Notes on Double Sourcing

An online appendix to "Becoming the Neighbor Bidder: Endogenous Winner's Curse in Dynamic Mechanisms" (2014)

Alejandro Francetich

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In these notes, we consider allocations in period $t = 1, 2$ as profiles of shares $\theta_t := (\theta_{at}, \theta_{bt}) \in [0,1]^2$ such that $\theta_a + \theta_b \leq 1$, where θ_{it} denotes agent i's share. The gathering of proprietary information about the asset is endogenous, and it is linked to the allocation as follows: If the profile of shares is θ_1, each agent observes v, privately and independently, with probabilities θ_{a1}, θ_{b1}, respectively.

I explore efficiency and implementation. The second-period portion of the second-best allocation rule can be implemented by a second-price or English auction. The notes that follow focus on the first-period problem. To simplify notation, I will write $\theta := \theta_1$. Given an allocation rule $\theta : [\underline{w}, \overline{w}]^2 \to [0,1]^2$, total welfare is:

$$S^\omega(\theta) = E\left[V + \theta_a(W_1)W_{a1} + \theta_b(W_1)W_{b1} + \delta S_2^{\omega 0} \right.$$
$$\left. + \delta\left[2\theta_a(W_1)\theta_b(W_1) - \theta_a(W_1) - \theta_b(W_1)\right]\Delta^\omega\right].$$

The constrained-efficient allocation of shares is given by the maximizer of $g(\theta; w_1) := \theta_a w_{a1} + \theta_b w_{b1} + (2\theta_a\theta_b - \theta_a - \theta_b)\delta\Delta^\omega$ on $[0,1]^2$, subject to the restriction that the shares add up to at least one, allowing for the possibility that less than all the shares are allocated: $\theta_a + \theta_b \leq 1$. In particular, we want to allow for trade not to take place, which is equivalent to allocating 0 shares. The maximizer is:

$$\theta^*(w_1) :=$$
$$\begin{cases} (0,1) & \min\{w_{a1}, w_{b1}\} \geq \delta\Delta^\omega,\ w_{a1} - w_{b1} < -2\delta\Delta^\omega; \\ \left(\frac{1}{2} + \frac{w_{a1}-w_{b1}}{4\delta\Delta^\omega}, \frac{1}{2} + \frac{w_{b1}-w_{a1}}{4\delta\Delta^\omega}\right) & \min\{w_{a1}, w_{b1}\} \geq \delta\Delta^\omega,\ |w_{a1} - w_{b1}| \leq 2\delta\Delta^\omega; \\ (1,0) & \min\{w_{a1}, w_{b1}\} \geq \delta\Delta^\omega,\ w_{a1} - w_{b1} > 2\delta\Delta^\omega; \\ (0,1) & w_{a1} < \delta\Delta^\omega < w_{b1}; \\ (1,0) & w_{a1} > \delta\Delta^\omega > w_{b1}; \\ (0,0) & \max\{w_{a1}, w_{b1}\} < \delta\Delta^\omega. \end{cases}$$

If both agents' first-period types are sufficiently low, then no shares at all are allocated. If only one of the agents has a valuation above this threshold, she gets all the shares for the unit. When both agents' types are sufficiently high, then the second-best allocation of shares dictates deviating from the 50-50 split by an amount that is directly proportional to the excess surplus generated by the beneficiary, and inversely proportional to the welfare cost of implementability, Δ^ω. See Figure 1.

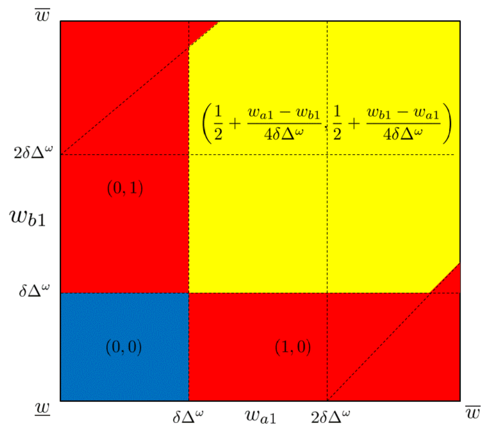

Figure 1: Second-best shares, θ^*.

