ω' possible at state ω (i.e., $\omega' \in b_{B_i}(\omega)$), state ω' is at least as informative as state ω according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.

Remark 9. Assume the Kripke property for B_i. Player i is certain of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff possibility implies informativeness (i.e., if $\omega' \in b_{B_i}(\omega)$ then ω' is at least as informative as ω according to $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$).

The proof of Remark 9 is in Appendix A.1.1. Three additional remarks are in order. First, when $\mathcal{X}$ is not necessarily closed under complementation, Definition 2 does not take into account the collection of observational contents $\{F \in \mathcal{X} \mid x(\omega) \notin F\}$ that do not hold at ω. In contrast, when $\mathcal{X}$ is closed under complementation, it can be seen that if ω is at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then ω and ω' are equally informative.

Second, suppose that B_i satisfies Consistency, Positive Introspection, and Negative Introspection. If ω is at least as informative as ω' according to a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$, then ω and ω' are equally informative.[19]

Third, under the assumption that $x^{-1}(\{x(\omega)\}) \in \mathcal{D}$ for each $\omega \in \Omega$, the equivalence relation of equal informativeness coincides with the one induced by the partition $\{x^{-1}(\{x(\omega)\}) \mid \omega \in \Omega\}$: ω and ω' are equally informative iff $x(\omega) = x(\omega')$.

Next, I apply the notion of informativeness to i's qualitative-type mapping $t_i : \Omega \to M(\Omega)$ with respect to $\{\beta_E \mid E \in \mathcal{D}\}$. That is, suppose that player i is reasoning about the underlying states based on her possession of beliefs. For states ω and ω' in Ω, ω is at least as informative as ω' to i (precisely, according to $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$) iff $t_i(\omega')(\cdot) \leq t_i(\omega)(\cdot)$ (i.e., $t_i(\omega')(E) \leq t_i(\omega)(E)$ for all $E \in \mathcal{D}$). Likewise, states ω and ω' are equally informative according to i iff $t_i(\omega) = t_i(\omega')$.

Fix $\omega \in \Omega$, and let $(\uparrow t_i(\omega)) := \{\omega' \in \Omega \mid t_i(\omega)(\cdot) \leq t_i(\omega')(\cdot)\}$ be the set of states that are at least as informative to i as ω. Also, define $(\downarrow t_i(\omega)) := \{\omega' \in \Omega \mid t_i(\omega')(\cdot) \leq t_i(\omega)(\cdot)\}$ and $[t_i(\omega)] := \{\omega' \in \Omega \mid t_i(\omega) = t_i(\omega')\}$. If $\omega' \in [t_i(\omega)]$ then ω and ω' are indistinguishable to player i in that her qualitative-types (and thus the collections of events that she believes) are exactly the same at these states. Put differently, the equal informativeness is translated into the indistinguishability. Thus, the collection $\{[t_i(\omega)] \mid \omega \in \Omega\}$ forms a partition of Ω generated by the qualitative-type mapping t_i. Note that $(\uparrow t_i(\omega))$, $(\downarrow t_i(\omega))$, and $[t_i(\omega)]$ may not necessarily be an event.

Now, I examine the sense in which a player is certain of her qualitative-type mapping by studying how introspective properties imply the relations between informativeness and possibility.

[19] The proof goes as follows. Suppose to the contrary that there are $\omega, \omega' \in \Omega$ such that $\{F \in \mathcal{X} \mid \omega' \in B_i(x^{-1}(F))\} \subsetneq \{F \in \mathcal{X} \mid \omega \in B_i(x^{-1}(F))\}$. Then, there is $F \in \mathcal{X}$ with the following properties: $\omega \in B_i(x^{-1}(F)) \subseteq B_i B_i(x^{-1}(F))$ (by Positive Introspection) and $\omega' \in (\neg B_i)(x^{-1}(F)) \subseteq B_i(\neg B_i)(x^{-1}(F))$ (by Negative Introspection), a contradiction to Consistency.

Proposition 2A. *Let $\overrightarrow{\Omega}$ be a belief model. Let $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$ be player i's qualitative-type mapping.*

1. *(a) B_i satisfies Truth Axiom iff $(\uparrow t_{B_i}(\cdot)) \subseteq b_{B_i}(\cdot)$.*

   (b) If B_i satisfies Positive Introspection, then $b_{B_i}(\cdot) \subseteq (\uparrow t_{B_i}(\cdot))$. If B_i satisfies the Kripke property, the converse also holds.

   (c) If B_i satisfies Negative Introspection, then $b_{B_i}(\cdot) \subseteq (\downarrow t_{B_i}(\cdot))$. If B_i satisfies the Kripke property, the converse also holds.

2. *(a) If B_i satisfies Truth Axiom and Positive Introspection, then $(\uparrow t_{B_i}(\cdot)) = b_{B_i}(\cdot)$. If t_{B_i} satisfies the Kripke property, the converse also holds.*

   (b) If B_i satisfies Truth Axiom, (Positive Introspection), and Negative Introspection, then $(\uparrow t_{B_i}(\cdot)) = [t_{B_i}(\cdot)] = b_{B_i}(\cdot)$. If B_i satisfies the Kripke property, the converse also holds.

Part (1a) states that, under Truth Axiom, informativeness implies possibility. In Part (1b), since player i is certain of her qualitative-type mapping t_i with respect to $\{\beta_E \mid E \in \mathcal{D}\}$, the notion of possibility that comes from her beliefs is already encoded in the notion of informativeness. That is, Part (1b) states that possibility implies informativeness when player i is certain of her qualitative-type mapping $t_i : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$. Hence, when player i's qualitative-type mapping satisfies Truth Axiom and Positive Introspection as in a reflexive-and-transitive possibility correspondence model (see footnote 11), the notions of informativeness and possibility coincide: $b_{B_i}(\cdot) = (\uparrow t_{B_i}(\cdot))$.

Part (1b) and (1c) jointly state that, under Positive Introspection and Negative Introspection, if player i considers ω' possible at ω then the states ω and ω' are equally informative.

In a model of knowledge in which player i's qualitative-type mapping satisfies Truth Axiom, (Positive Introspection,) and Negative Introspection, either notion of informativeness or possibility induces the same partition $\{b_{B_i}(\omega) \mid \omega \in \Omega\} = \{[t_{B_i}(\omega)] \mid \omega \in \Omega\}$ of Ω with the following property: if $\omega' \in [t_i(\omega)] = b_{t_i}(\omega)$, then, for any event, she knows it at ω iff she knows it at ω'. In a model of qualitative belief in which player i's qualitative-type mapping satisfies Consistency, Positive Introspection, and Negative Introspection, $\emptyset \neq b_{B_i}(\cdot) \subseteq [t_{B_i}(\cdot)] = (\uparrow t_{B_i}(\cdot))$.

Next, I apply the notion of informativeness to player i's probabilistic-type mapping $\tau_i : \Omega \to \Delta(\Omega)$ with respect to $\{\beta_E^p \mid (E, p) \in \mathcal{D} \times [0, 1]\}$. That is, player i is reasoning about the underlying states based on her possession of p-beliefs. Since the notion of possibility comes from qualitative beliefs, I start with a model that has both qualitative and probabilistic beliefs (the case in which player i has only probabilistic beliefs will be discussed shortly).

A state ω is at least as informative as a state ω' to i (precisely, according to $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\beta_E^p \mid (E, p) \in \mathcal{D} \times [0, 1]\}))$ iff $\tau_i(\omega')(\cdot) \leq \tau_i(\omega)(\cdot)$. However, since each $\tau_i(\cdot)$ is (countably-)additive, it follows that ω is at least as informative as ω' to i iff ω and ω' are equally informative: $\omega' \in [\tau_i(\omega)] := \{\omega'' \in \Omega \mid \tau_i(\omega'') = \tau_i(\omega)\}$. If player i does not believe an event E with probability at least p at a state, then she does believe E^c with probability at least $1-p$. Since player i is able to reason about the possession of beliefs for any event and any probability, she is also able to reason about the lack of beliefs when her probabilistic beliefs are (countably-)additive.[20]

Proposition 2B. *Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. *Either Positive Certainty $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ or Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ yields $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$: possibility implies (equal) informativeness.*

2. *Under the Kripke property of B_i, conversely, $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$ implies Positive Certainty $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ and Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$.*

3.4.2 Harsanyi Property

Next, I move on to studying the notion of informativeness in a type space. To that end, player i's type mapping $\tau_i : \Omega \to \Delta(\Omega)$ satisfies the *Harsanyi property* if $[\tau_i(\omega)] \subseteq E$ implies $\omega \in B^1_{\tau_i}(E)$ for any $(\omega, E) \in \Omega \times \mathcal{D}$. That is, whenever an event E is implied by the set of states $[\tau_i(\omega)]$ indistinguishable from ω, player i believes E with probability one at ω (e.g., Meier, 2008, 2012; Mertens and Zamir, 1985).

Under the regularity condition $[\tau_i(\cdot)] \in \mathcal{D}$, since each $\tau_i(\omega)$ satisfies Monotonicity, the Harsanyi property is equivalent to $\tau_i(\omega)([\tau_i(\omega)]) = 1$ for each $\omega \in \Omega$. It states that, at each state, player i assign probability one to the set of states indistinguishable from that state. In fact, in the type space literature, the informal assumption that each player is certain of her own type is formally represented as the condition on the type mapping to put probability one on the set of types indistinguishable from its own (Mertens and Zamir, 1985; Vassilakis and Zamir, 1993).

In a type space, I show that the Harsanyi property characterizes the idea that a player is certain of her own type mapping with respect to the beliefs that she could have been able to possess.

Proposition 3B. *Let $(\Omega, \mathcal{D})$ be a measurable space, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping.*

1. *Suppose that $[\tau_i(\cdot)] \in \mathcal{D}$. The type mapping τ_i satisfies the Harsanyi property iff player i is certain of $\tau_i : \Omega \to \Delta(\Omega)$ with respect to her realized beliefs $\{\{\tau_i(\omega)\} \mid \omega \in \Omega\}$.*

2. *Let $\mathcal{D}$ be generated from a countable algebra. The following are all equivalent.*

[20] While one can obtain a nuanced understanding of the relation between the informativeness and certainty of a type mapping τ_i when each $\tau_i(\cdot)$ is a non-additive measure, I focus on studying the sense in which player i is certain of her countably-additive type mapping τ_i. Recall footnote 8.

(a) *The type mapping* τ_i *satisfies the Harsanyi property.*

(b) *Player* i *is certain of* $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\{\tau_i(\omega)\} \mid \omega \in \Omega\})$.

(c) *Player* i *is certain of* $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$.

(d) *Player* i *is certain of* $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\beta_E^p \mid (E, p) \in \mathcal{D} \times [0,1]\})$.

(e) *Player* i *is certain of* $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\neg\beta_E^p \mid (E, p) \in \mathcal{D} \times [0,1]\})$.

Part (2) states that, under the regularity condition $[\tau_i(\cdot)] \in \mathcal{D}$, the Harsanyi property is equivalent to $B_{\tau_i}^p(\cdot) \subseteq B_{\tau_i}^1 B_{\tau_i}^p(\cdot)$ or $(\neg B_{\tau_i}^p)(\cdot) \subseteq B_{\tau_i}^1(\neg B_{\tau_i}^p)(\cdot)$.

Next, I define the analogue of the Harsanyi property for qualitative belief: player i believes an event E at ω if E is implied by the set $[t_{B_i}(\omega)]$ of states indistinguishable from ω. The proposition below shows that the analogue of the Harsanyi property characterizes the certainty of the qualitative-type mapping in the strongest sense.

Proposition 3A. *Let* $\overrightarrow{\Omega}$ *be a belief model such that* $[t_{B_i}(\cdot)] \in \mathcal{D}$.

1. *The following are equivalent.*

   (a) *Player* i *is certain of her qualitative-type mapping* $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \{\{t_{B_i}(\omega)\} \mid \omega \in \Omega\})$.

   (b) *For any* $(\omega, E) \in \Omega \times \mathcal{D}$ *with* $[t_{B_i}(\omega)] \subseteq E$, $\omega \in B_i(E)$.

2. *If* B_i *satisfies the Kripke property, Positive Introspection, and Negative Introspection, then player* i *is certain of her qualitative-type mapping* $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \{\{t_{B_i}(\omega)\} \mid \omega \in \Omega\})$.

3. *Under either condition in Part (1), Truth Axiom yields the Kripke property.*

Part (1) is similar to Proposition 3B: under the regularity condition, the Harsanyi property states that a player is certain of her own type mapping in the strongest sense. Part (2) states that, in a possibility correspondence model, if a player's belief is fully introspective then she is certain of her qualitative-type mapping (or her possibility correspondence) in the strongest sense.

Finally, I remark that, under Entailment $B_i(\cdot) \subseteq B_{\tau_i}^1(\cdot)$ in a qualitative belief model, the Kripke property implies the Harsanyi property. Suppose qualitative and probabilistic beliefs satisfy Entailment. Suppose also that player i is certain of her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\beta_E^p \mid (E, p) \in \mathcal{D} \times [0,1]\})$. Thus, $b_{B_i}(\cdot) \subseteq [\tau_i(\cdot)]$ holds as in Proposition 2B. Then, the Kripke property yields the Harsanyi property: if $[\tau_i(\omega)] \subseteq E$ then it follows from $b_{B_i}(\omega) \subseteq E$ that $\omega \in B_i(E) \subseteq B_{\tau_i}^1(E)$.