Given a mechanism $(\theta, \tau) : [\underline{w}, \overline{w}]^2 \to [0,1] \times \mathbb{R}^2$, continuation payoffs are:

$$\begin{aligned}
U_{a2}\left(\widetilde{w}_{a1}, w_{b1}\right) :=\ & S_2^0 + \theta_a\left(\widetilde{w}_{a1}, w_{b1}\right)\left(\overline{S}_2 - S_2^0\right) - \theta_b\left(\widetilde{w}_{a1}, w_{b1}\right)\left(S_2^0 - \underline{S}_2\right) \\
& - \theta_a\left(\widetilde{w}_{a1}, w_{b1}\right)\theta_b\left(\widetilde{w}_{a1}, w_{b1}\right)\left(\overline{S}_2 + \underline{S}_2 - 2S_2^0\right), \\
U_{b2}\left(w_{a1}, \widetilde{w}_{b1}\right) :=\ & S_2^0 + \theta_b\left(w_{a1}, \widetilde{w}_{b1}\right)\left(\overline{S}_2 - S_2^0\right) - \theta_a\left(w_{a1}, \widetilde{w}_{b1}\right)\left(S_2^0 - \underline{S}_2\right) \\
& - \theta_b\left(w_{a1}, \widetilde{w}_{b1}\right)\theta_a\left(w_{a1}, \widetilde{w}_{b1}\right)\left(\overline{S}_2 + \underline{S}_2 - 2S_2^0\right);
\end{aligned}$$

total payoffs are:

$$\begin{aligned}
U_a\left(\widetilde{w}_{a1}, w_{b1}; w_{a1}\right) &:= \theta_a\left(\widetilde{w}_{a1}, w_{b1}\right)\left[E(V) + w_{a1}\right] - \tau_a\left(\widetilde{w}_{a1}, w_{b1}\right) + \delta U_{a2}\left(\widetilde{w}_{a1}, w_{b1}\right), \\
U_b\left(\widetilde{w}_{b1}, w_{a1}; w_{b1}\right) &:= \theta_b\left(w_{a1}, \widetilde{w}_{b1}\right)\left[E(V) + w_{b1}\right] - \tau_b\left(w_{a1}, \widetilde{w}_{b1}\right) + \delta U_{b2}\left(\widetilde{w}_{b1}, w_{a1}\right).
\end{aligned}$$

We want to identify a mechanism that implements θ^*. Truthful continuation payoffs are given by:

$$\begin{aligned}
U_{a2}\left(w_1\right) =\ & S_2^0 + \theta_a^*\left(w_1\right)\left(\overline{S}_2 - S_2^0\right) - \theta_b^*\left(w_1\right)\left(S_2^0 - \underline{S}_2\right) \\
& - \theta_a^*\left(w_1\right)\theta_b^*\left(w_1\right)\left(\overline{S}_2 + \underline{S}_2 - 2S_2^0\right), \\
U_{b2}\left(w_1\right) =\ & S_2^0 + \theta_b^*\left(w_1\right)\left(\overline{S}_2 - S_2^0\right) - \theta_a^*\left(w_1\right)\left(S_2^0 - \underline{S}_2\right) \\
& - \theta_b^*\left(w_1\right)\theta_a^*\left(w_1\right)\left(\overline{S}_2 + \underline{S}_2 - 2S_2^0\right).
\end{aligned}$$

In the truthful equilibrium, we have:

$$\begin{aligned}
U_a\left(w_1\right) &:= U_a\left(w_{a1}, w_{b1}; w_{a1}\right) = \theta_a^*\left(w_1\right)\left[E(V) + w_{a1}\right] - \tau_a\left(w_1\right) + \delta U_{a2}\left(w_1\right), \\
U_b\left(w_1\right) &:= U_b\left(w_{b1}, w_{a1}; w_{b1}\right) = \theta_b^*\left(w_1\right)\left[E(V) + w_{b1}\right] - \tau_b\left(w_1\right) + \delta U_{b2}\left(w_1\right).
\end{aligned}$$