4 When are the Players Commonly Certain of a Belief Model?

With the analyses in Section 3 in mind, I formalize the sense in which the players are commonly certain of a belief model itself: the players are commonly certain of the profile of their (qualitative- or probabilistic-)type mappings. By Remark 7, it is sufficient to ask when every player i is certain of each player j's (qualitative- or probabilistic-)type mapping.

I start with a qualitative belief model. Proposition 1A applies to the case in which player i is certain of player j's qualitative-type mapping. For example, if player i is certain of player j's qualitative-type mapping $t_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, then $B_j(\cdot) \subseteq B_i B_j(\cdot)$ and $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$ hold. Thus, Proposition 1A implies:

Remark 10A. Let $\overrightarrow{\Omega}$ be a belief model in which player i's belief operator B_i satisfies Monotonicity, and let $t_{B_j} : \Omega \to M(\Omega)$ be player j's qualitative-type mapping.

1. (a) Player i is certain of t_{B_j} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ iff $B_j(\cdot) \subseteq B_i B_j(\cdot)$.
   (b) Player i is certain of t_{B_j} with respect to $\{\neg\beta_E \mid E \in \mathcal{D}\}$ iff $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$.
   (c) If player i is certain of $t_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, then $B_j(\cdot) \subseteq B_i B_j(\cdot)$ and $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$.
2. (a) Let B_i satisfy Truth Axiom. Player i is certain of $t_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff ($B_j(\cdot) \subseteq B_i B_j(\cdot)$ and) $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$.
   (b) Let B_i satisfy Consistency and Countable Conjunction. Player i is certain of $t_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff $B_j(\cdot) \subseteq B_i B_j(\cdot)$ and $(\neg B_j)(\cdot) \subseteq B_i(\neg B_j)(\cdot)$.

Roughly, Remark 10A states that player i is certain of player j's qualitative-type mapping t_{B_j} iff (i) whenever player j believes an event E at ω, player i believes that player j believes E at ω; and (ii) whenever player j does not believe an event E at ω, player i believes that player j does not believe E at ω.

Now, I move to one of the main questions of this paper: I ask when the players are commonly certain of the qualitative-type mappings in a belief model.

Theorem 1A. *Let $\overrightarrow{\Omega}$ be a belief model in which every B_i satisfies Monotonicity, and let $t_{B_i} : \Omega \to M(\Omega)$ be player i's qualitative-type mapping for each $i \in I$.*

1. *Assume Truth Axiom for every B_i. The players are commonly certain of the profile of qualitative-type mappings $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff, for every $i, j \in I$, $B_i = B_j$, (Positive Introspection $B_i(\cdot) \subseteq B_i B_i(\cdot)$), and Negative Introspection $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$. In particular, $B_i = C$ for each $i \in I$.*

2. *Assume Consistency and Countable Conjunction for every B_i. The players are commonly certain of the profile of qualitative-type mappings $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff $B_i(\cdot) \subseteq CB_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq C(\neg B_i)(\cdot)$ for every $i \in I$. In particular, $C = B_I$.*

While Part (1) studies a knowledge model, Part (2) does a belief model. I start with discussing three implications of Part (2). In words, Part (2) states that the players are commonly certain of their qualitative-type mappings iff (i) for any event E which some player i believes at some state ω, it is commonly believed that player i believes E at ω; and (ii) for any event E which some player i does not believe at some state ω, it is commonly believed that player i does not believe E at ω.[21]

In Part (2), the mutual belief and common belief operators coincide if the players are commonly certain of their qualitative-type mappings, under the mild conditions of Consistency and Countable Conjunction.[22] This is because, for any event E which everybody believes at some state ω, it is commonly believed that everybody believes E at ω: $B_I(\cdot) \subseteq CB_I(\cdot)$. Intuitively, in a model of which the players are commonly certain, if everybody believes an event E then it is common belief that everybody believes E. Thus, if everybody believes E then everybody believes that everybody believes E. Hence, the first-order mutual belief itself implies any higher-order mutual beliefs, and thus the mutual and common beliefs coincide.

Part (1) provides a contrast between knowledge and belief. In a knowledge model with Truth Axiom, for the players to be commonly certain of the model, it is *necessary* that their knowledge coincides with each other. In contrast, in a belief model without Truth Axiom, it may be the case that the players' beliefs are different but they are commonly certain of their qualitative-type mappings. The following simple example illustrates this point.

Example 3. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Let B_1 be induced by a possibility correspondence b_1 given as follows: $b_1(\omega_1) = \{\omega_1\}$ and $b_1(\omega_2) = b_1(\omega_3) = \{\omega_2\}$. Let B_2 be induced by a possibility correspondence b_2 given as follows: $b_2(\omega_1) = \{\omega_1\}$ and $b_2(\omega_2) = b_2(\omega_3) = \{\omega_2, \omega_3\}$. By inspection, each b_i is serial, transitive, and Euclidean. Thus, each B_i satisfies Consistency, Positive Introspection, and Negative Introspection in addition to the Kripke property (which implies Monotonicity, Countable Conjunction, and Necessitation). Table 1 depicts B_1, B_2, and $C = B_I$ (note that $B_I = B_I^n$).

By inspection, it can be seen from the table that $B_i = CB_i$ and $(\neg B_i) = C(\neg B_i)$ for each $i \in I = \{1, 2\}$. Thus, the players are commonly certain of the profile of qualitative-type mappings $(t_{B_i})_{i \in I}$. However, $B_1 \neq B_2$.

[21] In fact, since each B_i satisfies Consistency, it can be seen that C also satisfies Consistency. Then, it can also be seen that $B_i = CB_i$ and $(\neg B_i) = C(\neg B_i)$ for every $i \in I$.

[22] The converse does not hold, i.e., $C = B_I$ does not necessarily imply that the players are certain of the profile of their qualitative-type mappings. As a simple example, let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$. Let $B_1(E) = E$ and $B_2(E) = E^c$ for all $E \in \mathcal{D}$. Then, $B_I(\cdot) = C(\cdot) = \emptyset$, and the players are not commonly certain of their qualitative-type mappings.

E	$\emptyset$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω
$B_1(E)$	$\emptyset$	$\{\omega_1\}$	$\{\omega_2,\omega_3\}$	$\emptyset$	Ω	$\{\omega_1\}$	$\{\omega_2,\omega_3\}$	Ω
$B_2(E)$	$\emptyset$	$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3\}$	Ω
$B_I(E)$	$\emptyset$	$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3\}$	Ω
$C(E)$	$\emptyset$	$\{\omega_1\}$	$\emptyset$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2,\omega_3\}$	Ω

Table 1: Inidvidual and Common Beliefs: B_1, B_2, B_I, and C (Example 3)

Moving on to probabilistic beliefs, Proposition 1B applies to the case in which player i is certain of player j's probabilistic-type mapping:

Remark 10B. Let $\overrightarrow{\Omega}$ be a belief model in which player i's belief operator B_i satisfies Monotonicity, and let $\tau_j : \Omega \to \Delta(\Omega)$ be player j's type mapping.

1. (a) Player i is certain of τ_j with respect to $\{\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$.
   (b) Player i is certain of τ_j with respect to $\{\neg\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.
   (c) If player i is certain of $\tau_j : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$ and $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.
2. (a) Let B_i satisfy Truth Axiom and Negative Introspection. Player i is certain of $\tau_j : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$ iff $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.
   (b) Let B_i satisfy Consistency, Countable Conjunction, Positive Introspection, and Negative Introspection. Player i is certain of $\tau_j : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff $B_{\tau_j}^p(\cdot) \subseteq B_i B_{\tau_j}^p(\cdot)$ and $(\neg B_{\tau_j}^p)(\cdot) \subseteq B_i(\neg B_{\tau_j}^p)(\cdot)$.

As in Remark 10A, Remark 10B roughly states that player i is certain of player j's probabilistic-type mapping iff (i) whenever player j p-believes an event E at ω, player i believes that player j p-believes the event E at ω; and (ii) whenever player j does not p-believe an event E at ω, player i believes that player j does not p-believe the event E at ω.

Now, I ask one of the main questions of this paper in the context of probabilistic-type mappings: when are the players in a belief model commonly certain of their probabilistic-type mappings?

Theorem 1B. *Let $\overrightarrow{\Omega}$ be a belief model, and let $\tau_i : \Omega \to \Delta(\Omega)$ be player i's type mapping for each $i \in I$. Assume Monotonicity, Consistency, and Countable Conjunction for every B_i. The players are commonly certain of the profile of type mappings $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ iff $B_{\tau_i}^p(\cdot) \subseteq C B_{\tau_i}^p(\cdot)$ and $(\neg B_{\tau_i}^p)(\cdot) \subseteq C(\neg B_{\tau_i}^p)(\cdot)$ for every $(i,p) \in I \times [0,1]$. If $B_i = B_{\tau_i}^1$ is taken for every $i \in I$, then $C^1 = B_I^1$, where $B_I^1(\cdot) := \bigcap_{i \in I} B_{\tau_i}^1(\cdot)$ is the mutual 1-belief operator.*

Roughly, Theorem 1B states: the players are commonly certain of their probabilistic-type mappings iff (i) for any event E which some player i p-believes at some state ω, it is commonly believed that player i p-believes E at ω; and (ii) for any event E which some player i does not p-believe at some state ω, it is commonly believed that player i does not p-believe E at ω. If each player's belief operator B_i in the belief model is taken as probability-one belief operator $B^1_{\tau_i}$, then the probability-one common belief operator reduces to the probability-one mutual belief operator.

To conclude this section, I provide two further remarks on Theorems 1A and 1B. First, Theorems 1A and 1B are impossibility results in the following sense. In Theorem 1A, every player's knowledge operator coincides. In Theorem 1B, the mutual and common belief operators coincide. In this regard, at an informal level, Theorem 1A has some similarity with the following two results.

The first result is the impossibility of agreeing-to-disagree (Aumann, 1976): (under a common prior) if two players have common knowledge of their posteriors then the posteriors coincide. Here, if the players are commonly certain of their knowledge-generating mappings, then their knowledge operators coincide. Note that the underlying forces behind the two results are different. For instance, Bayes updating and a common prior, which play a crucial role in the agreement theorem, are not even present in Theorem 1A.

The second result is the characterization of rational-expectations equilibria in an interactive epistemic model. At a rational-expectations equilibrium, the informational contents of players' strategies are revealed so that their strategies are common knowledge, and if such information revelation is fine enough then the differences in players' knowledge may be muted (see, for instance, McKelvey and Page (1986) and Minelli and Polemarchakis (2003) in the context of interactive knowledge models).[23] My results are related to this strand of literature in that the meta-certainty of a model may mute differences in players' beliefs.

Second, I study an implication of Theorems 1A and 1B (the "common meta-certainty" of a belief model) to the certainty of a signal. Suppose that the players are commonly certain of a belief model. If player i is certain of a signal $x : \Omega \to X$, then is player j also certain of the signal x? While the players' beliefs may not be homogeneous, the proposition below shows that this is the case.

Proposition 4. *Let $\overrightarrow{\Omega}$ be a belief model such that each B_i satisfies Monotonicity and Consistency. Let $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ be a signal mapping such that, for any $F \in \mathcal{X}$, there exists a sub-collection $(F_\lambda)_{\lambda \in \Lambda}$ of $\mathcal{X}$ with $F^c = \bigcup_{\lambda \in \Lambda} F_\lambda$.*

A. i. If player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ and if player j is certain of player i's qualitative-type mapping $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$, then player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.

[23]In fact, the framework of this paper allows one to formally define what means by the statement that players commonly know their strategies.

ii. Suppose that the players are commonly certain of the profile of their qualitative-type mappings $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$. Then, player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.

B. i. Let $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ be player i's (probabilistic-)type mapping, and assume Entailment: $B_i(\cdot) \subseteq B^1_{\tau_i}(\cdot)$. If player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ and if player j is certain of player i's type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$, then player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.

ii. Suppose that the players are commonly certain of the profile of their type mappings $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$. Suppose Entailment for every player i: $B_i(\cdot) \subseteq B^1_{\tau_i}(\cdot)$ Then, player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ iff player j is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$.

While Part (A) asks the certainty of qualitative-type mappings, Part (B) does that of probabilistic-type mappings. The meta-common-certainty assumption states that if player i is certain of her own strategy and if player j is certain of player i's type mapping then player j is certain of player i's strategy. In particular, if the players are commonly certain of the profile of their type mappings and if each player is certain of her own strategy, then it follows that the players are commonly certain of the strategy profile. In the next section, I clarify the role of such meta-certainty assumptions on game-theoretic solution concepts.

Third, one can also ask whether the players are commonly certain of the qualitative-type mapping t_C that represents common belief. Since the common belief operator C satisfies Positive Introspection, the players are commonly certain of $t_C : (\Omega, \mathcal{D}) \to (M(\Omega), \{\beta_E \mid E \in \mathcal{D}\})$, which is equivalent to $C(\cdot) \subseteq B_i C(\cdot)$. Now:

Remark 11. Let $\overrightarrow{\Omega}$ be a belief model, and let $t_C : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ be the qualitative-type mapping that represents the common belief operator C. Suppose that each belief operator B_i satisfies Monotonicity, Consistency, and Countable Conjunction. The following are equivalent.

1. The players are commonly certain of $t_C : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$.
2. C satisfies Negative Introspection: $(\neg C)(\cdot) \subseteq C(\neg C)(\cdot)$.
3. $(\neg C)(\cdot) \subseteq B_i(\neg C)(\cdot)$ for each $i \in I$.

Since each B_i satisfies Consistency and Countable Conjunction, so does the common belief operator C. Then, the players are commonly certain of $t_C : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$ iff $C(\cdot) \subseteq B_i C(\cdot)$ (which always holds) and $(\neg C)(\cdot) \subseteq B_i(\neg C)(\cdot)$ for every $i \in I$. It can be seen that the latter condition is equivalent to Negative Introspection of C, establishing the remark.

5 What Role Does the "Meta-Certainty" of a Model Play in Game-theoretic Analyses?

Section 4 has examined when the players are commonly certain of a belief model. Moving on to the second objective of the paper, I examine the role that the "meta-certainty" assumption plays in game-theoretic analyses of solution concepts.

5.1 Iterated Elimination of Strictly Dominated Actions

This section considers the solution concept of iterated elimination of strictly dominated actions (IESDA) in a strategic game, one of the most important solution concepts in game theory. Informally, an epistemic characterization of IESDA states that, in a strategic game, if the (i) "logical" players are (ii) "commonly (meta-)certain of the game" and if they (iii) commonly believe their rationality, then their resulting actions survive IESDA. Formally, in the context of the framework of this paper, Fukuda (2020) shows that if the players commonly believe each player's rationality and if each of them correctly believes their own rationality, then their resulting actions survive IESDA, without assuming any property on individual players' beliefs.

This paper connects these two statements as follows: first, suppose that the players are logical in that their beliefs satisfy Monotonicity, Consistency, and Finite Conjunction. Second, suppose that each of them is certain of their own qualitative-type mapping and strategy. Third, suppose that the players commonly believe their rationality. Then, their resulting actions survive IESDA. In what follows, I show that the certainty (of her own strategy and type mapping) allows her to correctly believe her own rationality.

5.1.1 A Strategic Game, a Model of a Game, and Rationality

A *(strategic) game* is a tuple $\Gamma = \langle (A_i)_{i \in I}, (\succsim_i)_{i \in I} \rangle$: A_i is a non-empty finite set of player i's actions, and $\succsim_i$ is i's (complete and transitive) preference relation on $A := \bigtimes_{i \in I} A_i$.[24] Denote by $\sim_i$ and $\succ_i$ the indifference and strict relations, respectively.

A (belief) *model* of the game Γ is a tuple $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C, (\sigma_i)_{i \in I} \rangle$ (abusing the notation, denote it by $\overrightarrow{\Omega}$) with the following two properties. First, $\langle (\Omega, \mathcal{D}), (B_i)_{i \in I}, C \rangle$ is a belief model. Second, $\sigma_i : \Omega \to A_i$ is a *strategy* of player i satisfying the measurability condition that $\sigma_i^{-1}(\{a_i\}) \in \mathcal{D}$ for all $a_i \in A_i$. Denote $[\sigma_i(\omega)] := \sigma_i^{-1}(\{\sigma_i(\omega)\})$ for each $\omega \in \Omega$.

Denote by $[a'_i \succsim_i a_i] := \{\omega' \in \Omega \mid (a'_i, \sigma_{-i}(\omega')) \succsim_i (a_i, \sigma_{-i}(\omega'))\} \in \mathcal{D}$ for any $a_i, a'_i \in A_i$. In words, $[a'_i \succsim_i a_i]$ is the event that player i prefers taking action a'_i to a_i given the opponents' strategies σ_{-i}. The set $[a'_i \succsim_i a_i]$ is an event because

[24] The assumption on the cardinality of each action set A_i is to simplify the analysis. See Fukuda (2024b) for an epistemic characterization of IESDA when there is no cardinality restriction.

$[a'_i \not\succcurlyeq_i a_i] = \sigma_{-i}^{-1}(\{a_{-i} \in A_{-i} \mid (a'_i, a_{-i}) \not\succcurlyeq_i (a_i, a_{-i})\}) \in \mathcal{D}$. Define $[a'_i \succ_i a_i]$ and $[a'_i \sim_i a_i]$ analogously.

Denote by RAT_i the event that player i is *rational* (see, e.g., Bonanno, 2008, 2015; Chen, Long, and Luo, 2007):

$$\begin{aligned}\mathrm{RAT}_i :=& \{\omega \in \Omega \mid \omega \in B_i([a'_i \succ_i \sigma_i(\omega)]) \text{ for no } a'_i \in A_i\} \\ =& \bigcap_{a_i \in A_i} \left((\sigma_i^{-1}(\{a_i\}))^c \cup \bigcap_{a'_i \in A_i} (\neg B_i)([a'_i \succ_i a_i]) \right) \in \mathcal{D}.\end{aligned}$$

Let $\mathrm{RAT}_I := \bigcap_{i \in I} \mathrm{RAT}_i$. Player i is *rational* at $\omega \in \Omega$ if there is no action $a'_i \in A_i$ such that player i believes that playing a'_i is strictly better than playing $\sigma_i(\omega)$ given the opponents' strategies σ_{-i}. In other words, player i is rational at ω if, for any action a'_i, she always considers it possible that playing $\sigma_i(\omega)$ is at least as good as playing a'_i given the opponents' strategies σ_{-i}: $\omega \in (\neg B_i)(\neg[\sigma_i(\omega) \not\succcurlyeq_i a'_i])$ for any $a'_i \in A_i$.

Now, the epistemic characterization of IESDA is stated as follows. Suppose that each player i correctly believes her own rationality: $B_i(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$ for every $i \in I$. If every player's rationality is common belief at ω, i.e., $\omega \in \bigcap_{i \in I} C(\mathrm{RAT}_i)$, then the resulting actions $(\sigma_i(\omega))_{i \in I} \in A$ survive IESDA.[25]

Finally, in this section, player i is *certain of her own strategy* σ_i if she is certain of $\sigma_i : (\Omega, \mathcal{D}) \to (A_i, \{\{a_i\} \mid a_i \in A_i\})$. Especially, $[\sigma_i(\cdot)] \subseteq B_i([\sigma_i(\cdot)])$. Note that, under Monotonicity and Consistency, if player i is certain of her own strategy then $B_i([\sigma_i(\cdot)]) = [\sigma_i(\cdot)]$, $[\sigma_i(\cdot)]^c = B_i([\sigma_i(\cdot)]^c)$, and $B_i(\Omega) = \Omega$.[26]

5.1.2 A Counterexample without Certainty of Strategies

The standard assumptions on qualitative belief (i.e., Consistency, Positive Introspection, Negative Introspection, and the Kripke property) guarantee that $B_i(\mathrm{RAT}_i) = \mathrm{RAT}_i$ (e.g., Bonanno, 2008, 2015).[27] In such case, if players commonly believe their rationality at a state, their actions at that state survive IESDA.

Here, I provide a counterexample in which if players are not necessarily logical and are not necessarily certain of their qualitative-type mappings and strategies, then the prediction under common belief in rationality may not necessarily capture IESDA.

[25] The "converse" also holds. For any action profile hat survives IESDA, there exist a belief model and a state such that each player correctly believes her own rationality, the players commonly believe each player's rationality, and they take the given actions at the state.

[26] Under Consistency and Monotonicity of B_i, it can be shown that the certainty of own strategy implies that if player i is rational at ω, then she never takes a strictly dominated action at ω (if she takes a strictly dominated action, then her belief violates Necessitation $B_i(\Omega) = \Omega$).

[27] It can be seen that Consistency, Positive Introspection, and the Kripke property in addition to the certainty of i's own strategy yield $B_i(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$. Likewise, Negative Introspection and the Kripke property in addition to the certainty of i's own strategy yield $\mathrm{RAT}_i \subseteq B_i(\mathrm{RAT}_i)$.

	a	b	c
a	$0,0$	$0,1$	$0,0$
b	$1,0$	$1,1$	$1,2$
c	$0,0$	$2,1$	$2,2$

Table 2: A Strategic Game (Example 4)

Example 4. Consider a two-player strategic game represented by Table 2. The set of players is $I = \{1, 2\}$. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mathcal{D} = \mathcal{P}(\Omega)$. Suppose that each B_i is defined as follows:

$$B_i(E) = \begin{cases} \{\omega_1\} & \text{if } E = \{\omega_1\} \\ \{\omega_2\} & \text{if } E = \{\omega_2, \omega_3, \omega_4\} \\ \{\omega_1, \omega_2, \omega_3\} & \text{if } E \in \{\{\omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_3\}\} \\ \Omega & \text{if } E = \Omega \\ \emptyset & \text{otherwise} \end{cases}.$$

Each player's belief operator B_i does not satisfy Monotonicity (e.g., $B_i(\{\omega_1, \omega_2\}) \not\supseteq B_i(\{\omega_1\})$). Since B_i also violates Positive Introspection (e.g., $B_i(\{\omega_2, \omega_3, \omega_4\}) \not\subseteq B_i B_i(\{\omega_2, \omega_3, \omega_4\})$), player i is not certain of her own qualitative-type mapping.

Since $\mathcal{J}_{B_I} = \{\emptyset, \{\omega_1, \omega_2, \omega_3\}, \Omega\}$, it follows from Expression (1) that the common belief operator C satisfies:

$$C(E) = \begin{cases} \{\omega_1, \omega_2, \omega_3\} & \text{if } E \in \{\{\omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_3\}\} \\ \Omega & \text{if } E = \Omega \\ \emptyset & \text{otherwise} \end{cases}.$$

Suppose that each player's strategy is given by $(\sigma_i(\omega))_{\omega \in \Omega} = (a, b, b, c)$. Each player is not certain of her own strategy σ_i. For example, at state ω_4 at which player i takes c, she does not believe that she takes c.

One can show $\mathrm{RAT}_i = \{\omega_3, \omega_4\}$.[28] Thus, each player i does not have correct belief in her rationality: $B_i(\mathrm{RAT}_i) \not\subseteq \mathrm{RAT}_i$. In fact, since $C(\{\omega_3, \omega_4\}) = \{\omega_1, \omega_2, \omega_3\}$, at $\omega_3 \in \left(\bigcap_{i \in I} \mathrm{RAT}_i\right) \cap \left(\bigcap_{i \in I} C(\mathrm{RAT}_i)\right)$ at which every player is rational and it is common belief that each player is rational, the players' actions are (b, b), which is not consistent with the unique prediction under IESDA, which is (c, c).

[28] Roughly, the proof goes as follows. At ω_1, player i believes that playing b is better than playing $\sigma_i(\omega_1) = a$. At ω_2 and only at ω_2, player i believes that playing c is better than playing $\sigma_i(\omega_2) = b$ given σ_{-i}. At ω_3, player i does not believe that playing a_i' is better than playing $\sigma_i(\omega_3) = b$ given σ_{-i}. At ω_4, player i does not believe that playing a_i' is better than playing $\sigma_i(\omega_4) = c$ given σ_{-i}, because only at ω_1, player i believes that playing b is better than playing c given σ_{-i}.

The rest of this section studies a sufficient condition in terms of a player's reasoning about her own beliefs under which the player correctly believes her own rationality. In particular, when each player is logical and correctly believes her own rationality, then common belief in rationality captures the predictions under IESDA.

5.1.3 The Role of Meta-certainty in Correctly Believing One's Own Rationality

I ask under what conditions player i correctly believes her own rationality: $B_i(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$. Here, I provide a compatibility condition on belief with informativeness, under which a player correctly believes her own rationality.

Definition 3. Player i's belief (operator B_i) is *compatible with informativeness* (derived from the qualitative-type mapping t_{B_i}) if $(\uparrow t_{B_i}(\omega)) \cap E \neq \emptyset$ for any $E \in \mathcal{D}$ with $\omega \in B_i(E)$.

In words, player i's beliefs are compatible with informativeness if, for any event E which player i believes at some ω, there exists a state ω' in E which is at least as informative as ω. If player i's belief operator B_i satisfies the Kripke property, Consistency, and Positive Introspection, then B_i is compatible with informativeness.[29] The compatibility with informativeness does not necessarily imply the Kripke property (and vice versa).[30] Thus, the compatibility with informativeness allows for the failure of Monotonicity. If player i's belief operator B_i is compatible with informativeness, then it satisfies $B_i(\emptyset) = \emptyset$. Thus, under Finite Conjunction, if B_i is compatible with informativeness, then it satisfies Consistency.

One can also apply the compatibility with informativeness to the context of probabilistic beliefs. Namely, player i's 1-belief operator $B^1_{\tau_i}$ is compatible with informativeness derived from the type mapping τ_i. Proposition 5 (B) below implies that player i's belief operator $B^1_{\tau_i}$ is compatible with informativeness under the Harsanyi property.

The following proposition states that the compatibility of beliefs with informativeness is implied by the certainty of a type mapping.

Proposition 5. *Let $\overrightarrow{\Omega}$ be a belief model.*

A. *Assume: (i) $(\uparrow t_{B_i}(\cdot)) \in \mathcal{D}$; (ii) B_i satisfies Monotonicity, Consistency, and Finite Conjunction; and that (iii) player i is certain of $t_{B_i} : \Omega \to M(\Omega)$ with respect to $\{\{\mu \in M(\Omega) \mid \mu(\cdot) \geq t_{B_i}(\omega)(\cdot)\} \mid \omega \in \Omega\}$. Then, B_i is compatible with informativeness.*

[29] The proof goes as follows. For any $E \in \mathcal{D}$ with $\omega \in B_i(E)$, $\emptyset \neq E \cap b_{B_i}(\omega) \subseteq E \cap (\uparrow t_{B_i}(\omega))$.