By the envelope formula for payoffs, we have:

$$\begin{aligned}
U_a\left(w_1\right) &= U_a\left(\underline{w}, w_{b1}\right) + \int_{\underline{w}}^{w_{a1}} \theta_a^*(\epsilon, w_{b1})\lambda(d\epsilon), \\
U_b\left(w_1\right) &= U_b\left(\underline{w}, w_{a1}\right) + \int_{\underline{w}}^{w_{b1}} \theta_b^*(\epsilon, w_{a1})\lambda(d\epsilon).
\end{aligned}$$

Combining these two ways to express payoffs gives a formula for expected transfers $\tau_a^*(w_1)$ that implement the constrained-efficient share-allocation rule:

$$\begin{cases} 0 & w_{a1} < \delta\Delta^\omega; \\ 0 & \min\{w_{a1}, w_{b1}\} \geq \delta\Delta^\omega, \\ & w_{a1} < w_{b1} - 2\delta\Delta^\omega; \\ \theta^*(w_1)\left[E(V) + \frac{w_{a1}+w_{b1}}{2} + \delta\left(S_2^0 - \underline{S}_2 - \Delta^\omega\right) + \theta^*(w_1)\delta\left(\overline{S}_2 + \underline{S}_2 - 2S_2^0\right)\right] & \min\{w_{a1}, w_{b1}\} \geq \delta\Delta^\omega, \\ & |w_{a1} - w_{b1}| \leq 2\delta\Delta^\omega; \\ E(V) + w_{b1} + \delta\left(\overline{S}_2 - S_2^0\right) & \min\{w_{a1}, w_{b1}\} \geq \delta\Delta^\omega, \\ & w_{a1} > w_{b1} + 2\delta\Delta^\omega; \\ \delta\left(\overline{S}_2 - S_2^0 + \Delta^\omega\right) & w_{a1} \geq \delta\Delta^\omega, \\ & w_{b1} < \delta\Delta^\omega; \end{cases}$$

for b, we have $\tau_b^*(w_{a1}, w_{b1}) := \tau_a^*(w_{b1}, w_{a1})$.

The second-best allocation rule can be implemented by means of the following dynamic auction. As before, the second-period unit is allocated by means of a second-price or English auction. In the first period, bidders are asked to pay deposits r^0 each, to participate. These deposits are returned either totally or partially according to the outcome of the auction, and also act as floors for admissible bids.

In the event in which both bidders participate, the outcome is as follows. If the profile of bids is $\beta := (\beta_a, \beta_b)$, if $\beta_a \vee \beta_b > \beta_a \wedge \beta_b + 2\delta\Delta^\omega$, the highest bidder wins and pays her opponent's bid net of the deposit; the loser gets her deposit back and ends up empty-handed. If $|\beta_a - \beta_b| \leq 2\delta\Delta^\omega$, then bidder a receives $q(\beta) := \frac{1}{2} + \frac{\beta_a - \beta_b}{4\delta\Delta^\omega}$ shares, while bidder b gets $1 - q(\beta)$ shares. If bidder i receives q shares, she pays

$$\tau(q, \beta) := q\left[\frac{\beta_a + \beta_b}{2} + q\delta\left(\overline{S}_2 + \underline{S}_2 - 2S_2^0\right) - \delta\left(\Delta^\omega + \overline{S}_2 + \underline{S}_2 - 2S_2^0\right)\right]$$

minus the deposit. See Figure 2.

Under τ, for each of her shares, an agent pays a non-linear price. This price is the average bid, minus a discount, plus an extra charge that is proportional to the number of shares already allocated to the agent. By allocating an additional share to an agent, we are subtracting this marginal share from her opponent. Moreover, larger shares make it more for likely for an agent and, simultaneously, less likely for her opponent, to have access to v.