[30] Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2\}, \mathcal{P}(\Omega))$. Define B_i as $B_i(E) = E$ if $E \neq \Omega$; and $B_i(\Omega) = \emptyset$. While B_i is compatible with informativeness, it does not satisfy the Kripke property. Define B_j as $B_j(E) = E$ if $E \in \{\emptyset, \Omega\}$; and $B_j(E) = E^c$ if $E \in \{\{\omega_1\}, \{\omega_2\}\}$. While B_j satisfies the Kripke property, it is not compatible with informativeness.

B. *Let* $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ *be player* i*'s probabilistic-type mapping in the belief model* $\overrightarrow{\Omega}$*. Assume (i) the Harsanyi property; and (ii) Entailment:* $B_i(\cdot) \subseteq B^1_{\tau_i}(\cdot)$*. Then,* B_i *is compatible with informativeness.*

Part (A) states that, under the regularity condition (i), if player i is logical (in that her belief operator satisfies Monotonicity, Consistency, and Finite Conjunction) and if she is certain of her qualitative-type mapping, then her beliefs are compatible with informativeness. Theorem 2 below establishes that if player i's beliefs are compatible with informativeness then she correctly believes her rationality, which is a part of the preconditions of the epistemic characterization of IESDA.

Part (B) states that the Harsanyi property implies the compatibility with informativeness. Recall Proposition 2B: under the regularity condition $[\tau_i(\cdot)] \in \mathcal{D}$, the Harsanyi property holds iff player i is certain of her probabilistic-type mapping.

Now, I present the main result of this subsection. The theorem says that a player correctly believes her own rationality if: (i) she is certain of her own strategy; (ii) her belief is compatible with the informativeness; and if (iii) her belief is (finitely) conjunctive so that she can simultaneously reason about her own strategy and her own rationality.

Theorem 2. *Suppose that player* i *is certain of her own strategy. Also, let* B_i *be compatible with informativeness and satisfy Finite Conjunction. Then, player* i *correctly believes her own rationality:* $B_i(\mathrm{RAT}_i) \subseteq \mathrm{RAT}_i$.

Proposition 5 and Theorem 2 imply that player i correctly believes her own rationality if she is logical and if she is certain of her own type mapping and strategy. Theorem 2 states that, for the role of the meta-certainty assumption of a belief model on IESDA, it is not necessary that each player is certain of the profile of type mappings but it is sufficient that each player is certain of her own type mapping. In fact, one can incorporate the assumptions that each player is certain of her own qualitative type-mapping and strategy into the condition that she is certain of the part of the model of a game $\langle (\Omega, \mathcal{D}), (t_{B_i}, \sigma_i) \rangle$ that dictates her beliefs and strategy.

In Theorem 2, as shown in Example 5 below, the assumptions of the compatibility with informativeness and Finite Conjunction cannot be dropped.

Example 5. Consider the two-player coordination game represented by the left panel of Table 3. Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \omega_3\}, \mathcal{P}(\Omega))$. Let $(\sigma_1(\omega))_{\omega\in\Omega} = (a, a, b)$ and $(\sigma_2(\omega))_{\omega\in\Omega} = (a, a, c)$. Suppose that B_1 and B_2 are given by the right panel of Table 3. It can be seen that B_1 violates Finite Conjunction (as $B_1(\{\omega_1, \omega_2\}) \cap B_1(\{\omega_2, \omega_3\}) \not\subseteq B_1(\{\omega_2\})$), and that B_2 is not consistent with informativeness (e.g., while $\omega_1 \in B_2(\{\omega_2, \omega_3\})$, $(\uparrow t_{B_2})(\omega_1) \cap \{\omega_2, \omega_3\} = \emptyset$).

It can be seen that $\mathrm{RAT}_i = \{\omega_1, \omega_2\}$. For instance, at ω_3, player i believes that playing a is better than playing $\sigma_i(\omega_3)$ given σ_{-i}. Then, $B_i(\mathrm{RAT}_i) \not\subseteq \mathrm{RAT}_i$ for each $i \in \{1, 2\}$.

	a	b	c
a	$1,1$	$0,0$	$0,0$
b	$0,0$	$1,1$	$0,0$
c	$0,0$	$0,0$	$1,1$

E	$B_1(E)$	$B_2(E)$
$\emptyset$	$\emptyset$	$\emptyset$
$\{\omega_1\}$	$\emptyset$	$\emptyset$
$\{\omega_2\}$	$\emptyset$	$\emptyset$
$\{\omega_3\}$	$\emptyset$	$\emptyset$
$\{\omega_1, \omega_2\}$	$\{\omega_2, \omega_3\}$	$\{\omega_3\}$
$\{\omega_1, \omega_3\}$	$\{\omega_1\}$	$\{\omega_2\}$
$\{\omega_2, \omega_3\}$	$\{\omega_2\}$	$\{\omega_1\}$
Ω	Ω	Ω

Table 3: A Counterexample for Theorem 2. The left panel depicts the coordination game while the right panel depicts the players' beliefs on $(\Omega, \mathcal{D})$.

5.2 Discussions on Other Solution Concepts

This subsection briefly discusses other solution concepts. First, similar analyses and conclusion to Section 5.1 would be obtained for other "rationalizability" solution concepts such as Börgers (1993) (see also Bonanno and Tsakas, 2018; Hillas and Samet, 2020).

Second, another important solution concept is Nash equilibria. One of the well-known epistemic characterizations of a pure-strategy Nash equilibrium is that if, at a state ω, if every player is certain of the strategy profile σ (in the formal sense of this paper) and if every player is rational (i.e., $\omega \in \mathrm{RAT}_I$), then the resulting actions $\sigma(\omega)$ taken at that state constitute a pure-strategy Nash equilibrium.[31] The common certainty of the model would not be needed.

Third, I consider mixed-strategy Nash equilibria. A *(strategic) game* is a tuple $\Gamma = \langle (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$, where each A_i is a non-empty finite set of player i's actions, and $u_i : A \to \mathbb{R}$ is player i's von-Neumann Morgenstern utility function. I simply focus on players' probabilistic beliefs. Thus, a (belief) *model* of the game Γ is a tuple $\langle (\Omega, \mathcal{D}), (\tau_i)_{i \in I}, C^1, (\sigma_i)_{i \in I} \rangle$: each $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ is player i's type mapping, C^1 is the common 1-belief operator, and $\sigma_i : (\Omega, \mathcal{D}) \to (A_i, \mathcal{P}(A_i))$ is player i's strategy. In this context, player i is *rational* at ω if

$$\int u_i(\sigma(\tilde{\omega}))\tau_i(\omega)(d\tilde{\omega}) \geq \int u_i(a_i, \sigma_{-i}(\tilde{\omega}))\tau_i(\omega)(d\tilde{\omega}) \text{ for all } a_i \in A_i.$$

I start with the two-players case: $I = \{1, 2\}$. One of the well-known epistemic characterizations of mixed-strategy Nash equilibria is stated as follows (e.g., Aumann and Brandenburger, 1995; Stalnaker, 1994): if each player i is certain of the other's beliefs about i's strategy choice at ω (i.e., player i is certain of the conjecture $\tau_j(\omega) \circ \sigma_i^{-1} \in \Delta(A_i)$ where j is the opponent) and if each player i 1-believes that the other is

[31] See Aumann and Brandenburger (1995) and Stalnaker (1994) for pioneering papers in epistemic characterizations of Nash equilibria.

	L	R
U	1, 1, 1	0, 0, 0
D	0, 0, 0	0, 0, 0

A

	L	R
U	0, 0, 0	0, 0, 0
D	0, 0, 0	1, 1, 1

B

Table 4: Three-players Coordination Game (Example 6)

rational at ω, then the resulting pair of conjectures $(\tau_2(\omega)\circ\sigma_1^{-1}, \tau_1(\omega)\circ\sigma_2^{-1})$ constitutes a mixed-strategy Nash equilibrium. In this statement, the common certainty of the model is not required.[32] In this epistemic characterization, each player i is certain not of the other's type mapping τ_j but of the conjecture (j's beliefs about i's actions).

Next, consider the case in which $I = \{1, 2, \ldots, n\}$. Suppose that each player's type mapping τ_i is induced from a common prior μ (recall Expression (2) with respect to $\mu_i = \mu$). For ease of exposition, restrict attention to the case in which Ω is finite and the common prior puts positive probability to every state. If the players mutually 1-believe that they are rational at ω and if they are commonly certain of their conjectures at ω, then, for each player j, all the conjectures of players $i \in I \backslash \{j\}$ induce the same conjecture $\phi_j \in \Delta(A_j)$, and $(\phi_j)_{j\in I}$ is a mixed-strategy Nash equilibrium.

Now, I provide a simple example which illustrates that the common certainty of the model is not required for the above epistemic characterizations of mixed-strategy Nash equilibria.

Example 6. Consider the following three-players coordination game depicted by Table 4. Player 1 chooses a row, player 2 does a column, and player 3 does a matrix.

Let $(\Omega, \mathcal{D}) = (\{\omega_1, \omega_2, \ldots, \omega_8\}, \mathcal{P}(\Omega))$. Assume that there is a uniform common prior $\mu = (\frac{1}{8}, \frac{1}{8}, \ldots, \frac{1}{8})$. For player 1, let

$$\tau_1(\omega) = \begin{cases} (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0) & \text{if } \omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ (0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) & \text{if } \omega \in \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases}.$$

For player 2, let

$$\tau_2(\omega) = \begin{cases} (\frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0) & \text{if } \omega \in \{\omega_1, \omega_2, \omega_5, \omega_6\} \\ (0, 0, \frac{1}{4}, \frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}) & \text{if } \omega \in \{\omega_3, \omega_4, \omega_7, \omega_8\} \end{cases}.$$

For player 3, let

$$\tau_3(\omega) = \begin{cases} (\frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}0) & \text{if } \omega \in \{\omega_1, \omega_3, \omega_5, \omega_7\} \\ (0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}, 0, \frac{1}{4}) & \text{if } \omega \in \{\omega_2, \omega_4, \omega_6, \omega_8\} \end{cases}.$$

[32]I will provide an example in the context of $|I| \geq 3$, which requires a tighter condition on the players' beliefs.

Thus, the players are not commonly certain of the model (i.e., the profile of type mappings). Let

$$\sigma_1 = (U, U, U, U, D, D, D, D),$$
$$\sigma_2 = (L, L, R, R, L, L, R, R), \text{ and}$$
$$\sigma_3 = (A, B, A, B, A, B, A, B).$$

Now, it can be seen that the conditions for the above epistemic characterization are met, and their conjectures satisfy the following, which constitute a mixed-strategy Nash equilibrium in which every player is mixing with probability $\frac{1}{2}$:

$$\phi_1 = \tau_j(\omega) \circ \sigma_1^{-1} = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ on } (U, D) \text{ for each } j \neq 1,$$
$$\phi_2 = \tau_j(\omega) \circ \sigma_2^{-1} = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ on } (L, R) \text{ for each } j \neq 2, \text{ and}$$
$$\phi_3 = \tau_j(\omega) \circ \sigma_3^{-1} = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ on } (A, B) \text{ for each } j \neq 3.$$

5.3 Communication Leading to Agreement or No Trade

In epistemic game theory, there is a strand of literature studying whether communication between players lead to eliminating differences in private information. The pioneering papers are Geanakoplos and Polemarchakis (1982) and Sebenius and Geanakoplos (1983).

Geanakoplos and Polemarchakis (1982) show that "if two agents have the same priors and if they know the type of information the other is capable of having obtained (they know each other's "information partitions"), then simply the *terse* announcement of probability assessments back and forth *will* leads to a common judgment."

Sebenius and Geanakoplos (1983) demonstrate that "if the two parties share priors and their information partitions are common knowledge, simple discussion of the acceptability of any proposed bet [reveals] enough about the parties' private information to render the bet unacceptable."

To illustrate that the common certainty assumption would not be needed to derive these results, this subsection studies Hart and Tauman (2004)'s ingenious example through the lens of this paper. Hart and Tauman (2004) recast one of the examples of Geanakoplos and Polemarchakis (1982) to illustrate that sudden changes in market behavior can come from traders' endogenous information processing.

Example 7. Suppose that there are two traders A (Alice) and B (Bob). Day after day, Alice is selling and Bob buying, until when Bob switches from buying to selling.

Let $\Omega = \{1, 2, \ldots, 9\}$ and $\mathcal{D} = \mathcal{P}(\Omega)$, and let μ be the uniform common prior. Each state denotes an economic outcome.

At date 1, Alice's belief (indeed, knowledge) operator B_A^1 is induced by the possibility correspondence b_A^1 given by $b_A^1(1) = b_A^1(2) = b_A^1(3) = \{1,2,3\}$, $b_A^1(4) = b_A^1(5) = b_A^1(6) = \{4,5,6\}$, and $b_A^1(7) = b_A^1(8) = b_A^1(9) = \{7,8,9\}$. With some abuse of notation, I denote b_A^1 by the partition $b_A^1 = \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$. Bob's belief (indeed, knowledge) operator B_B^1 is induced by the possibility correspondence $b_B^1 = \{\{1,2,3,4\},\{5,6,7,8\},\{9\}\}$.

I denote by b_i^t each trader i's possibility correspondence at time t. At time t, then, for each trader $i \in I = \{A, B\}$, i's type mapping τ_i^t is given by

$$\tau_i^t(\omega)(E) = \mu(E \mid b_i^t(\omega)) \text{ for each } (\omega, E) \in \Omega \times \mathcal{D}.$$

Thus, at each date t, each trader is certain of her/his type mapping τ_i^t.

Assume that the true state is $\omega^* = 1$. Let $E^* = \{1,5,9\}$, which is interpreted as the event of a "bad" outcome. At each date t, each trader i is assumed to behave according to the following rule:

$$x_i^t(\omega) = \begin{cases} \text{Sell,} & \text{if the trader's belief in } E^* \text{ at } \omega \text{ is 0.3 or more} \\ \text{Buy,} & \text{if the trader's belief in } E^* \text{ at } \omega \text{ is less than 0.3} \end{cases}.$$

The behavior rule of trader i at date t depends on i's possibility correspondence at that date.

At the beginning of date 1, Alice's trading rule x_A^1 is

$$x_A^1(\omega) = \text{Sell for all } \omega \in \Omega.$$

This is because $\tau_A^1(\cdot)(E^*) = \frac{1}{3} > 0.3$. In contrast, Bob's trading rule x_B^1 is

$$x_B^1(\omega) = \begin{cases} \text{Sell} & \text{if } \omega = 9 \\ \text{Buy} & \text{if } \omega \neq 9 \end{cases}.$$

This is because $\tau_B^1(9)(E^*) = 1 > 0.3$ and $\tau_B^1(\omega)(E^*) = \frac{1}{4} < 0.3$ if $\omega \neq 9$. While Alice's action at ω^* is Sell, Bob's Buy.

At the end of date t, each trader i updates the possibility correspondence b_i^{t+1} so that, at the beginning of date $t+1$, trader i is certain of i's type mapping $\tau_i^{t+1} : \Omega \to \Delta(\Omega)$ with respect to $\{[\tau_i^t(\omega)]\}_{\omega\in\Omega} \cup \{[x_j^t(\omega)]\}_{\omega\in\Omega}$, where j denotes the opponent. In words, at the beginning of date $t+1$, each trader i updates her/his knowledge and probabilistic belief in ways such that trader i has observed the trading rule of the opponent at the previous date and that trader i is certain of her/his type mapping.

Thus, at the beginning of date 2, Alice's possibility correspondence is given by the coarsest partition that is finer than b_A^1 and $\{[x_B^1(\omega)]\}_{\omega\in\Omega}$: $b_A^2 = \{\{1,2,3\},\{4,5,6\},\{7,8\},\{9\}\}$. Bob's possibility correspondence is given by $b_B^2 = b_B^1$, because no new information is revealed from Alice's trading rule at date 1, i.e., $[x_A^1(\cdot)] = \Omega$.

At date 2, accordingly, while $x_B^2 = x_B^1$,

$$x_A^2(\omega) = \begin{cases} \text{Sell} & \text{if } \omega \notin \{7,8\} \\ \text{Buy} & \text{if } \omega \in \{7,8\} \end{cases}.$$

While Alice's action at ω^* is Sell, Bob's Buy.

At date 3, while Alice's possibility correspondence is given by $b_A^3 = b_A^2$, Bob's possibility correspondence is given by: $b_B^3 = \{\{1,2,3,4\},\{5,6\},\{7,8\},\{9\}\}$. Accordingly, while $x_A^3 = x_A^2$,

$$x_B^3(\omega) = \begin{cases} \text{Sell} & \text{if } \omega \in \{5,6,9\} \\ \text{Buy} & \text{if } \omega \notin \{5,6,9\} \end{cases}.$$

While Alice's action at ω^* is Sell, Bob's Buy.

At date 4, while Bob's possibility correspondence is given by $b_B^4 = b_B^3$, Alice's possibility correspondence is given by: $b_A^4 = \{\{1,2,3\},\{4\},\{5,6\},\{7,8\},\{9\}\}$. Accordingly, while $x_B^4 = x_B^3$,

$$x_A^4(\omega) = \begin{cases} \text{Sell} & \text{if } \omega \notin \{4,7,8\} \\ \text{Buy} & \text{if } \omega \in \{4,7,8\} \end{cases}.$$

While Alice's action at ω^* is Sell, Bob's Buy.

At date 5, while Alice's possibility correspondence is given by $b_A^5 = b_A^4$, Bob's possibility correspondence is given by: $b_A^5 = \{\{1,2,3\},\{4\},\{5,6\},\{7,8\},\{9\}\}$. Accordingly, while $x_A^5 = x_A^4$,

$$x_B^5(\omega) = \begin{cases} \text{Sell} & \text{if } \omega \notin \{4,7,8\} \\ \text{Buy} & \text{if } \omega \in \{4,7,8\} \end{cases}.$$

Now, Alice and Bob both sell at ω^*.

On the one hand, in the literature, the common knowledge of a model (i.e., the players' partitions) are informally assumed. On the other hand, in Example 7, according to the definition of this paper, Alice and Bob are not commonly certain of the model (i.e., the profile of their type mappings). While each trader is certain of her/his own type mapping, she/he is not certain of the opponent's type mapping. In fact, if both traders are commonly certain of their type mappings, then each trader would be able to utilize the opponent's information. This would lead to belief updating unless otherwise each trader is at least as informed as the other. Thus, if the traders are commonly certain of their type mappings, then their possibility correspondences would be given by $b_A^1 = b_B^2 = \{\{1,2,3\},\{4\},\{5,6\},\{7,8\},\{9\}\}$ at the outset. This suggests that, in the sense of Definition 1, what is really needed is not the common certainty of the model (where the model is construed as the profile of the type mappings) but the common certainty of the behavior rules.[33] The analysis of this subsection shows that the common certainty of the behavior rules can indeed be formalized.

6 Conclusion

This paper asked two questions. First, what does it mean by the statement that the players in a belief model are commonly (meta-)certain of the model itself? Second, what role does such meta-certainty assumption play in epistemic characterizations of game-theoretic solution concepts? The paper started with expanding the objects of the players' beliefs from events to signals (functions) defined on the underlying states. A player is certain of the value of a signal x at a state if, the player believes, at the state, any observational content that holds at the state. If she is certain of the value of the signal x at every state, then she is certain of the signal. The common certainty of the signal was analogously defined: the players are commonly certain of the value of the signal x if the players commonly believe, at the state, any observational content that holds at the state. The players are commonly certain of the signal x if they are commonly certain of its value at every state. Then, the players' belief-generating maps (i.e., type mappings) and strategies became objects of their beliefs. A player is certain of her own type mapping iff her belief satisfies the positive and negative introspective properties. For probabilistic beliefs, the Harsanyi property is the strongest form of the certainty of own type mapping.

The main result regarding the first question is: the players are commonly certain of the profile of the players' type mappings (i.e., the belief model) iff, for any event E which some player i believes at some state, it is common belief that player i believes E at that state. I summarize two implications. First, the common belief operator collapses into the mutual belief operator when the players are commonly certain of the model. This is because, whenever everybody believes an event, everybody believes that everybody believes the event. Second, if the players are commonly certain of their type mappings and if each player is certain of her own strategy, then the players are commonly certain of their strategies. The negative results have some similarity with Aumann (1976)'s impossibility to agree-to-disagree result and with information revelation at a rational-expectations equilibrium.

Using the formalization of certainty of signals, the second objective was to elucidate the role of the common meta-certainty assumption on epistemic characterizations of game-theoretic solution concepts. The paper studied the solution concept of iterated elimination of strictly dominated actions (IESDA). Informally, if the players are "logical," if they are (meta-)certain of a game, and if they commonly believe their

[33]In the example (i.e., Hart and Tauman, 2004), each player's behavior rule associates, with underlying states, the Sell/Buy actions. In Sebenius and Geanakoplos (1983), it associates, with underlying states, the Yes/No announcements. In Geanakoplos and Polemarchakis (1982), it associates, with underlying states, values of posterior beliefs.

rationality, then their resulting actions survive IESDA. Formally, the paper showed: if the players' beliefs satisfy Monotonicity, Consistency, and Finite Conjunction, if each player is certain of her qualitative-type mapping (or if each player's beliefs are compatible with informativeness), and if the players commonly believe their rationality, then their resulting actions survive IESDA. Together with other applications, the paper demonstrated the positive result that the common meta-certainty assumption may not be needed.

Regarding the first question of the paper, as this paper is the first paper that systematically studies the meta-certainty of a belief model, there may be other formulations of the meta-certainty of a belief model that can also be used in game-theoretic analyses. Such formalizations would be interesting future research avenues. Regarding the second question of the paper, applications to other solution concepts in economic models such as rational expectation equilibria would be interesting avenues for future research.

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A Appendix

A.1 Proofs

A.1.1 Section 3

Proof of Remark 5. Since the conditions in Part (2) imply those in Part (1), it suffices to show Part (1). First, it follows from Consistency, Finite Conjunction, and Monotonicity that $\emptyset = B_i(E) \cap B_i(E^c) = B_i(E \cap E^c) = B_i(\emptyset) \in \mathcal{B}_i$. Second, $\mathcal{B}_i$ is closed under countable intersection because Countable Conjunction and Monotonicity imply $\bigcap_{n\in\mathbb{N}} B_i(E_n) = B_i(\bigcap_{n\in\mathbb{N}} E_n) \in \mathcal{B}_i$. Third, I show that $\mathcal{B}_i$ is closed under complementation by proving $(\neg B_i)(\cdot) = B_i(\neg B_i)(\cdot)$. Negative Introspection, Consistency, and Positive Introspection imply $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot) \subseteq (\neg B_i)B_i(\cdot) \subseteq (\neg B_i)(\cdot)$. □

Proof of Remark 6. I prove only Part (1), as the proof for Part (2) is similar. Suppose $B_i(\Omega) = \Omega$. Take any constant signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. Fix $\omega \in \Omega$. For any $F \in \mathcal{X}$ with $x(\omega) \in F$, $x^{-1}(F) = \Omega$. Thus, $x^{-1}(F) \subseteq \Omega = B_i(E)$ for any $E \in \mathcal{D}$ with $\Omega \subseteq E$ (i.e., $E = \Omega$). Conversely, take $\omega \in \Omega$, and consider the constant signal $x : (\Omega, \mathcal{D}) \to (\{\omega\}, \{\{\omega\}\})$. Since player i is certain of it, $\Omega = x^{-1}(\{\omega\}) \subseteq B_i(x^{-1}(\{\omega\})) = B_i(\Omega)$. □

Proof of Remark 7. Assume that player i is certain of every $x_\alpha : (\Omega, \mathcal{D}) \to (X_\alpha, \mathcal{X}_\alpha)$. Observe that for any $\pi_\alpha^{-1}(F_\alpha)$ with $F_\alpha \in \mathcal{X}_\alpha$, $x^{-1}(\pi_\alpha^{-1}(F_\alpha)) = x_\alpha^{-1}(F_\alpha) \in \mathcal{D}$. Thus, if $\omega \in x^{-1}(\pi_\alpha^{-1}(F_\alpha)) \subseteq E$, then $\omega \in B_i(E)$.

Conversely, suppose that player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. Since $x_\alpha = \pi_\alpha \circ x$, for any $F_\alpha \in \mathcal{X}_\alpha$, $x_\alpha^{-1}(F_\alpha) = (\pi_\alpha \circ x)^{-1}(F_\alpha) = x^{-1}(\pi_\alpha^{-1}(F_\alpha)) \in \mathcal{D}$. If $\omega \in x_\alpha^{-1}(F_\alpha) \subseteq E$, then $\omega \in B_i(E)$. □

Proof of Proposition 1A. 1. For (1a), i is certain of t_{B_i} with respect to $\{\beta_E \mid E \in \mathcal{D}\}$ iff $t_{B_i}^{-1}(\beta_E) = B_i(E)$ is a basis to i: if $B_i(E) \subseteq F$ then $B_i(E) \subseteq B_i(F)$ (note that $B_{t_{B_i}} = B_i$). When B_i satisfies Monotonicity, the latter condition reduces to $B_i(E) \subseteq B_iB_i(E)$.

Likewise, for (1b), i is certain of t_{B_i} with respect to $\{\neg\beta_E \mid E \in \mathcal{D}\}$ iff $\neg t_{B_i}^{-1}(\beta_E) = (\neg B_i)(E)$ is a basis to i: $(\neg B_i)(E) \subseteq F$ implies $(\neg B_i)(E) \subseteq B_i(F)$. When B_i satisfies Monotonicity, the latter condition reduces to $(\neg B_i)(E) \subseteq B_i(\neg B_i)(E)$. Then, (1c) follows from the previous two parts.

2. I prove only (2b), as the proof for (2a) is similar. Also, it suffices to show the "if" part, as the "only if" part follows from the previous assertion. It follows from the discussions on Remark 5 that, under the assumptions on B_i, $\mathcal{B}_i = \{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ is a sub-σ-algebra of $\mathcal{D}$. Since B_i satisfies Positive Introspection, $\mathcal{B}_i \subseteq \mathcal{J}_{B_i}$. Since $t_i^{-1}(\beta_E) = B_i(E) \in \mathcal{B}_i$ and since $\mathcal{B}_i$ is a σ-algebra, $t_i^{-1}(\mathcal{D}_M) = \sigma(\{t_i^{-1}(\beta_E) \in \mathcal{D} \mid E \in \mathcal{D}\}) \subseteq \sigma(\mathcal{B}_i) = \mathcal{B}_i \subseteq \mathcal{J}_{B_i}$.

□

Proof of Proposition 1B. 1. For (1a), player i is certain of τ_i with respect to $\{\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}$ iff, for any $F \in \mathcal{D}$ with $B_{\tau_i}^p(E) \subseteq F$, $B_{\tau_i}^p(E) \subseteq B_i(F)$, as $B_{\tau_i}^p(E) = \tau_i^{-1}(\beta_E^p)$. When B_i satisfies Monotonicity, the latter condition reduces to $B_{\tau_i}^p(E) \subseteq B_iB_{\tau_i}^p(E)$.