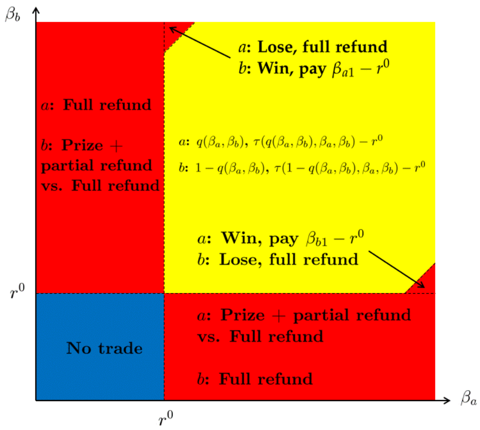

Figure 2: Second-best auction, $t = 1$.

Theorem 1. *Define* $r^0 := \delta\left[\Delta^\omega + \Delta^S\right]$ *and* $r^1 := r^0 - \delta\left(S_2^0 - \underline{S}_2\right)$. *The second-best allocation rule can be implemented by the following sequential auctions. In the first period, bidders are asked to pay (simultaneously) a deposit of* r^0. *If only one bidder pays, she is given the option to get the full first-period unit and a partial refund on her deposit of* $r^0 - r^1 = \delta\left(S_2^0 - \underline{S}_2\right)$, *or to get the full refund. If both pay, they (simultaneously) submit bids. Admissible bids are not lower than* r^0. *If the difference between the bids is at least* $2\delta\Delta^\omega$, *the highest bidder wins the full unit and pays the difference between the lowest bid and* r^0; *the loser gets her deposit back. If the bids are within* $2\delta\Delta^\omega$ *of each other, then bidder* a *receives* $q(\beta) := \frac{1}{2} + \frac{\beta_a - \beta_b}{4\delta\Delta^\omega}$ *shares, while bidder* b *gets* $1 - q(\beta)$ *shares, where* $\beta = (\beta_a, \beta_b)$ *is the profile of bids; payments are, respectively,* $\tau(q(\beta), \beta)$ *and* $\tau(1 - q(\beta), \beta)$. *The first-period unit is withheld if no bidder pays the deposit.*

***Proof*.** Let $h^{(i)}$ denote the history in which agent i, and only agent i, has observed v. Similarly, let $h^{(0)}$ denote the history in which neither has observed v, while $h^{(*)}$ represents the history under which both agents observe v.

Consider bidder i's problem of type w_{i1}, w_{i2}, if bidder $-i$ of type w_{-i1}, w_{-i2} adopts the following strategy:

- At history $h^{(0)}$, bid $w_{-i2} + E(V)$.
- At history $h^{(*)}$, bid u_{-i2}.
- At history $h^{(-i)}$, if v is the realized common component, bid $u_{-i2} = v + w_{i2}$.
- At history $h^{(i)}$, bid $h^{-1}(w_{-i2})$.
- If both bidders have paid the deposit, bid $w_{-i1} + E(V) + \delta\Delta^S$.
- If only $-i$ has paid the deposit, accept if $w_{-i1} + E(V) + \delta\overline{S}_2 - r^1 \geq \delta S_2^0$.
- Pay the deposit if $w_{-i1} \geq \underline{w}^* := \delta\Delta^\omega - E(V)$.

As first-period signals are uninformative of v, beliefs for agent i at histories $h^{(0)}$ and h^{-i} are given by the priors. At histories $h^{(0)}$ and $h^{(*)}$, the symmetric equilibrium in the second-price auction has both bidders bidding their (expected) valuations. Straightforward bidding gives a bid of $u_{i2} = w_{i2} + v$ at history $h^{(i)}$, while bidder i bids $h^{-1}(w_{i2})$ at history $h^{(-i)}$.