For (1b), player i is certain of τ_i with respect to $\{\neg\beta_E^p \mid (E,p) \in \mathcal{D}\times[0,1]\}$ iff, for any $F \in \mathcal{D}$ with $(\neg B_{\tau_i}^p)(E) \subseteq F$, $(\neg B_{\tau_i}^p)(E) \subseteq B_i(F)$, as $(\neg B_{\tau_i}^p)(E) = \tau_i^{-1}(\neg\beta_E^p)$. When B_i satisfies Monotonicity, the latter condition reduces to $(\neg B_{\tau_i}^p)(E) \subseteq B_i(\neg B_{\tau_i}^p)(E)$. Then, (1c) follows from the previous two parts.

2. (a) If player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ then B_i satisfies Positive Certainty. Conversely, let B_i satisfy Positive Certainty. By (1a), $\tau_i^{-1}(\{\beta_E^p \mid (E,p) \in \mathcal{D} \times [0,1]\}) \subseteq \mathcal{J}_{B_i}$. Since B_i satisfies Truth Axiom and Negative Introspection, $\mathcal{J}_{B_i}$ is a sub-σ-algebra of $\mathcal{D}$. Thus, $\tau_i^{-1}(\mathcal{D}_\Delta) \subseteq \mathcal{J}_{B_i}$. Hence, player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$.

   Next, I show that, since B_i satisfies Truth Axiom and Negative Introspection, Positive Certainty is equivalent to Negative Certainty. Assume Positive Certainty. Then, $(\neg B_{\tau_i}^p) = (\neg B_i)B_{\tau_i}^p = B_i(\neg B_i)B_{\tau_i}^p = B_i(\neg B_{\tau_i}^p)$. The first and third equalities follow from Positive Certainty and Truth Axiom, and the second from Negative Introspection and Truth Axiom.

   Conversely, assume Negative Certainty. Then, $B_{\tau_i}^p = (\neg B_i)(\neg B_{\tau_i}^p) = B_i(\neg B_i)(\neg B_{\tau_i}^p) = B_iB_{\tau_i}^p$. The first and third equalities follow from Negative Certainty and Truth Axiom, and the second from Negative Introspection and Truth Axiom.

   (b) It is sufficient to prove the "if" part. First, it follows from the assumptions and the discussions on Remark 5 that, under the assumptions on B_i, $\mathcal{B}_i =$

$\{B_i(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ is a sub-σ-algebra of $\mathcal{D}$.

Second, since B_i satisfies Monotonicity and Positive Introspection, $\mathcal{B}_i \subseteq \mathcal{J}_{B_i}$. Third, I show that Positive Certainty, Negative Certainty, and Consistency of B_i imply $B^p_{\tau_i}(E) = B_i B^p_{\tau_i}(E)$. The "$\subseteq$" part is Positive Certainty. Conversely, it follows from Negative Certainty and Consistency that $(\neg B^p_{\tau_i})(E) \subseteq B_i(\neg B^p_{\tau_i})(E) \subseteq (\neg B_i)B^p_{\tau_i}(E)$. Then, $B_i B^p_{\tau_i}(E) \subseteq B^p_{\tau_i}(E)$.

Fourth, since $\tau_i^{-1}(\beta^p_E) = B^p_{\tau_i}(E) = B_i B^p_{\tau_i}(E) \in \mathcal{B}_i$ and since $\mathcal{B}_i$ is a σ-algebra, $\tau_i^{-1}(\mathcal{D}_\Delta) = \sigma(\{\tau_i^{-1}(\beta^p_E) \in \mathcal{D} \mid (E,p) \in \mathcal{D} \times [0,1]\}) \subseteq \sigma(\mathcal{B}_i) = \mathcal{B}_i \subseteq \mathcal{J}_{B_i}$.

(c) It suffices to prove the "if" part. First, I show below that $\mathcal{B}^1_{\tau_i} := \{B^1_{\tau_i}(E) \in \mathcal{D} \mid E \in \mathcal{D}\}$ is a sub-σ-algebra of $\mathcal{D}$. Second, since B_i satisfies Monotonicity and Positive Certainty, $\mathcal{B}^1_{\tau_i} \subseteq \mathcal{J}_{B_i}$. Third, I show below that Positive Certainty, Negative Certainty, and Consistency of $B^1_{\tau_i}$ (i.e., $B^1_{\tau_i}(E) \subseteq (\neg B^1_{\tau_i})(E^c)$) imply $B^p_{\tau_i}(E) = B^1_{\tau_i} B^p_{\tau_i}(E)$. Fourth, since $\tau_i^{-1}(\beta^p_E) = B^p_{\tau_i}(E) = B^1_{\tau_i} B^p_{\tau_i}(E) \in \mathcal{B}^1_{\tau_i}$ and since $\mathcal{B}^1_{\tau_i}$ is a σ-algebra, $\tau_i^{-1}(\mathcal{D}_\Delta) = \sigma(\{\tau_i^{-1}(\beta^p_E) \in \mathcal{D} \mid (E,p) \in \mathcal{D} \times [0,1]\}) \subseteq \sigma(\mathcal{B}^1_{\tau_i}) = \mathcal{B}^1_{\tau_i} \subseteq \mathcal{J}_{B_i}$.

Hence, I show the first statement that $\mathcal{B}^1_{\tau_i}$ is a sub-σ-algebra of $\mathcal{D}$. First, since $\tau_i(\cdot)(\emptyset) = 0$, $\emptyset = B^1_{\tau_i}(\emptyset) \in \mathcal{B}^1_{\tau_i}$. Second, since $B^1_{\tau_i}$ satisfies Monotonicity and Countable Conjunction, $\mathcal{B}^1_{\tau_i}$ is closed under countable intersection. Third, as in the proof of Remark 5, to prove that $\mathcal{B}^1_{\tau_i}$ is closed under complementation, it is sufficient to show $(\neg B^1_{\tau_i})(\cdot) = B^1_{\tau_i}(\neg B^1_{\tau_i})(\cdot)$. However, this property follows from $B^1_{\tau_i}(E) \subseteq (\neg B^1_{\tau_i})(E^c)$, $B^1_{\tau_i}(\cdot) \subseteq B^1_{\tau_i} B^1_{\tau_i}(\cdot)$, and $(\neg B^1_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^1_{\tau_i})(\cdot)$. Indeed, $(\neg B^1_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^1_{\tau_i})(\cdot) \subseteq (\neg B^1_{\tau_i}) B^1_{\tau_i}(\cdot) \subseteq (\neg B^1_{\tau_i})(\cdot)$.

Next, I show the third statement $B^p_{\tau_i}(E) = B^1_{\tau_i} B^p_{\tau_i}(E)$. It follows from Positive Certainty and Entailment that $B^p_{\tau_i}(E) \subseteq B_i B^p_{\tau_i}(E) \subseteq B^1_{\tau_i} B^p_{\tau_i}(E)$. Conversely, it follows from Negative Certainty and Entailment that $(\neg B^p_{\tau_i})(E) \subseteq B_i(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(E)$. Then, it follows from Consistency of $B^1_{\tau_i}$ that $B^1_{\tau_i} B^p_{\tau_i}(E) \subseteq (\neg B^1_{\tau_i})(\neg B^p_{\tau_i})(E) \subseteq B^p_{\tau_i}(E)$.

$\square$

Proof of Remark 8. First, since each player i is certain of her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ within a model $\langle (\Omega, \mathcal{D}), (B^1_{\tau_i})_{i \in I} \rangle$, it follows from Proposition 1B that Negative Certainty $(\neg B^p_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(\cdot)$ holds. Second, I show below that $B^p_{\tau_i} B^p_{\tau_i}(\cdot) \subseteq B^p_{\tau_i}(\cdot)$. Third, I show that the mutual p-beliefs also satisfies $B^p_I B^p_I(\cdot) \subseteq B^p_I(\cdot)$, where $B^p_I(\cdot) := \bigcap_{i \in I} B^p_{\tau_i}(\cdot)$ as in Section 2.1. It means that the chain of mutual p-beliefs is decreasing. Fourth, since mutual p-beliefs are preserved for a decreasing sequence of events (i.e., if $E_n \downarrow E$ then $B^p_I(E_n) \downarrow B^p_I(E)$), the common p-belief operator $C^p : \mathcal{D} \to \mathcal{D}$ defined according to Expression (1) reduces to the iteration of mutual p-beliefs (see Monderer and Samet, 1989).

Thus, it suffices to prove the second and third statements. I show the second statement. If $p = 0$ then $B^p_{\tau_i} B^p_{\tau_i}(\cdot) = \Omega = B^p_{\tau_i}(\cdot)$. Thus, let $p > 0$. Let $\omega \in B^p_{\tau_i} B^p_{\tau_i}(E)$.

Suppose to the contrary that $\omega \in (\neg B^p_{\tau_i})(E)$. Then, $\omega \in (\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(E)$. Then, $\tau_i(\omega)((\neg B^p_{\tau_i})(E)) = 1$ and $\tau_i(\omega)(B^p_{\tau_i}(E)) \geq p > 0$, and thus $1 = \tau_i(\omega)(B^p_{\tau_i}(E) \cup (\neg B^p_{\tau_i})(E)) = 1 + p > 1$, a contradiction.

Turning to the third statement, Monotonicity of $B^p_{\tau_i}$ implies that $B^p_{\tau_i} B^p_I(\cdot) \subseteq B^p_{\tau_i} B^p_{\tau_i}(\cdot) \subseteq B^p_{\tau_i}(\cdot)$. Taking the intersection over all $i \in I$, $B^p_I B^p_I(\cdot) \subseteq B^p_I(\cdot)$. $\square$

Proof of Remark 9. For the "only if" part, suppose that player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$. Take $\omega' \in b_{B_i}(\omega)$. For any $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$, it follows from the supposition that $\omega \in B_i(x^{-1}(F))$. By the definition of b_{B_i}, I have $\omega' \in b_{B_i}(\omega) \subseteq x^{-1}(F)$. For the "if" part, assume the Kripke property. Suppose that possibility implies informativeness. Take any $\omega \in \Omega$ and $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$. To show $\omega \in B_i(x^{-1}(F))$, it is enough to show $b_{B_i}(\omega) \subseteq x^{-1}(F)$. Now, if $\omega' \in b_{B_i}(\omega)$, then it follows from the supposition that $\omega' \in x^{-1}(F)$. $\square$

Proof of Proposition 2A. Observe that if ω' is at least as informative to i as ω according to t_{B_i} (i.e., $\omega' \in (\uparrow t_{B_i}(\omega))$), then

$$b_{B_i}(\omega') = \bigcap \{E \in \mathcal{D} \mid t_{B_i}(\omega')(E) = 1\} \subseteq \bigcap \{E \in \mathcal{D} \mid t_{B_i}(\omega)(E) = 1\} = b_{B_i}(\omega).$$

Moreover, if t_{B_i} satisfies the Kripke property, then the converse also holds: $b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$ implies $\omega' \in (\uparrow t_{B_i}(\omega))$. This is because, if $t_{B_i}(\omega)(E) = 1$ then $b_{B_i}(\omega') \subseteq b_{B_i}(\omega) \subseteq E$ and thus $t_{B_i}(\omega')(E) = 1$.

1. (a) Since Truth Axiom yields $\omega' \in b_{B_i}(\omega')$ for all $\omega' \in \Omega$, it follows that $\omega' \in b_{B_i}(\omega') \subseteq b_{B_i}(\omega)$ for all $\omega' \in (\uparrow t_{B_i}(\omega))$. Conversely, Truth Axiom follows from $\omega \in (\uparrow t_{B_i}(\omega)) \subseteq b_{B_i}(\omega)$ for all $\omega \in \Omega$.

   (b) Suppose $\omega' \in b_{B_i}(\omega)$. For any $F \in \mathcal{D}$ with $t_{B_i}(\omega)(F) = 1$, it follows from Positive Introspection that $t_{B_i}(\omega)(t^{-1}_{B_i}(\beta_F)) = 1$. By the supposition, $\omega' \in b_{B_i}(\omega) \subseteq t^{-1}_{B_i}(\beta_F)$, and hence $t_{B_i}(\omega')(F) = 1$. Thus, $\omega' \in (\uparrow t_{B_i}(\omega))$.

   Conversely, let B_i satisfy the Kripke property, and assume $b_{B_i}(\cdot) \subseteq (\uparrow t_{B_i}(\cdot))$. Suppose $\omega \in B_i(E)$. In order to show $\omega \in B_i B_i(E)$, it is enough to prove $\omega' \in B_i(E)$ for all $\omega' \in b_{B_i}(\omega)$. Take any $\omega' \in b_{B_i}(\omega)$. Since $\omega' \in (\uparrow t_{B_i}(\omega))$ and $\omega \in B_i(E)$, it follows that $\omega' \in B_i(E)$.

   (c) The proof is analogous to Part (1b). Suppose $\omega' \in b_{B_i}(\omega)$. Suppose to the contrary that $\omega' \notin (\downarrow t_{B_i}(\omega))$, i.e., $t_{B_i}(\omega)(F) = 0 < 1 = t_{B_i}(\omega')(F)$ for some $F \in \mathcal{D}$. By Negative Introspection, $t_{B_i}(\omega)(\neg t^{-1}_{B_i}(\beta_F)) = 1$, and thus $\omega' \in b_{B_i}(\omega) \subseteq \neg t^{-1}_{B_i}(\beta_F)$, and hence $t_{B_i}(\omega')(F) = 0$, a contradiction.