Turn to the first period. If bidder i is the only one who paid the deposit, she will be offered the first-period unit and the rebate. This offer is accepted by all types w_{i1} such that $w_{i1} + E(V) + \delta\overline{S}_2 - r^1 \geq \delta S_2^0$, or, equivalently, $w_{i1} \geq \delta\Delta^\omega - E(V) = \underline{w}^*$.

When both bidders are active in the auction, the payoff to type w_{i1} of i when she bids b and $-i$ bids b' is:

$$
s_{i1}(b, b'; w_{i1}) = \delta\underline{S}_2 +
\begin{cases}
0 & b < b' - 2\delta\Delta^\omega \\
\left(\frac{1}{2} + \frac{b-b'}{4\delta\Delta^\omega}\right)\left[E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2 + \Delta^\omega\right) - \frac{b+b'}{2}\right] & |b - b'| \leq 2\delta\Delta^\omega \\
E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2\right) - b' & b > b' + 2\delta\Delta^\omega.
\end{cases}
$$

On the range $[b' - 2\delta\Delta^\omega, b' + 2\delta\Delta^\omega]$, the payoff function is a strictly concave function of bidder i's bid, with first and second derivatives given by:

$$
\frac{\partial s_{i1}(b, b'; w_{i1})}{\partial b} = \frac{E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2\right)}{4\delta\Delta^\omega} - \frac{b}{4\delta\Delta^\omega},
$$

$$\frac{\partial^2 s_{i1}(b, b'; w_{i1})}{\partial b^2} = -\frac{1}{4\delta\Delta^\omega}.$$

If $E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2\right) > b' + 2\delta\Delta^\omega$, $s_{i1}(b, b'; w_{i1})$ attains its maximum on the range $b > b' + 2\delta\Delta^\omega$; by the usual argument for second-price auctions, a weakly best response is for bidder i to bid exactly $E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2\right)$. Now, if $E(V)+w_{i1}+\delta\left(\overline{S}_2 - \underline{S}_2\right) < b'-2\delta\Delta^\omega$, then $s_{i1}(b, b'; w_{i1})$ is non-positive; any bid below $b' - 2\delta\Delta^\omega$, such as $E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2\right)$, is a best response. Finally, if $E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2\right)$ lies in $[b' - 2\delta\Delta^\omega, b' + 2\delta\Delta^\omega]$, $s_{i1}(b, b'; w_{i1})$ attains its unique maximizer at $E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2\right)$. Hence, any type w_{i1} finds bidding $E(V) + w_{i1} + \delta\left(\overline{S}_2 - \underline{S}_2\right)$ a weakly dominant strategy.

Finally, consider the participation problem. Start with the case $w_{i1} < \underline{w}^*$. If $w_{-i1} < \underline{w}^*$ and bidder i pays the deposit, she will be the only one to do so and can get the first-period unit for a net payment of r^1, her outside option being δS_2^0. However, this is not profitable: $w_{i1} + E(V) + \delta\overline{S}_2 - r^1 < \delta S_2^0$. An opponent of type $w_{-i1} \geq \underline{w}^*$ would pay the deposit and take the offer if unopposed. Hence, i is better off not paying the deposit:

$$w_{i1} + E(V) + \delta\overline{S}_2 - \beta_1(w_{-i1}) \leq w_{i1} + E(V) + \delta\overline{S}_2 - r^0 < \delta\underline{S}_2.$$

In either case, type $w_{i1} < \underline{w}^*$ cannot do better than not paying the deposit.

If $w_{i1} \geq \underline{w}^*$ and her opponent is of type $w_{-i1} < \underline{w}^*$, she will be the only bidder in the auction, should she choose to participate. She can get the first-period unit for a net payment of r^1, which yields a payoff of at least δS_2^0. This payoff is exactly her outside option, so she cannot profit by withholding the deposit. When her opponent's type is also above $\underline{w}^*$, her outside option is $\delta\underline{S}_2$. If she participates in the auction, both bidders will be present and the (interim) payoff to i will be $\max\{w_{i1} - w_{-i1}, 0\} + \delta\underline{S}_2 \geq \delta\underline{S}_2$. □