   Conversely, let B_i satisfy the Kripke property, and suppose $b_{B_i}(\cdot) \subseteq (\downarrow t_{B_i}(\cdot))$. If $\omega \notin B_i(E)$, then $b_{B_i}(\omega) \cap E^c \neq \emptyset$. In order to establish $\omega \in B_i(\neg B_i)(E)$, it is enough to show that $b_{B_i}(\omega') \cap E^c \neq \emptyset$ for all $\omega' \in b_{B_i}(\omega)$. Take any $\omega' \in b_{B_i}(\omega)$. Since $\omega' \in (\downarrow t_{B_i}(\omega))$ and since $t_{B_i}(\omega)(E) = 0$, it follows $t_{B_i}(\omega')(E) = 0$, i.e., $b_{B_i}(\omega') \cap E^c \neq \emptyset$.

2. The assertion follows from Part (1).

□

Proof of Proposition 2B. 1. It can be seen that

$$[\tau_i(\omega)] = \bigcap_{(E,p)\in\mathcal{D}\times[0,1]:\omega\in B^p_{\tau_i}(E)} B^p_{\tau_i}(E) = \bigcap_{(E,p)\in\mathcal{D}\times[0,1]:\omega\in(\neg B^p_{\tau_i})(E)} (\neg B^p_{\tau_i})(E). \quad \text{(A.1)}$$

Now, $B^p_{\tau_i}(\cdot) \subseteq B_i B^p_{\tau_i}(\cdot)$ implies $b_{B_i}(\omega) \subseteq B^p_{\tau_i}(E)$ for any $(E,p) \in \mathcal{D}\times[0,1]$ with $\omega \in B^p_{\tau_i}(E)$. Likewise, $(\neg B^p_{\tau_i})(\cdot) \subseteq B_i(\neg B^p_{\tau_i})(\cdot)$ implies $b_{B_i}(\omega) \subseteq (\neg B^p_{\tau_i})(E)$ for any $(E,p) \in \mathcal{D}\times[0,1]$ with $\omega \in (\neg B^p_{\tau_i})(E)$. In either case, $b_{B_i}(\omega) \subseteq [\tau_i(\omega)]$.

2. Take $(E,p) \in \mathcal{D}\times[0,1]$. Since $b_{B_i}(\omega) \subseteq [\tau_i(\omega)] \subseteq B^p_{\tau_i}(E)$ for any $\omega \in B^p_{\tau_i}(E)$, it follows from the Kripke property that $B^p_{\tau_i}(E) \subseteq B_i B^p_{\tau_i}(E)$. Likewise, since $b_{B_i}(\omega) \subseteq [\tau_i(\omega)] \subseteq (\neg B^p_{\tau_i})(E)$ for any $\omega \in (\neg B^p_{\tau_i})(E)$, it follows from the Kripke property that $(\neg B^p_{\tau_i})(E) \subseteq B_i(\neg B^p_{\tau_i})(E)$.

□

Proof of Proposition 3B. 1. Let τ_i satisfy the Harsanyi property. For any $\omega \in \Omega$ and $\tau_i^{-1}(\{\tau_i(\omega)\}) = [\tau_i(\omega)] \subseteq E$, if $\omega' \in \tau_i^{-1}(\{\tau_i(\omega)\}) = [\tau_i(\omega)]$ then $\tau_i(\omega')(E) = \tau_i(\omega)(E) = 1$, i.e., $\omega' \in B^1_{\tau_i}(E)$. Thus, player i is certain of $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \{\{\tau_i(\omega)\} \mid \omega \in \Omega\})$. Conversely, for any $E \in \mathcal{D}$ with $\tau_i^{-1}(\{\tau_i(\omega)\}) = [\tau_i(\omega)] \subseteq E$, $\omega \in B^1_{\tau_i}(E)$, i.e., $\tau_i(\omega)(E) = 1$.

2. Let $\mathcal{D}$ be generated by a countable algebra $\mathcal{A}$. Let $[0,1]_{\mathbb{Q}} := [0,1]\cap\mathbb{Q}$. Similarly to Expression (A.1), one can show:

$$[\tau_i(\omega)] = \bigcap_{(E,p)\in\mathcal{A}\times[0,1]_{\mathbb{Q}}:\omega\in B^p_{\tau_i}(E)} B^p_{\tau_i}(E) = \bigcap_{(E,p)\in\mathcal{A}\times[0,1]_{\mathbb{Q}}:\omega\in(\neg B^p_{\tau_i})(E)} (\neg B^p_{\tau_i})(E) \in \mathcal{D}.$$

Then, it follows from Part (1) that (2a) and (2b) are equivalent. Part (2b) implies (2c), and (2c) implies (2d) and (2e).

Now, I show that (2d) implies (2a). Assume (2d). Fix $\omega \in \Omega$. For any $(E,p) \in \mathcal{A}\times[0,1]_{\mathbb{Q}}$ with $\omega \in B^p_{\tau_i}(E)$, I have $B^p_{\tau_i}(E) \subseteq B^1_{\tau_i} B^p_{\tau_i}(E)$. Since $\mathcal{A}\times[0,1]_{\mathbb{Q}}$ is countable, take the intersection over all $(E,p) \in \mathcal{A}\times[0,1]_{\mathbb{Q}}$ with $\omega \in B^p_{\tau_i}(E)$ to obtain:

$$[\tau_i(\omega)] = \bigcap_{(E,p)} B^p_{\tau_i}(E) \subseteq \bigcap_{(E,p)} B^1_{\tau_i} B^p_{\tau_i}(E) \subseteq B^1_{\tau_i}\left(\bigcap_{(E,p)} B^p_{\tau_i}(E)\right) = B^1_{\tau_i}([\tau_i(\omega)]).$$

Thus, (2a) holds.

Likewise, I show that (2e) implies (2a). Assume (2e). Fix $\omega \in \Omega$. For any $(E,p) \in \mathcal{A}\times[0,1]_{\mathbb{Q}}$ with $\omega \in (\neg B^p_{\tau_i})(E)$, I have $(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(E)$.

Since $\mathcal{A} \times [0,1]_{\mathbb{Q}}$ is countable, take the intersection over all $(E,p) \in \mathcal{A} \times [0,1]_{\mathbb{Q}}$ with $\omega \in (\neg B^p_{\tau_i})(E)$ to obtain:

$$[\tau_i(\omega)] = \bigcap_{(E,p)} (\neg B^p_{\tau_i})(E) \subseteq \bigcap_{(E,p)} B^1_{\tau_i}(\neg B^p_{\tau_i})(E) \subseteq B^1_{\tau_i}\left(\bigcap_{(E,p)} (\neg B^p_{\tau_i})(E)\right) = B^1_{\tau_i}([\tau_i(\omega)]).$$

Hence, (2a) holds.

□

Proof of Proposition 3A. 1. Suppose (1a). If $[t_{B_i}(\omega)] \subseteq E$, then $\omega \in [t_{B_i}(\omega)]$ implies $\omega \in B_i(E)$. Conversely, suppose (1b). If $\omega' \in [t_{B_i}(\omega)] = t^{-1}_{B_i}(\{t_{B_i}(\omega)\}) \subseteq E$, then $t_{B_i}(\omega)(E) = t_{B_i}(\omega')(E) = 1$, and thus $\omega \in B_i(E)$.

2. Suppose $\omega' \in [t_{B_i}(\omega)] \subseteq E$. Since B_i satisfies Positive Introspection and Negative Introspection, Proposition 2A implies that $b_{B_i}(\omega') \subseteq [t_{B_i}(\omega')] = [t_{B_i}(\omega)] \subseteq E$. By the Kripke property, $\omega' \in B_i(E)$.

3. Without loss, assume (1b). If $b_{B_i}(\omega) \subseteq E$, then it follows from Truth Axiom of B_i and Proposition 2A that $[t_{B_i}(\omega)] \subseteq b_{B_i}(\omega) \subseteq E$. Then, $\omega \in B_i(E)$.

□

A.1.2 Section 4

Proof of Theorem 1A. 1. Suppose that the players are commonly certain of the profile of qualitative-type mappings. Since player i is certain of her own qualitative-type mapping, it follows from Proposition 1A that Positive Introspection and Negative Introspection hold: $B_i(\cdot) \subseteq B_iB_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq B_i(\neg B_i)(\cdot)$. Next, since player i is certain of player j's qualitative-type mapping, it follows from Remark 10A that $B_j(\cdot) \subseteq B_iB_j(\cdot)$. Since B_j satisfies Truth Axiom and since B_i satisfies Monotonicity, $B_iB_j(\cdot) \subseteq B_i(\cdot)$. Thus, $B_j(\cdot) \subseteq B_i(\cdot)$. Since i and j are arbitrary, $B_i = B_j$. Conversely, invoking Remark 10A, it follows from the suppositions that each player i is certain of every player j's qualitative-type mapping $t_{B_j} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$.

Lastly, since $B_i = B_I$ for all $i \in I$ and since B_i satisfies Positive Introspection, it follows $B_i = C$ for each $i \in I$.

2. Suppose that the players are commonly certain of the profile of qualitative-type mappings. Since player j is certain of player i's qualitative-type mapping, it follows from Remark 10A that $B_i(\cdot) \subseteq B_jB_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq B_j(\neg B_i)(\cdot)$. Since j is arbitrary, $B_i(\cdot) \subseteq B_IB_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq B_I(\neg B_i)(\cdot)$. Then, $B_i(\cdot) \subseteq CB_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq C(\neg B_i)(\cdot)$. Conversely, it follows from the supposition that $B_i(\cdot) \subseteq CB_i(\cdot) \subseteq B_jB_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq C(\neg B_i)(\cdot) \subseteq B_j(\neg B_i)(\cdot)$. Thus,

invoking Remark 10A, player j is certain of player i's qualitative-type mapping $t_{B_i} : (\Omega, \mathcal{D}) \to (M(\Omega), \mathcal{D}_M)$.

Lastly, since each B_i satisfies Countable Conjunction, C also satisfies it. Since $B_I(\cdot) \subseteq B_i(\cdot) \subseteq CB_i(\cdot)$ for each $i \in I$, $B_I(\cdot) \subseteq \bigcap_{i \in I} CB_i(\cdot) \subseteq CB_I(\cdot)$, where the last set inclusion follows from Countable Conjunction of C. Then, each $B_I(\cdot)$ itself is a publicly-evident event implying the mutual belief, and thus $C = B_I$. □

Proof of Theorem 1B. Suppose that the players are commonly certain of the profile of type mappings. Since player j is certain of player i's type mapping, it follows from Remark 10B that $B^p_{\tau_i}(\cdot) \subseteq B_j B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B_j(\neg B^p_{\tau_i})(\cdot)$. Since j is arbitrary, $B^p_{\tau_i}(\cdot) \subseteq B_I B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B_I(\neg B^p_{\tau_i})(\cdot)$. Then, $B^p_{\tau_i}(\cdot) \subseteq CB^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq C(\neg B^p_{\tau_i})(\cdot)$. Conversely, it follows from the supposition that $B^p_{\tau_i}(\cdot) \subseteq CB^p_{\tau_i}(\cdot) \subseteq B_j B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq C(\neg B^p_{\tau_i})(\cdot) \subseteq B_j(\neg B^p_{\tau_i})(\cdot)$. Thus, player j is certain of player i's type mapping.

Lastly, C^1 satisfies Countable Conjunction because each $B^1_{\tau_i}$ satisfies it. Since $B^1_I(\cdot) \subseteq B^1_{\tau_i}(\cdot) \subseteq C^1 B^1_{\tau_i}(\cdot)$ for each $i \in I$, $B_I(\cdot) \subseteq \bigcap_{i \in I} C^1 B_{\tau_i}(\cdot) \subseteq C^1 B^1_I(\cdot)$, where the last set inclusion follows because C^1 satisfies Countable Conjunction. Then, each $B^1_I(\cdot)$ itself is a publicly-1-evident event implying the mutual 1-belief, and thus $C^1 = B^1_I$. □

Proof of Proposition 4. A. It suffices to prove (Ai). Take $F \in \mathcal{X}$. By assumption, $x^{-1}(F) \subseteq B_i(x^{-1}(F))$. It suffices to show $x^{-1}(F) \subseteq B_j(x^{-1}(F))$.

It follows from Remark 10A and Consistency of B_j that $B_i = B_j B_i$. In fact, $B_i(\cdot) \subseteq B_j B_i(\cdot)$ and $(\neg B_i)(\cdot) \subseteq B_j(\neg B_i)(\cdot)$ follow from Remark 10A. Then, since B_j satisfies Consistency, $B_j B_i(\cdot) \subseteq (\neg B_j)(\neg B_i)(\cdot) \subseteq B_i(\cdot)$.

Take $(F_\lambda)_{\lambda \in \Lambda}$ from $\mathcal{X}$ with $F^c = \bigcup_{\lambda \in \Lambda} F_\lambda$. It can be seen that $\neg x^{-1}(F) = \bigcup_{\lambda \in \Lambda} x^{-1}(F_\lambda)$. Then,

$$\neg x^{-1}(F) = \bigcup_{\lambda \in \Lambda} x^{-1}(F_\lambda) \subseteq \bigcup_{\lambda \in \Lambda} B_i(x^{-1}(F_\lambda)) \subseteq B_i(x^{-1}(F^c)) \subseteq (\neg B_i)(x^{-1}(F)),$$

implying $x^{-1}(F) = B_i(x^{-1}(F))$. It follows from Remark 10A that

$$x^{-1}(F) = B_i(x^{-1}(F)) = B_j B_i(x^{-1}(F)) = B_j(x^{-1}(F)).$$

B. It suffices to prove (Bi). Take $F \in \mathcal{X}$. By assumption, $x^{-1}(F) \subseteq B_i(x^{-1}(F))$. It suffices to show $x^{-1}(F) \subseteq B_j(x^{-1}(F))$. It follows from Theorem 1B and Consistency of B_j that $B^p_{\tau_i} = B_j B^p_{\tau_i}$. Take $(F_\lambda)_{\lambda \in \Lambda}$ from $\mathcal{X}$ with $F^c = \bigcup_{\lambda \in \Lambda} F_\lambda$. It can be seen that $\neg x^{-1}(F) = \bigcup_{\lambda \in \Lambda} x^{-1}(F_\lambda)$. Then,

$$\begin{aligned}\neg x^{-1}(F) = \bigcup_{\lambda \in \Lambda} x^{-1}(F_\lambda) \subseteq \bigcup_{\lambda \in \Lambda} B_i(x^{-1}(F_\lambda)) \subseteq \bigcup_{\lambda \in \Lambda} B^1_{\tau_i}(x^{-1}(F_\lambda)) \\ \subseteq B^1_{\tau_i}(x^{-1}(F^c)) \subseteq (\neg B^1_{\tau_i})(x^{-1}(F)),\end{aligned}$$

implying $x^{-1}(F) = B^1_{\tau_i}(x^{-1}(F))$. Now, it follows from Theorem 1B that

$$x^{-1}(F) = B^1_{\tau_i}(x^{-1}(F)) = B_j B^1_{\tau_i}(x^{-1}(F)) = B_j(x^{-1}(F)).$$

$\square$

A.1.3 Section 5

Proof of Proposition 5. A. By (i) and (iii) and by observing $(\uparrow t_{B_i}(\omega)) = t_{B_i}^{-1}(\{\mu \in M(\Omega) \mid \mu(\cdot) \geq t_{B_i}(\omega)(\cdot)\})$, $\omega \in (\uparrow t_{B_i}(\omega)) \subseteq B_i(\uparrow t_{B_i}(\omega))$. If $(\omega, E) \in \Omega \times \mathcal{D}$ satisfies $\omega \in B_i(E)$, then it follows from Finite Conjunction that $\omega \in B_i(E \cap (\uparrow t_{B_i}(\omega)))$. By Consistency and Monotonicity, $B_i(\emptyset) = \emptyset$, because $B_i(\emptyset) = B_i(E \cap E^c) \subseteq B_i(E) \cap B_i(E^c) = \emptyset$. Then, it must be the case that $E \cap (\uparrow t_{B_i}(\omega)) \neq \emptyset$.

B. Take $E \in \mathcal{D}$ with $\omega \in B_i(E)$. By Entailment, $\omega \in B^1_{\tau_i}(E)$. If $E \cap [\tau_i(\omega)] = \emptyset$ (observe $(\uparrow \tau_i(\omega)) = [\tau_i(\omega)]$), then $[\tau_i(\omega)] \subseteq E^c$. By the Harsanyi property, $\omega \in B^1_{\tau_i}(E^c)$. However, this is a contradiction to Consistency. Thus, $E \cap [\tau_i(\omega)] \neq \emptyset$. $\square$

Proof of Theorem 2. Let $\omega \in B_i(\mathrm{RAT}_i)$. Since player i is certain of her strategy, it follows from $\omega \in [\sigma_i(\omega)]$ that $\omega \in B_i([\sigma_i(\omega)])$. Then, it follows from Finite Conjunction that $\omega \in B_i(\mathrm{RAT}_i \cap [\sigma_i(\omega)])$. Next, since B_i is compatible with informativeness, there is $\omega' \in \Omega$ such that $\omega' \in (\uparrow t_{B_i}(\omega)) \cap \mathrm{RAT}_i \cap [\sigma_i(\omega)]$. Now, suppose to the contrary that $\omega \notin \mathrm{RAT}_i$. Then, there is $a'_i \in A_i$ such that $\omega \in B_i([a'_i \succ_i \sigma_i(\omega)]) = B_i([a'_i \succ_i \sigma_i(\omega')])$, which implies $\omega' \in B_i([a'_i \succ_i \sigma_i(\omega')])$, i.e., $\omega' \notin \mathrm{RAT}_i$. This is a contradiction. Thus, $\omega \in \mathrm{RAT}_i$. $\square$

A.2 Difference between Mutual and Common Certainty at a State

Remark 2 in Section 3.1 states that if every player is certain of the value of a signal $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at every state then the players are commonly certain of the value of x at every state. This appendix shows through an example that the mutual and common certainty may differ if the players are certain of the value of the signal only at some state. This appendix also briefly discusses the higher-order certainty of a signal at a state.

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ with $m \geq 3$, and let $\mathcal{D} = \mathcal{P}(\Omega)$. Introduce the natural order on Ω based on the indices: $\omega_k \leq \omega_\ell$ iff $k \leq \ell$. Define each player's belief operator B_i as follows:

$$B_i(E) := \begin{cases} \emptyset & \text{if } E = \emptyset \\ \{\omega_1\} & \text{if } |E| = 1 \\ E \setminus \{\max E\} & \text{if } |E| \geq 2 \end{cases}.$$

E	$\emptyset$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω
$B_i(E)$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1,\omega_2\}$
$C(E)$	$\emptyset$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\{\omega_1\}$	$\emptyset$	$\{\omega_1\}$

Table 5: Inidvidual and Common Beliefs B_i and C

Then, the common belief operator C is written as:

$$C(E) = \begin{cases} \emptyset & \text{if } E = \emptyset \text{ or } (|E| \geq 2 \text{ and } \omega_1 \notin E) \\ \{\omega_1\} & \text{if } |E| = 1 \text{ or } (|E| \geq 2 \text{ and } \omega_1 \in E) \end{cases}.$$

For example, if $m = 3$ then the individual and common belief operators are depicted in Table 5.

Let $(X, \mathcal{X}) = (\{x_1, x_2\}, \mathcal{P}(X))$, and define $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ as follows:

$$x(\omega) = \begin{cases} x_1 & \text{if } \omega = \omega_1 \\ x_2 & \text{if } \omega \neq \omega_1 \end{cases}.$$

I show (i) that each player i is certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω_2 and (ii) that the players are not commonly certain of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω_2. Observe $F \in \{\{x_2\}, X\}$ satisfies $\omega_2 \in x^{-1}(F)$. Indeed, $x^{-1}(\{x_2\}) = \{\omega_2, \ldots, \omega_m\}$ and $x^{-1}(X) = \Omega$. Then,

$$B_I^k(x^{-1}(\{x_2\})) = \begin{cases} \{\omega_2, \ldots, \omega_{m-k}\} & \text{if } k \leq m-2 \\ \emptyset & \text{if } k > m-2 \end{cases} \text{ and } C(x^{-1}(\{x_2\})) = \emptyset.$$

Also,

$$B_I^k(x^{-1}(X)) = \begin{cases} \{\omega_1, \ldots, \omega_{m-k}\} & \text{if } k \leq m-2 \\ \{\omega_1\} & \text{if } k \geq m-1 \end{cases} \text{ and } C(x^{-1}(X)) = \{\omega_1\}.$$

In fact, one can define higher-order certainty as follows. Player i is *certain that player j is certain of the value of a signal* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at a state $\omega \in \Omega$ if, for any $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$ and for any $E \in \mathcal{D}$ with $x^{-1}(F) \subseteq E$, $\omega \in B_i B_j(E)$.

The players are *mutually certain of the value of* $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω if, for any $F \in \mathcal{X}$ with $\omega \in x^{-1}(F)$ and for any $E \in \mathcal{D}$ with $x^{-1}(F) \subseteq E$, $\omega \in B_I(E)$. One can analogously define higher-order mutual certainty of the value of the signal x.

If the mutual belief operator B_I satisfies Countable Conjunction in addition to Monotonicity, then the players are commonly certain of the value of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω iff they are mutually certain of the value of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω, they are mutually certain that they are mutually certain of the value of $x : (\Omega, \mathcal{D}) \to (X, \mathcal{X})$ at ω, and so forth *ad infinitum*.

A.3 Product Type Spaces

In the product type-space literature, a state space is given as a product type space. Formally, fix a measurable space $(S, \mathcal{S})$ of nature states. A *product type space* is a tuple $\langle (T_i, \mathcal{T}_i)_{i\in I}, (m_i)_{i\in I}\rangle$ such that each $(T_i, \mathcal{T}_i)$ is a measurable space of player i's types and that each $m_i : (T_i, \mathcal{T}_i) \to (\Delta(T_{-i}), (\mathcal{T}_{-i})_\Delta)$ is a measurable mapping, where $T_{-i} = S \times \prod_{j\in I\setminus\{i\}} T_j$, $\mathcal{T}_{-i}$ is the product σ-algebra on T_{-i}, and $(\mathcal{T}_{-i})_\Delta$ is the σ-algebra generated by $\{\mu \in \Delta(T_{-i}) \mid \mu(E) \geq p\}$ for some $(E, p) \in \mathcal{T}_{-i} \times [0, 1]$.

I show that a product type space $\langle (T_i, \mathcal{T}_i)_{i\in I}, (m_i)_{i\in I}\rangle$ is identified as a belief model $\langle (\Omega, \mathcal{D}), (\tau_i)_{i\in I}\rangle$ with certain properties. Let the state space Ω be the product space $\Omega := S \times \prod_{i\in I} T_i$. Let $\mathcal{D}$ be the product σ-algebra on Ω. Define each player i's type mapping $\tau_i : \Omega \to \Delta(\Omega)$ as follows: for each state $(s, (\omega_i)_{i\in I}) \in \Omega$, let $\tau_i(\omega)$ be the product measure $\tau_i(\omega) = m_i(\omega_i) \times \delta_{\omega_i}$ induced by the type $m_i(\omega_i)$ and the Dirac measure δ_{ω_i}. Observe that $m_i : (T_i, \mathcal{T}_i) \to (\Delta(T_{-i}), \mathcal{T}_{-i})$ and $\delta_i : (T_i, \mathcal{T}_i) \ni \omega_i \mapsto \delta(\omega_i) = \delta_{\omega_i} \in (\Delta(T_i), (\mathcal{T}_i)_\Delta)$ are measurable, and hence $m_i \times \delta_i : (T_i, \mathcal{T}_i) \ni \omega_i \mapsto (m_i \times \delta_i)(\omega_i) = m_i(\omega_i) \times \delta_{\omega_i} \in (\Delta(\Omega), \mathcal{D}_\Delta)$ is measurable. Then, $\tau_i : (\Omega, \mathcal{D}) \ni \omega \mapsto \tau_i(\omega) = (m_i \times \delta_i)(\omega_i) \in (\Delta(\Omega), \mathcal{D}_\Delta)$ is measurable.

Conversely, consider a belief model $\langle (\Omega, \mathcal{D}), (\tau_i)_{i\in I}\rangle$ with the following properties: the state space $(\Omega, \mathcal{D})$ is the product measurable space of $(S, \mathcal{S})$ and $((T_i, \mathcal{T}_i))_{i\in I}$; and each $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ satisfies

1. $\tau_i(s, (\omega_j)_{j\in I}) = \tau_i(\tilde{s}, (\omega_i, \tilde{\omega}_{-i}))$ for all $s, \tilde{s}, \omega_i, \omega_{-i}, \tilde{\omega}_{-i}$; and
2. $\tau_i(\omega) \circ \pi_i^{-1} = \delta_{\omega_i}$, where $\pi_i : \Omega \to T_i$ is the projection.

The first property states that τ_i depends only on ω_i. The second property states that the marginal of the player i's type on the player's own type set is the Dirac measure. Then, define $m_i : T_i \to \Delta(T_{-i})$ as $m_i(\omega_i) := \tau_i(\omega) \circ \pi_{-i}^{-1}$, where $\pi_{-i} : \Omega \to T_{-i}$ is the projection. It can be seen that $m_i : (T_i, \mathcal{T}_i) \to (\Delta(T_{-i}), (\mathcal{T}_{-i})_\Delta)$ is measurable.

Now, I formally show that a player is certain of her type mapping m_i in a product type space $\langle (T_i, \mathcal{T}_i)_{i\in I}, (m_i)_{i\in I}\rangle$ in the sense that she is certain of her type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$ in the corresponding belief model $\langle (\Omega, \mathcal{D}), (\tau_i)_{i\in I}\rangle$.

The proof goes as follows. Given a product type space $\langle (T_i, \mathcal{T}_i)_{i\in I}, (m_i)_{i\in I}\rangle$, take the corresponding belief model $\langle (\Omega, \mathcal{D}), (\tau_i)_{i\in I}\rangle$. Since $\tau_i(\omega)(E) \geq p$ implies $\tau_i(\omega)(\{\omega' \in \Omega \mid \tau_i(\omega')(E) \geq p\}) = 1$ and since $\tau_i(\omega)(E) < p$ implies $\tau_i(\omega)(\{\omega' \in \Omega \mid \tau_i(\omega')(E) < p\}) = 1$, it follows that $B^p_{\tau_i}(\cdot) \subseteq B^1_{\tau_i} B^p_{\tau_i}(\cdot)$ and $(\neg B^p_{\tau_i})(\cdot) \subseteq B^1_{\tau_i}(\neg B^p_{\tau_i})(\cdot)$. Hence, Proposition 1B (2c) implies that player i is certain of her own type mapping $\tau_i : (\Omega, \mathcal{D}) \to (\Delta(\Omega), \mathcal{D}_\Delta)$.